

Determining of and finding patterns in n -periodic points of the $\exp()$ -function

Abstract: it is known that for any natural number $n \geq 1$ there are infinitely many n -periodic points [B1990]. To answer some question in the math-forum MSE [He20c] I needed numerical examples of such n -periodic points with some given properties. From my little overview in the literature in that field I didn't find much helpful information, so I developed my own method to easily find n -periodic points [He20a].

The key is not to employ the (iterated) $\exp()$ -function itself (and/or its Newton-iteration) but to employ the $\log()$ -function instead and explicitly provide a branchindex k , and for iterated $\log()$ to provide a fixed vector \mathbf{K} of such branch indexes. With this it is possible to apply only iteration $z_{j+1} \leftarrow \log(z_j, \mathbf{K})$ (not necessarily polished by a following Newton-iteration) to find n -periodic points of any order n . Moreover, it seems, the sets P_n of the n -periodic points are exactly indexable by the vectors \mathbf{K}_n such that each fixed- or n -periodic point is uniquely defined by the vectors $\mathbf{K}_n = [k_1, k_2, \dots, k_n]$ (if all $k_i = 0$ there are the two conjugate 1-periodic points to be taken, see below).

*Without proof, I **conjecture** that this index addresses all existing n -periodic points, up to conjugacy of the primary fixed point..*

The work is a hobby-exploration, the author is a retired lecturer for statistics in quantitative social research and has been an active amateur in number-theory in various mathematical forums over many years, especially on the problem of fractional iteration of the exponential function ("tetration").

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1. The indexing of n -periodic points by iterated branched logarithm

1.1. Notations for iteration of $\exp()$ and $\log()$.

We denote $f: z \rightarrow \exp(z)$, and in the context of iteration we define & write

$$\begin{aligned} f^{\circ 0}(z) &:= z \\ f^{\circ 1}(z) &:= f(z) \\ f^{\circ h+1}(z) &:= f^{\circ h}(f^{\circ 1}(z)) \end{aligned}$$

We denote $g: z \rightarrow \log(z)$, and for supplying a branchindex $k \in \mathbb{Z}$ we define & write

$$g(z, k) := \log(z) + k \cdot \omega \quad \text{where} \quad \omega = 2\pi i$$

... for vector of branches

$$\begin{aligned} g(z, [k_1, k_2]) &:= \log(\log(z) + k_1 \cdot \omega) + k_2 \cdot \omega \\ g(z, [k_1, k_2, \dots, k_n]) &:= \log(\dots \log(\log(z) + k_1 \cdot \omega) + k_2 \cdot \omega) \dots) + k_n \cdot \omega \end{aligned}$$

... for iteration given one full branch-vector $\mathbf{K}_n = [k_i]_{i=1..n}$ of length n

$$\begin{aligned} g^{\circ 1}(z, \mathbf{K}_n) &:= g(z, \mathbf{K}_n) \quad \text{where} \quad \mathbf{K}_n = [k_1, k_2, \dots, k_n] \\ g^{\circ h+1}(z, \mathbf{K}_n) &:= g^{\circ h}(g^{\circ 1}(z, \mathbf{K}_n), \mathbf{K}_n) \end{aligned}$$

1.2. "Fixed points", "1-periodic points"

a) Notations and some known properties

If $z = f(z)$ then z is called a fixed-point, or better for this treatize a "1-periodic point".

For the text here, I denote them as p_1 or more precisely with $p_{1:k}$, where k is the branch-index such that

$$\begin{aligned} p_{1:k} &= f(p_{1:k}) \\ p_{1:k} &= g(p_{1:k}, k) \end{aligned}$$

and use

$$\mathcal{P}_1 := \{p_{1:k}\}_{k=-\infty.. \infty}$$

for the set of all 1-periodic points

Theorems in literature -as I have found it so far- are concerned with the question of existence, cardinality of the set of fixed points \mathcal{P}_1 , as well as about attraction and repulsion of its elements $p_{1:k}$. It is known (for instance by [B1990], pg 156), that

- f has infinitely many fixed points, and that all of them are complex
- the infinite set of fixed points \mathcal{P}_1 is countable
- all fixed points are repelling/"repulsive" over iteration of f .

