The first derivative proof of ***b^^x* , *x^^x*** and ***x^^f(x)*** by differentiation fundamental limits method.

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ABSTRACT

This paper is to find by proof the first derive of known tetration functions, fixed base iterated functions *b^^x* , general case for *b^^f(x)* and variable base with variable height iterated function *x^^x*. although the case of *b^^x* is already known by using the base change method but its derive function *f(x)* is still depend on the derive of *f(x-1)* which gives a shortcoming derivation. However, in the coming proofs, the resulted derivative functions are proved by applying differentiation elementary concepts step by step up to the final first derive ,but an unknown limit and a non-elementary product part of the resulted derivative function still needs study, Although I included approximation method for numerical solutions.

INTRODUCTION

Tetration derivation is still difficult process, it is more complicated than any other exponential function, nevertheless here is a method to determine the first derive of the mentioned functions in the title, their proof will depend on differentiation elementary sequence starting from delta x which refers to a very small amount approaching zero. Each proof is written separately but the second and third proof will point out to a certain steps in the first proof as a part of its evidence to avoid repeating. There are also some important points that should be carefully red so that proofs steps would be clearly understood.

IMPORTANT INPUTS

Here are some tetration properties, the logarithm limit and functions as shorthand which have to be considered before starting any of the next proofs.

Properties of tetration:

For base $b > 0$ in all cases.

1. $exp_b^{x+z}(1) = exp_b^x(exp_b^z(1))\ \ x, z\ are\ real$
2. $exp_b^0(a) = a$

3. $exp_b^x(1) = b^{exp_b^{x-1}(1)}\ for\ all\ real\ x$
4. $\lim_{\Delta x \to 0}(exp_b^{\Delta x}(1)) = 1$
5. $\lim_{\Delta x \to 0}(exp_b^{\Delta x-1}(1)) = 0$
6. $exp_b^{x-1}(1) = \frac{\ln(exp_b^x(1))}{\ln(b)}\ for\ all\ real\ x$

The logarithm limit:

It is known that natural logarithm can be expressed as:

$$\ln(a) = \lim_{\Delta x \to 0} \frac{(a)^{\Delta x} - 1}{\Delta x} \qquad (i)$$

From property No. 5 ,natural logarithm can be also shown as:

$$\ln(a) = \lim_{\Delta x \to 0} \frac{(a)^{exp_b^{\Delta x-1}(1)} - 1}{exp_b^{\Delta x-1}(1)} \qquad (ii)$$

which can also be expressed as:

$$\lim_{\Delta x \to 0} (a)^{exp_b^{\Delta x-1}(1)} = \lim_{\Delta x \to 0} exp_b^{\Delta x-1}(1) * \ln(a) + 1 \qquad (iii)$$

Functions as shorthand:

Here is an assumed iterated function and a super exponential limit (as shorthand) :

$$P_b(a) = \prod_{n=1}^{a} exp_b^n(1)\ ,when\ only\ \ a\ \in \mathbb{N}\ \ ,otherwise\ approximation\ will\ take\ place. (iv)$$

$$R_b(a) = \lim_{\Delta x \to 0, a \to 0} \frac{\ln(exp_b^a(1))}{\Delta x} \qquad (v)$$

In fact, $R_b(a)$ results from the following limit:

$$\lim_{\Delta x \to 0, a \to 0} \frac{exp_b^{a-1}(1)}{\Delta x} \qquad (vi)$$

The above limit will appear during proofs steps.

From tetration property No. 6, I obtain.

$$\lim_{\Delta x \to 0, a \to 0} \frac{exp_b^{a-1}(1)}{\Delta x} = \lim_{\Delta x \to 0, a \to 0} \frac{\frac{\ln\left(exp_b^a(1)\right)}{\ln(b)}}{\Delta x} = \lim_{\Delta x \to 0, a \to 0} \frac{\ln\left(exp_b^a(1)\right)}{\Delta x \ln(b)} = \frac{R_b(a)}{\ln(b)} \qquad (vii)$$

$P_b(a)$ and $R_b(a)$ are not solvable by elementary methods, however approximation solutions will be discussed after the next proofs…

DERIVE PROOF OF *B*^^*X*

The first proof will be for

$$y = exp_b^x(1) \;\; , b > 0 \tag{1}$$

By adding a very small value Δx to x will lead to:

$$y + \Delta y = exp_b^{x+\Delta x}(1) \tag{2}$$

From equation No. (1) and equation No. (2) I obtain:

$$\frac{\Delta y}{\Delta x} = \frac{exp_b^{x+\Delta x}(1) - exp_b^x(1)}{\Delta x} \tag{3}$$

Applying property No. 1.

$$\frac{\Delta y}{\Delta x} = \frac{exp_b^x\left(exp_b^{\Delta x}(1)\right) - exp_b^x(1)}{\Delta x} \tag{4}$$

Applying property No. 3.

$$\frac{\Delta y}{\Delta x} = \frac{exp_b^x\left(b^{exp_b^{\Delta x-1}(1)}\right) - exp_b^x(1)}{\Delta x} \tag{5}$$

By applying logarithm equation (*iii*) in equation No. *5*. :(let $a = b$)

$$\frac{\Delta y}{\Delta x} = \frac{exp_b^x\left(exp_b^{\Delta x-1}(1) * \ln(b) + 1\right) - exp_b^x(1)}{\Delta x} \tag{6}$$

$$\frac{\Delta y}{\Delta x} = \frac{exp_b^{x-1+1}\left(exp_b^{\Delta x-1}(1) * \ln(b) + 1\right) - exp_b^x(1)}{\Delta x} \tag{7}$$

from property No. 1 I get:

$$\frac{\Delta y}{\Delta x} = \frac{exp_b^{x-1}\left(b^{exp_b^{\Delta x-1}(1)*\ln(b)+1}\right) - exp_b^x(1)}{\Delta x} \tag{8}$$

$$\frac{\Delta y}{\Delta x} = \frac{exp_b^{x-1}\left(b * b^{exp_b^{\Delta x-1}(1)*\ln(b)}\right) - exp_b^x(1)}{\Delta x} \tag{9}$$

$$\frac{\Delta y}{\Delta x} = \frac{exp_b^{x-1}\left(b * \left(b^{\ln(b)}\right)^{exp_b^{\Delta x-1}(1)}\right) - exp_b^x(1)}{\Delta x} \tag{10}$$

By applying logarithm limit No.(*iii*) method.

$$\frac{\Delta y}{\Delta x} = \frac{exp_b^{x-1}\left(b(exp_b^{\Delta x-1}(1) * \ln\left(b^{\ln(b)}\right) + 1)\right) - exp_b^x(1)}{\Delta x} \tag{11}$$

$$\frac{\Delta y}{\Delta x} = \frac{exp_b^{x-1}\left(b * exp_b^{\Delta x-1}(1) * (\ln(b))^2 + b\ \right) - exp_b^x(1)}{\Delta x} \tag{12}$$

$$\frac{\Delta y}{\Delta x} = \frac{exp_b^{x-2}\left(b^{b*exp_b^{\Delta x-1}(1)*(\ln(b))^2+b}\right) - exp_b^x(1)}{\Delta x} \tag{13}$$

Repeating the same steps from equation No. (8) to (13)

$$\frac{\Delta y}{\Delta x} = \frac{exp_b^{x-2}\left(b^{(b\,(\ln(b))^2*exp_b^{\Delta x-1}(1)+b)}\right) - exp_b^x(1)}{\Delta x} \tag{14}$$

$$\frac{\Delta y}{\Delta x} = \frac{exp_b^{x-2}\left(b^b * b^{(b\,(\ln(b))^2*exp_b^{\Delta x-1}(1))}\right) - exp_b^x(1)}{\Delta x} \tag{15}$$

$$\frac{\Delta y}{\Delta x} = \frac{exp_b^{x-2}\left(b^b * \left(b^{b\,(\ln(b))^2}\right)^{exp_b^{\Delta x-1}(1)}\right) - exp_b^x(1)}{\Delta x} \tag{16}$$