b) How to find/how to index: the Lambert W -function

The standard tool to actually find some fixed point p_1 is nowadays surely the Lambert W -function, basically

$$p_1 = \exp(-W(-1))$$

Note: the "1" occurs as log of the base e of the exponentiation

Moreover **all** p_1 -points can be found using the Lambert W providing a branch-index $k^* \in \mathbb{Z}$

$$p_{1:k^*} = \exp(-W_{k^*}(-1)) \quad \text{where } k^* \in \mathbb{Z}$$

Note: notation for the branched W taken from Corless et. al. [CK1996]

The infinitude of the fixed points and relation to the index k^* of the branch in the W -function allows an indexed notation for all fixed points (or 1-periodic points) in this type in the analogous way as in \mathbf{a})

$$\mathcal{P}_1 := \{ p_{1:k^*} \}_{k^*=-\infty..00}$$

Note: unfortunately the indexes k and k^* are slightly different by the convention in the Lambert W -notation: the ambiguous $k=0$ is separated in $k^*=0$ or $k^*=1$, and for all other values we have for $k < 0$ $k^* = -k$ and for $k > 0$ $k^* = -k - 1$. However, the Lambert W -indexing shall not be of concern in the following because we don't use the Lambert W anymore. For completeness I've appended a comparing table in Appendix 4.1.

c) How to find/how to index : fixpoint-iteration over branched logarithm

Instead of using the W -function one can use simple fixpoint-iteration over the inverse function to f , because a fixed point which is repelling over iteration of f is attracting over iteration of its inverse g .

So we have above defined basically

$$g(z) := \log(z) \quad \text{which is also } = f^{-1}(z)$$

The logarithm in the complex numbers is multivalued; to make $g(z)$ a fully usable inverse to $f(z)$ we need to explicitly denote the index k for the branch used:

$$g(z, k) := \log(z) + k \cdot \omega \quad \text{where } \omega = i 2 \pi$$

and can then, for some z and the principal branch $k=0$ write

$$g(f(z), 0) = z$$

However, for some other $z = z' + k \cdot \omega$ we have to code the branch-index

$$g(f(z), k) = g(f(z' + k \cdot \omega), k) = g(f(z'), k) = z' + k \cdot \omega = z$$

which fully recovers the underlying value.

Note: perhaps a nicer and full exposition of this in Sykora [Syk16].

It is known, that for the exponential-function f all fixpoints are repelling (again for instance reported in [B1990]), thus fixpoint iteration on an initial value z_0 over the logarithm (principal branch) as well as over the logarithm with explicit branch-index is *attractive*. Moreover, it seems that the *basin of attraction* $p_{n:k}$ is the complex plane \mathbb{C} with some set of exceptions (moreover per branchindex k). For instance, if $k=0$, then the exceptions are $z_0 \notin \{0, 1, e, e^e, \dots\}$.

So the iteration with some chosen index k

$$\begin{aligned} z_0 &= 1+i && // \text{initializing with some value} \\ z_{j+1} &= g(z_j, k) && // \text{iterating} \\ p_{1:k} &= \lim_{j \rightarrow \infty} z_j \end{aligned}$$

approximates an 1-periodic point.

The infinite set \mathcal{P}_1 of 1-periodic points $p_{1:k}$ can now as well be said to be indexed by the branchindex $k \in \mathbb{Z}$.

1.3. n -periodic points

The idea of the "fixpoint-iteration using the branched logarithm" for finding and even for indexing of the 1-periodic points shall now be generalized to n -periodic points with $n \geq 2$.

For the sets of n -periodic points \mathcal{P}_n we know, again for instance by [B1990], that

for each $n \geq 2$

- the set \mathcal{P}_n is infinite
- all $p_{n:k} \in \mathcal{P}_n$ are repelling (over iteration of f)

We assume in the following: because all periodic points are repelling over iteration on f , we'll have again, that

- all periodic points are attracting over iteration on $g(\cdot)$.
- the basin of attraction for all n is \mathcal{C} in the same way as with the 1-periodic points.

To express the various branch indexes as iteration goes towards n we expand the definition for $g()$ one more time such that we allow a vector $\mathbf{K}_n = [k_1, k_2, \dots, k_n]$, $k_i \in \mathcal{Z}$ instead of a single branchindex k :

$$\begin{aligned} g(z, [k_1, k_2]) &:= g(g(z, k_1), k_2) && = \log(\log(z) + k_1 \cdot w) + k_2 \cdot w \\ g(z, [k_1, k_2, \dots, k_n]) &:= g(\dots g(g(z, k_1), k_2), \dots, k_n) \end{aligned}$$

Since we know that all periodic points must be attractive, we can always do:

$$\begin{aligned} z_0 &= \text{<some initial value>} \\ z_{j+1} &= g(z_j, \mathbf{K}_n) && // \text{iteration towards } n\text{-periodic point} \\ p_{n:k} &= \lim_{j \rightarrow \infty} z_j \end{aligned}$$

This way we can safely approximate the n -periodic point $p_{n:k}$ iteratively $j+1$ times up to z_j . If we append then a Newton-iteration on the function $g(z, \mathbf{K}_n)$ we get quadratic approximation rate towards $p_{n:k}$ to arbitrary precision.

The set of 2-periodic points for instance is then

$$\mathcal{P}_2 = \{p_{2:[k_1, k_2]}\}_{k_1, k_2 \in \mathcal{Z}}$$

By heuristics it seems, that this description of \mathcal{P}_2 indeed captures the whole set of 2-periodic points; each index \mathbf{K}_2 gave one 2-periodic point, and there were none else found when other methods were tried.

Note: I tried manual screening of a square area in \mathcal{C} around the origin in small steps using Newton-iteration over f and all so found n -periodic points had a valid \mathbf{K}_n -index.

Note: Y. Galidakis in a 2005-article described a method for finding p_2 -fixpoints with a function called "HyperW()" or "HW()" essentially searching for roots of the polynomials of the truncated power-series for $f(f(z)) - z$. Again all 2-periodic points found with this method had valid \mathbf{K}_2 -indexes.

Note: I worked with other bases for the exponential function instead of $e = \exp(1)$. I found generally results in the same style, but sometimes with few exceptions in small n and small k_i , for instance for base $b=i$ the vector $\mathbf{K}_3 = [0, 0, 0]$ gave an additional primitive 3-periodic cycle; $\mathbf{K}_3 = [-1, 0, 0]$ allows two different cycles, both attracting for $g(\cdot, \mathbf{K})$. The latter occurs the similar way for base $b=-1+i$; for another base $\mathbf{K}_3 = [1, 0, 0]$ didn't define a 3-periodic point, but $\mathbf{K}_6 = [1, 0, 0, 1, 0, 0]$ defined a primitive 6-periodic point instead. That examples are rare, but I've not yet even a usable part of a systematic table.

Anyway, none of such exceptions occurred with the base e as I discuss it in this treatise.

1.4. Conjecture about indexing the n -periodic points

My conjecture by this observations is the following:

For base $e=\exp(1)$ we have

- 1) All n -periodic points can uniquely be approximated by fixpoint iteration over $g(\mathbf{K}_n)$ with branch indexes \mathbf{K}_n independently of the initial value z_0 .

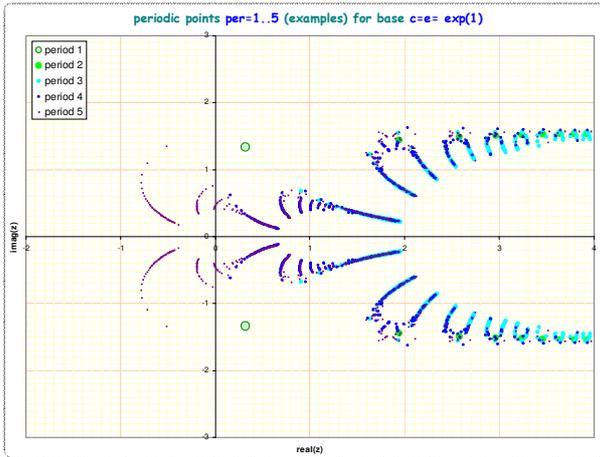
This has the exception for $\mathbf{K}_n=[0],[0,0],\dots$ where z_0 and $\text{conj}(z_0)$ taken as initial value lead to the conjugate primary fixpoints $p_{1:[0]}$ and $\text{conj}(p_{1:[0]})$ respectively.