$$\frac{\Delta y}{\Delta x} = \frac{exp_b^{x-2}\left(b^b * (exp_b^{\Delta x-1}(1) * \ln\left(b^{b\,(\ln(b))^2}\right) + 1)\ \right) - exp_b^x(1)}{\Delta x} \tag{17}$$

$$\frac{\Delta y}{\Delta x} = \frac{exp_b^{x-2}\left(b^b * (exp_b^{\Delta x-1}(1) * b\,(\ln(b))^3 + 1)\ \right) - exp_b^x(1)}{\Delta x} \tag{18}$$

$$\frac{\Delta y}{\Delta x} = \frac{exp_b^{x-2}\left(exp_b^{\Delta x-1}(1) * b\,b^b\,(\ln(b))^3 + b^b\ \right) - exp_b^x(1)}{\Delta x} \tag{19}$$

Repeat again.

$$\frac{\Delta y}{\Delta x} = \frac{exp_b^{x-3}\left(b^{exp_b^{\Delta x-1}(1)*b\,b^b\,(\ln(b))^3+b^b}\ \right) - exp_b^x(1)}{\Delta x} \tag{20}$$

$$\frac{\Delta y}{\Delta x} = \frac{exp_b^{x-3}\left(b^{b^b} b^{exp_b^{\Delta x-1}(1)*b\,b^b\,(\ln(b))^3}\ \right) - exp_b^x(1)}{\Delta x} \tag{21}$$

By repeating this process x times it will give:

$$\frac{\Delta y}{\Delta x} = \frac{exp_b^{x-x}\left(exp_b^x(1) * b^{exp_b^{\Delta x-1}(1)\prod_{n=1}^{x-1}(exp_b^n(1))\,(\ln(b))^x}\ \right) - exp_b^x(1)}{\Delta x} \tag{22}$$

$$\frac{\Delta y}{\Delta x} = \frac{exp_b^0\left(exp_b^x(1) * \ \left(b^{\prod_{n=1}^{x-1}(exp_b^n(1))\,(\ln(b))^x}\right)^{exp_b^{\Delta x-1}(1)}\ \right) - exp_b^x(1)}{\Delta x} \tag{23}$$

By applying equation No.(*iii*)

$$\frac{\Delta y}{\Delta x} = \frac{exp_b^0\left(exp_b^x(1)(exp_b^{\Delta x-1}(1) * \ln\left(b^{\prod_{n=1}^{x-1}(exp_b^n(1))\,(\ln(b))^x}\right) + 1)\ \right) - exp_b^x(1)}{\Delta x} \tag{24}$$

$$\frac{\Delta y}{\Delta x}=\frac{exp_b^0\left(exp_b^x(1)\left(exp_b^{\Delta x-1}(1)*\prod_{n=1}^{x-1}(exp_b^n(1))\,(\ln(b))^{x+1}+1\right)\right)-exp_b^x(1)}{\Delta x} \tag{25}$$

applying property No. 2.

$$\frac{\Delta y}{\Delta x}=\frac{exp_b^x(1)((exp_b^{\Delta x-1}(1)*\prod_{n=1}^{x-1}(exp_b^n(1))\,(\ln(b))^{x+1}+1)-exp_b^x(1)}{\Delta x} \tag{26}$$

$$\frac{\Delta y}{\Delta x}=\frac{exp_b^x(1)*exp_b^{\Delta x-1}(1)*\prod_{n=1}^{x-1}(exp_b^n(1))\,(\ln(b))^{x+1}+exp_b^x(1)-exp_b^x(1)}{\Delta x} \tag{27}$$

$$\frac{\Delta y}{\Delta x}=\frac{exp_b^x(1)*exp_b^{\Delta x-1}(1)*\prod_{n=1}^{x-1}(exp_b^n(1))\,(\ln(b))^{x+1}}{\Delta x} \tag{28}$$

$$\frac{\Delta y}{\Delta x}=exp_b^x(1)*\prod_{n=1}^{x-1}(exp_b^n(1))\,(\ln(b))^{x+1}*\frac{exp_b^{\Delta x-1}(1)}{\Delta x} \tag{29}$$