- 2) the set of n -periodic fixpoints \mathbf{P}_n is in bijection with the set of vectors \mathbf{K}_n , and may be called "indexed by \mathbf{K}_n " - with the only exception of \mathbf{K}_n as mentioned in 1)

There are a lot of more -but rather secondary- observations worth to be made into conjectures. Some examples with graphic visualizations are given in part 2.

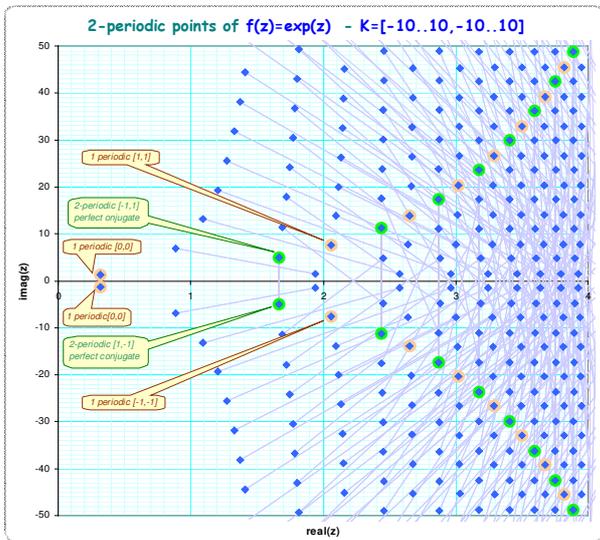
Gottfried Helms, May-Jun'2020

2. Graphical display of some interesting observations



Pic. 0: This is a picture of the periodic points which I found by screening the square around the origin of the complex plane for z in $-4.4 + -4..+4 \cdot I$ in steps of $1/20$. The initial values z were used for Newton-iteration over $f^{o1}(z)$, $f^{o2}(z)$, $f^{o3}(z)$, $f^{o4}(z)$ and $f^{o5}(z)$. The usable found values of the n -periodic points are plotted (only that which fit in the given box). The colors in the plot are indicating the period lengths.

I checked all usable periodic points and found that all had valid K -indexes.

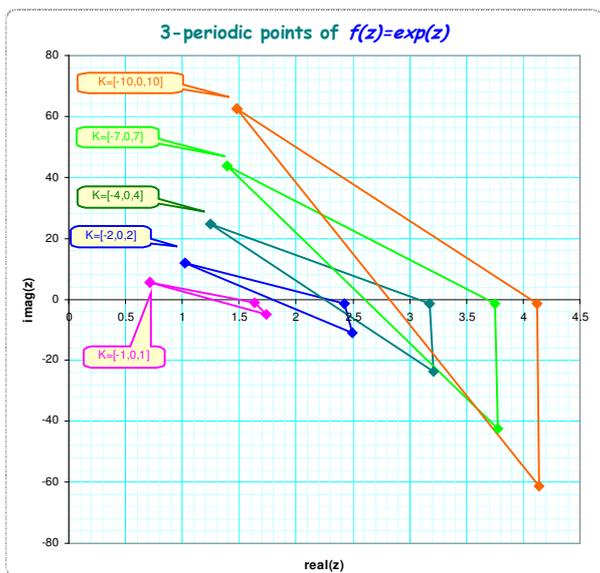


Pic. 1: These are 2-periodic points with indexes $K=[-10..10,-10..10]$ found using the fixpoint iteration as described in 1.1.3 and 1.2. (The picture was cutted to size $real(p_{2:k})=0..5$ and $imag(p_{2:k})=-50..50$ to keep readability).