$$\frac{\Delta y}{\Delta x}=\left(\prod_{n=1}^{x}exp_b^n(1)\right)(\ln(b))^{x+1}*\frac{exp_b^{\Delta x-1}(1)}{\Delta x} \tag{30}$$

$$\lim_{\Delta x\to 0}\frac{exp_b^{\Delta x-1}(1)}{\Delta x}=\frac{R_b(\Delta x)}{\ln(b)}\qquad ,see\ equation\ No.(vii)$$

Finally I have:

$$y'=P_b(x)*\ln(b)^x*R_b(\Delta x) \tag{31}$$

DERIVE PROOF OF *X*^^*X*

The second proof will be for

$$y=exp_x^x(1)\ \ ,x>0 \tag{1}$$

And by adding a very small value Δx to x will lead to:

$$y+\Delta y=exp_{x+\Delta x}^{x+\Delta x}(1) \tag{2}$$

$$\frac{\Delta y}{\Delta x}=\frac{exp_{x+\Delta x}^{x+\Delta x}(1)-exp_x^x(1)}{\Delta x} \tag{3}$$

From property No. 1.

$$\frac{\Delta y}{\Delta x}=\frac{exp_{x+\Delta x}^{x}\left(exp_{x+\Delta x}^{\Delta x}(1)\right)-exp_x^x(1)}{\Delta x} \tag{4}$$

let $(x+\Delta x)=b$ then I can continue with same steps in the previous proof up to step No. 27.

Hence.

$$\frac{\Delta y}{\Delta x} = \frac{exp_{x+\Delta x}^{x}(1) * exp_{x+\Delta x}^{\Delta x-1}(1) * \prod_{n=1}^{x-1}(exp_{x+\Delta x}^{n}(1))\,(\ln(x+\Delta x))^{x+1} + exp_{x+\Delta x}^{x}(1) - exp_{x}^{x}(1)}{\Delta x} \quad (5)$$

$$\frac{\Delta y}{\Delta x} = \frac{exp_{x+\Delta x}^{\Delta x-1}(1) * \prod_{n=1}^{x}(exp_{x+\Delta x}^{n}(1))\,(\ln(x+\Delta x))^{x+1} + exp_{x+\Delta x}^{x}(1) - exp_{x}^{x}(1)}{\Delta x} \quad (6)$$

$$\frac{\Delta y}{\Delta x} = \left(\prod_{n=1}^{x} exp_{x+\Delta x}^{n}(1)\right)(\ln(x+\Delta x))^{x+1} * \frac{exp_{x+\Delta x}^{\Delta x-1}(1)}{\Delta x} + \frac{exp_{x+\Delta x}^{x}(1) - exp_{x}^{x}(1)}{\Delta x} \quad (7)$$

$$y' = \left(\prod_{n=1}^{x} exp_{x}^{n}(1)\right)(\ln(x))^{x+1} * \lim_{\Delta x\to 0}\frac{exp_{x+\Delta x}^{\Delta x-1}(1)}{\Delta x} + \lim_{\Delta x\to 0}\frac{exp_{x+\Delta x}^{x}(1) - exp_{x}^{x}(1)}{\Delta x} \quad (8)$$

The following limit,

$$\lim_{\Delta x\to 0}\frac{exp_{x+\Delta x}^{x}(1) - exp_{x}^{x}(1)}{\Delta x} \quad (a)$$

Is the same differentiation limit of fixed height power tower.