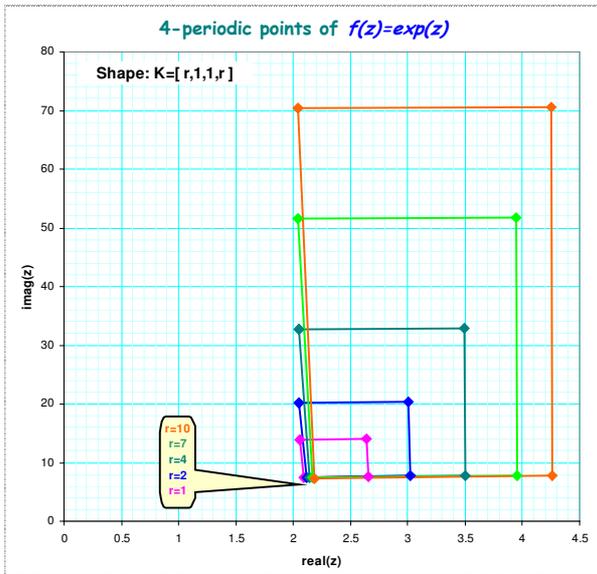
We see, that for $K_2=[k_1,k_2]$ with $k_1=k_2$ we have 1-periodic points and for $k_1=-k_2$ we get 2-periodic points with the two perfect conjugate values in one cycle.

For $[k_1,k_1]$ and $[-k_1,-k_1]$ we get 1-periodic points which are conjugates of each other.

If $k_1=0$ then $-k_1=k_1$ and the conjugate of the primary fixed point must be found by using conjugate initial values for the fixpoint-iteration.

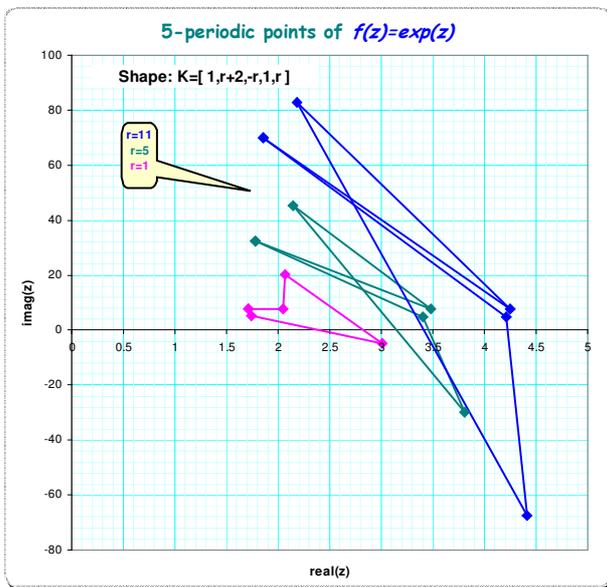


Pic. 2: For display of some examples of 3-periodic points (found according to 1.2) I thought it might be interesting to show various exemplars of a *shape-family*. With this I mean that the index-vector K is modified in a way which keeps its internal structure rather intact, for instance scaling all but one of its entries by an overall scaling. (The opposite: keeping all but one constant can be seen in the picture for the 31-periodic points below)

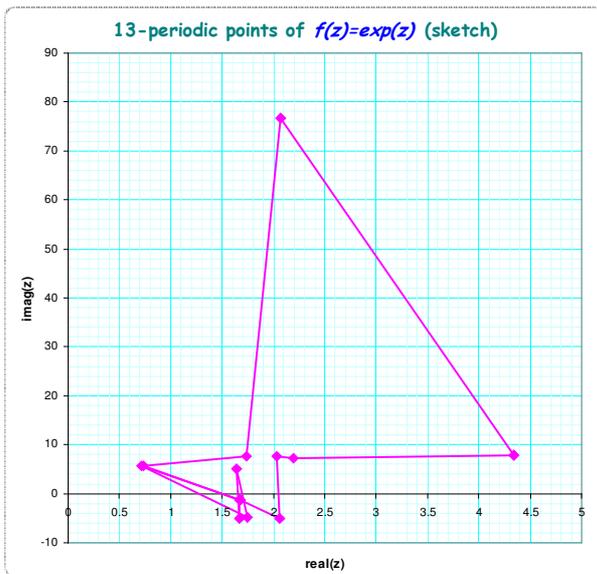


Pic. 3: A similar idea of showing a shape-family with variation in one subset of indexes k .

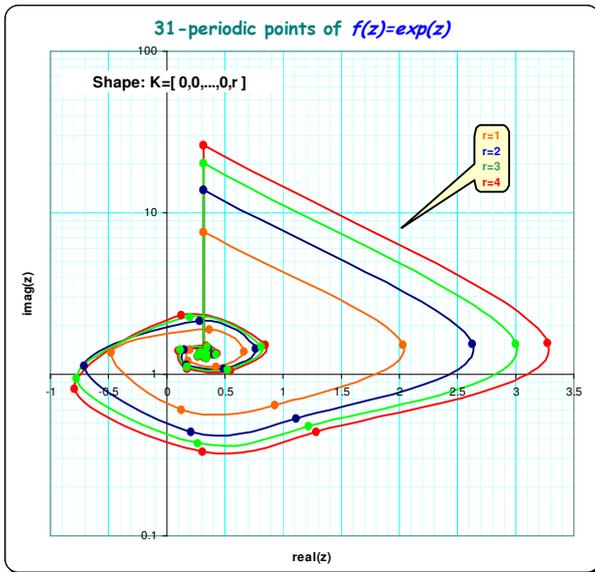
This plot shows 5 exemplars of some 4-periodic points with a certain shape.



Pic. 4: Similarly for some exemplars of 5-periodic points.



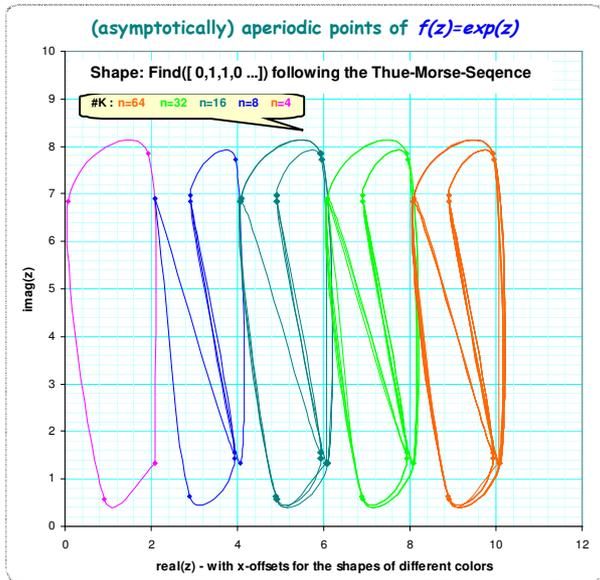
Pic. 5: See that the method can easily find n -periodic points with another n : $n=13$



Pic. 6: An especially interesting shape is the one of $K_{31}=[0,0,\dots,0,r]$ because this is nearly the fix-point iteration towards the primary fixed-point. Only that - after the well known attracting orbit has arrived near the fixed point - the iteration jumps by the parameter $k_{31}=r$ back to the eccentric initial point.

Having a period of length 31 and a visible spiral towards the primary fixed point, this picture gives a very instructive view into the internal mechanics of the branched iterated logarithm.

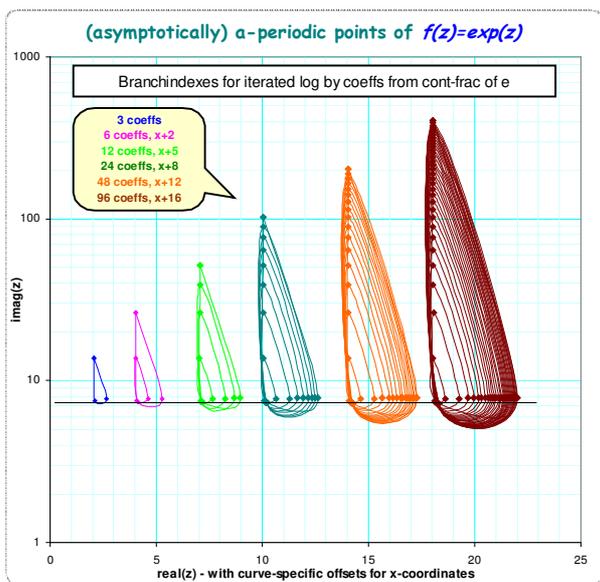
Outlook: A-periodic points?