The fixed height here $= x$

So, from fix height derivation I find.

$$\lim_{\Delta x\to 0}\frac{exp_{x+\Delta x}^{x}(1) - exp_{x}^{x}(1)}{\Delta x} = \frac{1}{x} * \sum_{m=1}^{x}\left(lin(x)^{m-1} * \prod_{k=0}^{m} exp_{x}^{x-k}(1)\right) \quad (b)$$

And also the by substituting following limit:

$$\lim_{\Delta x\to 0}\frac{exp_{x+\Delta x}^{\Delta x-1}(1)}{\Delta x} = \frac{R_{x+\Delta x}(\Delta x)}{\ln(x)} \qquad , see\ equation\ No.(vii)$$

Finally I have.

$$y' = P_x(x)\,(\ln(x))^{x} * R_{x+\Delta x}(\Delta x) + \frac{1}{x} * \sum_{m=1}^{x}\left(lin(x)^{m-1} * \prod_{k=0}^{m} exp_{x}^{x-k}(1)\right) \quad (9)$$

Another simplified formula

$$y' = \left(exp_{b}^{x}(1)\right)' * \frac{R_{x+\Delta x}(\Delta x)}{R_x(\Delta x)} + \left(exp_{x}^{b}(1)\right)' \qquad replace\ b\ with\ x\ after\ derivation. \quad (10)$$

GENERAL CASE FOR *B^^F(X)*

The third proof ,

By using the above methods , derive of general case of *b^^f(x)* can be found as below:

$$y = exp_b^{f(x)}(1) \ \ , b > 0 \tag{1}$$

Which its derive limit is:

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{exp_b^{f(x+\Delta x)}(1) - exp_b^{f(x)}(1)}{\Delta x} \tag{2}$$

Let

$$f(x + \Delta x) = \ f_1(x) + \ f_2(x, \Delta x) \tag{3.1}$$

Provided that

$$\lim_{\Delta x \to 0} f_2(x, \Delta x) = 0 \tag{3.2}$$

Let

$$z = f_1(x) \ \ , \Delta z \ = f_2(x, \Delta x) \tag{3.3}$$

Hence.

$$\frac{\Delta y}{\Delta x} = \frac{exp_b^{z+\Delta z}(1) - exp_b^{z}(1)}{\Delta x} \tag{4}$$

It is the same step No. 3. in the previous proof of *b^^x* and also the same procedure can be applied up to step No. 30.

$$\frac{\Delta y}{\Delta x} = \left(\prod_{n=1}^{z} exp_b^{n}(1)\right)(\ln(b))^{z+1} * \frac{exp_b^{\Delta z-1}(1)}{\Delta x} \tag{5}$$

then

$$\frac{\Delta y}{\Delta x} = \left(\prod_{n=1}^{f_1(x)} exp_b^{n}(1)\right)(\ln(b))^{f_1(x)+1} * \frac{exp_b^{f_2(x,\Delta x)-1}(1)}{\Delta x} \tag{6}$$

And also the by substituting following limit:

$$\lim_{\Delta x \to 0} \frac{exp_b^{f_2(x,\Delta x)-1}(1)}{\Delta x} = \ \frac{R_b\big(f_2(x, \Delta x)\big)}{\ln(b)} \qquad , see\ equation\ No.(vii)$$

Finally I have.

$$y' = P_b\big(f_1(x)\big) * \ln(b)^{f_1(x)} * R_b\big(f_2(x, \Delta x)\big) \tag{7}$$

Examples for the general case:

1) $y = exp_b^{x^n}(1)$, $b > 0$

Which its derive limit is:

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{exp_b^{(x+\Delta x)^n}(1) - exp_b^{x^n}(1)}{\Delta x}$$

$$(x + \Delta x)^n = x^n + \frac{nx^{n-1}\Delta x}{1!} + \frac{n(n-1)x^{n-2}\Delta x^2}{2!} + \cdots + \Delta x^n$$

Let

$$f_1(x) = x^n \ and \ \ f_2(x, \Delta x) = \frac{nx^{n-1}\Delta x}{1!} + \frac{n(n-1)x^{n-2}\Delta x^2}{2!} + \cdots + \Delta x^n$$

$\lim_{\Delta x \to 0} f_2(x, \Delta x) = \lim_{\Delta x \to 0}(\frac{nx^{n-1}\Delta x}{1!} + \frac{n(n-1)x^{n-2}\Delta x^2}{2!} + \cdots + \Delta x^n) = 0$,So it is ok.

From the general case formula:

$$y' = P_b\big(f_1(x)\big) * \ln(b)^{f_1(x)} * R_b\big(f_2(x, \Delta x)\big)$$

The first derive of $exp_b^{x^n}(1)$ will be:

$$y' = P_b(x^n) * \ln(b)^{x^n} * R_b\left(\frac{nx^{n-1}\Delta x}{1!} + \frac{n(n-1)x^{n-2}\Delta x^2}{2!} + \cdots + \Delta x^n\right)$$

A Second example,

2) $y = exp_b^{\sin x}(1)$, $b > 0$

Which its derive limit is:

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{exp_b^{\sin(x+\Delta x)}(1) - exp_b^{\sin x}(1)}{\Delta x}$$

$$\sin(x + \Delta x) = \sin x \cos \Delta x + \cos x \sin \Delta x = \sin x \,(1 - 2\sin^2(\frac{\Delta x}{2})\,) \ + \cos x \sin \Delta x$$

$$\sin(x + \Delta x) = \sin x - 2 \sin x \sin^2(\frac{\Delta x}{2}) \ + \cos x \sin \Delta x$$

Let

$$f_1(x) = \sin x \ \ and \ f_2(x, \Delta x) = \cos x \sin \Delta x - 2 \sin x \sin^2(\frac{\Delta x}{2})$$

$\lim_{\Delta x\to 0} f_2(x,\Delta x) = \lim_{\Delta x\to 0}(\cos x \sin\Delta x - 2\sin x \sin^2(\frac{\Delta x}{2})) = 0$,So it is ok.

From the general case formula:

$$y' = P_b\big(f_1(x)\big) * \ln(b)^{f_1(x)} * R_b\big(f_2(x,\Delta x)\big)$$

Then the first derive of $exp_b^{\sin x}(1)$ is:

$$y' = P_b(\sin x) * \ln(b)^{\sin x} * R_b\left(\cos x \sin\Delta x - 2\sin x \sin^2(\frac{\Delta x}{2})\right)$$

APPROXIMATION SOLUTION FOR $P_b(a)$ AND $R_b(a)$

The goal of approximation in this paper is just to show that it is possible to find approximation formulas for $P_b(x)$ and $R_b(x)$ although the approximations here does not have a precise numerical calculations or relationships proofs. But we should know that these approximation are not almost useful especially in mathematical proofs at last.

1) Approximating $P_b(x)$ for non-integer values of (x)

$$P_b(x) = \prod_{n=1}^{x} exp_b^n(1)$$

$$let\ x = u + v\ , where\ u \in \mathbb{N}^0\ , 0 < v < 1$$

$$P_b(x) = \prod_{n=1}^{u+v} exp_b^n(1)$$

$$exp_b^{u+v}(1) = exp_b^v\big(exp_b^u(1)\big) \approx \big(exp_b^v(1)\big)^{exp_b^u(1)}$$

$$P_b(x) = \prod_{n=1}^{u+v} exp_b^n(1) \approx \big(exp_b^v(1)\big)^{exp_b^u(1)} \prod_{n=1}^{u} exp_b^n(1)$$

The approximation is exactly equal to $P_b(x)$ for $\min(v)$ and $\max(v)$ as below:

For $v = 0$,then $x = u + v = u$

$$\big(exp_b^v(1)\big)^{exp_b^u(1)} = \left(exp_b^0(1)\right)^{exp_b^u(1)} = (1)^{exp_b^u(1)} = 1$$

For $v = 1$,then $x = u + v = u + 1$

$$\big(exp_b^v(1)\big)^{exp_b^u(1)} = \left(exp_b^1(1)\right)^{exp_b^u(1)} = (b)^{exp_b^u(1)} = exp_b^{u+1}(1)$$

2) Approximating $R_b(x)$ limit.

$$R_b(\Delta x) = \lim_{\Delta x \to 0} \frac{\ln\left(exp_b^{\Delta x}(1)\right)}{\Delta x}$$