Pic. 7: Having seen that it is easily possible to find 31-periodic points (and even 255-periodic points) simply by blindly increasing the length of the index-vector K I was interested, how the approximation to some infinite, **aperiodic** vector K would evolve. One very well known infinite sequence - only using numbers 0 and 1, being very naturally distributed, and being still aperiodic- is the Thue-Morse sequence.

Here are the n -periodic points defined by K of increasing length where K contains the leading n numbers from the Thue-Morse sequence.

The shapes overlay very dense, so I add some x -offset to each shape to avoid this overlap. It seems, that increasing the length of K (by doubling) the basic shape stays constant and only refines at the edges with very small disturbances.



Pic. 8: The indexes k_i in K in the previous example are only from the set $\{0,1\}$. To see one example, where the indexes grow unboundedly (but in a tame pattern) I used the coefficients from the continued fraction of $\exp(1)$. Two of three stay constant at 1, and one of three grows unboundedly.

The members of the shape-family have a very common form, only the size of the shape grows with the length of the K -index.

Closing words:

The latter two examples of approaching a-periodic points is so far only experimental and I don't have further ideas how to make something out of it.

3. Literature and links

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- [Syk16] Sykora, S.: Fixed points of the mappings $\exp(z)$ and $-\exp(z)$ in \mathbb{C}
 Stan's Library, Vol.VI, Oct 2016,
 DOI: [10.3247/SL6Math16.002](https://doi.org/10.3247/SL6Math16.002)
<http://ebyte.it/library/docs/math16/FixedPointsExp.html> .

Links to some author's questions and answers in math.stackexchange.com and mathoverflow.net

- [He20a] Helms, Gottfried: How to find examples of periodic points of the (complex) exponential-function $z \mapsto \exp(z)$? - *presenting the initial idea*
 URL (version: 2020-05-14): <https://math.stackexchange.com/q/3674391>
 (<https://math.stackexchange.com/users/1714/gottfried-helms>)
- [He20b] Helms, Gottfried: Is my ansatz for finding n -periodic-points of the exponential-function exhaustive? - *and asking for help for proving*
 URL (version: 2020-06-21): <https://mathoverflow.net/q/361136>
 (<https://mathoverflow.net/users/7710/gottfried-helms>)
- [He20c] Helms, Gottfried: Is there a way to calculate the zeros of $f(z,w) = w - z^{z^w}$? - Answer 1
 URL (version: 2020-06-08): <https://math.stackexchange.com/q/3707910>
 (<https://math.stackexchange.com/users/1714/gottfried-helms>)
- [He20d] Helms, Gottfried: Is there a way to calculate the zeros of $f(z,w) = w - z^{z^w}$? - Answer 2
 URL (version: 2020-06-19): <https://math.stackexchange.com/q/3713978>
 (<https://math.stackexchange.com/users/1714/gottfried-helms>)

4. Appendix

4.1. Relation between the indexing by Lambert W and by iterated branched logarithm

Table 1: fixed points $p_{w:k^*} = \exp(p_{w:k^*})$ according to the branches in the Lambert- W -function and the branchindexes k for the iterated branched logarithm

k^*	$p_{w:k^*} = p_{1:k}$	k
...
-5	$3.28777+26.5805*I$	4
-4	$3.02024+20.2725*I$	3
-3	$2.65319+13.9492*I$	2
-2	$2.06228+7.58863*I$	1
-1	$0.31813+1.33724*I$	0
0	$0.31813-1.33724*I$	0
1	$2.06228-7.58863*I$	-1
2	$2.65319-13.9492*I$	-2
3	$3.02024-20.2725*I$	-3
4	$3.28777-26.5805*I$	-4
5	$3.49852-32.8807*I$	-5
...

Let us use $k^*=-4$ and thus $z_0=p_{w:-4} \sim 3.02+20.27 \hat{i}$. Then we have

$$\log(z_0)+k \cdot w = z_0 \quad \text{\textbackslash\textbackslash by fixed point property}$$

$$k = (z_0 - \log(z_0))/w$$

$$k = 3$$

$$\implies g(z_0, 3) = z_0$$

$$\implies z_0 = p_{1:[3]} \text{ is fixed point with index } \mathbf{K}_1=[3]$$