Applying l'Hospital's rule

$$\lim_{\Delta x \to 0} \frac{\ln\left(exp_b^{\Delta x}(1)\right)}{\Delta x} = \frac{(\ln\left(exp_b^{x}(1)\right))'}{(x)'} = \frac{\frac{\left(exp_b^{x}(1)\right)'}{exp_b^{x}(1)}}{1} = \frac{\left(exp_b^{x}(1)\right)'}{exp_b^{x}(1)}$$

As known already that

$$exp_b^{x}(1) = 1\ , when\ x = 0$$

$$\left(exp_b^{x}(1)\right)' = P_b(x) * \ln(b)^x * R_b(\Delta x)$$

$$\left(exp_b^{0}(1)\right)' = P_b(0) * \ln(b)^0 * R_b(\Delta x)$$

Then

$$\lim_{\Delta x \to 0} \frac{\ln\left(exp_b^{\Delta x}(1)\right)}{\Delta x} = \frac{P_b(0) * \ln(b)^0 * R_b(\Delta x)}{exp_b^{0}(1)} = R_b(\Delta x)$$

But l'Hospital's rule led us to the first step so it would not help but anyway it verifies that the relationship is correct.

Anyway, there are many approximation methods for tetration, but here I will use linear one as it is a simple and clear method:

On the interval [-1, 0], we have $exp_b^{x}(1) \approx x + 1$. This approximation is continuous everywhere, but generally not differentiable at integers.

In this case I assume that Δx is a very small amount less than zero so that it will be inside the provided interval.

By substituting Δx to x

$$\lim_{\Delta x \to 0} \frac{\ln\left(exp_b^{\Delta x}(1)\right)}{\Delta x} \approx \lim_{\Delta x \to 0} \frac{\ln(\Delta x + 1)}{\Delta x}$$

By applying l'Hospital's rule again on the elementary limit I finally obtain.

$$\lim_{\Delta x \to 0} \frac{\ln\left(exp_b^{\Delta x}(1)\right)}{\Delta x} \approx 1$$

RESULTS

Derive of *b^^x* , *x^^x* and the general case of *x^^f(x)* are shown per the following in order:

$$(exp_b^x(1))' = P_b(x) * \ln(b)^x * R_b(\Delta x)$$

$$\left(exp_x^x(1)\right)' = P_x(x) * \ln(x)^x * R_{x+\Delta x}(\Delta x) + \frac{1}{x} * \sum_{m=1}^{x}\left(lin(x)^{m-1} * \prod_{k=0}^{m} exp_x^{x-k}(1)\right)$$

$$(exp_b^{f(x)}(1))' = P_b\big(f_1(x)\big) * \ln(b)^{f_1(x)} * R_b\big(f_2(x,\Delta x)\big)$$

Where,

1. $f(x+\Delta x) = f_1(x) + f_2(x,\Delta x)$
2. $\lim_{\Delta x \to 0} f_2(x,\Delta x) = 0$
3. $P_b(a) = \prod_{n=1}^{a} exp_b^n(1)$ $, when\ only\ a \in \mathbb{N}$
4. $R_b(a) = \lim_{\Delta x \to 0, a \to 0} \frac{\ln(exp_b^a(1))}{\Delta x}$

the above $P_b(a)$ and $R_b(a)$ assumed formulas are important parts of the derivative functions, but they are not yet in the simple form that gives clear output values.

A concluded result from the first proof that gives a new relation between the $R_b(a)$ limit and the derive of $exp_b^0(1)$ as below:

It is already known that the first derive of $exp_b^x(1)$ is

$$\left(exp_b^x(1)\right)' = \left(exp_b^0(1)\right)' P_b(x)\ln(b)^x$$

So, from the first proof it is clear that is:

$$\left(exp_b^0(1)\right)' = \lim_{\Delta x \to 0} \frac{\ln(exp_b^{\Delta x}(1))}{\Delta x}$$

CONCLUSION

It was Essentially that this paper focused only on the previous three proofs to confirm that it is possible to determine the first derive from elementary solutions of differentiation fundamentals.

The proofs give the possibilities of defining the derive of many tetration functions by using differentiation elementary concepts and set up new differentiation lows for those functions.

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