Composition, bullet notation and the general role of categories: *the softest introduction ever made*

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Jan 30 2021, Composition, bullet notation and the general role of categories: the softest introduction ever made

*Exchange had at the Tetration forum. Thread n° XXX - **"Composition, bullet notation and the general role of categories: the softest introduction ever made"***

MphLee, 30 January 2021

> But if I did all this bullet stuff with ∘-that's not really how ∘ is usually used, so I'd be overriding the meaning of an existent symbol within this context. Better to use a new symbol and be fresh. This is especially beneficial when we talk about $ds \bullet z$ which is almost like a differential form. Writing $ds \circ z$ would be going a step too far I think.

It is clear to me where you are coming from. Your solution is pleasant, pretty, comfortable and the notation, as usually happens with good notation, hints at new developments, e.g. the differential forms. Even if I don't get n-forms yet and exterior algebra feels alien to me I feel it's a similarity worth considering: for this and another reason, I like your choice.

But said that you shouldn't be overly confident about the variable vs function distinction:

$$f \circ g \circ z$$

> Wtf is that nonsense? lol

What nonsense is this? It's abstract nonsense!

In category theory, a land where only composition and arrows make the whole ontology, writing $f \circ g \circ z$ makes perfect sense, not only that, it means exactly what you expect it should.

More than that: **I claim that the natural home for general iterated compositions are categories!** In the last part I'll offer moral reasons for that but first let's inspect your "nonsensical" composition.

About evaluation. In general categories morphisms are just abstract arrows, not functions, and evaluation of them doesn't generally make sense because not every object can be conceived as a bag of something, e.g. points. The philosophy of category theory is exactly this: ignore what's inside, the inner structure of things, and solely observe how your things interact with each other.

There are some very special categories where objects are indeed made of points, e.g. the category of topological spaces, of vector spaces, of abelian groups or the category of bare sets: with this I mean that some categories have among all the objects a special "point object" $*$.

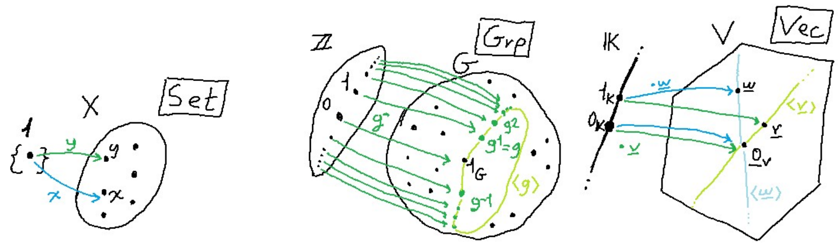
In these particular categories a morphism from this ***objectified abstract point*** to an ***arbitrary object*** X can be thought as (a choice of) ***a point*** x in X

$$* \xrightarrow{x} X$$

therefore defining the set points of X to be the (hom-)set of arrows

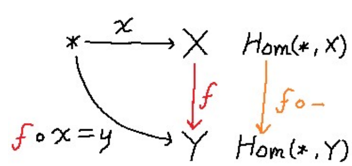
$$\text{Points}(X) \coloneqq \text{Hom}(*, X) = \{x \,|\, * \xrightarrow{x} X\}$$

For example for a bare set X we have $X^1 \simeq X$ where 1 is a singleton; the set of group homomorphisms $\text{Hom}(\mathbb{Z}, G) \simeq G$ is isomorphic to the set of group's elements; the linear applications from a field $\mathbb{K}$ to a vector space V are in bijection with vectors of V, i.e. $\text{Hom}(\mathbb{K}, V) \simeq V$



and all of this without actually being able to look inside our objects ***nor knowing what set membership is***!

In the case you make, given an abstract arrow $X \xrightarrow{f} Y$ and a point $* \xrightarrow{x} X$ we can evaluate f at x composing the two and producing a new point $* \xrightarrow{y} Y$

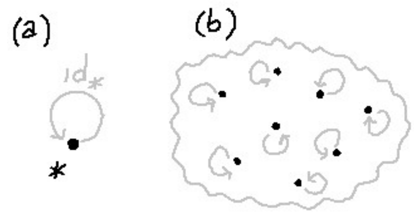


On categories: the softest introduction. Habitually, the prototypical example of category made is "sets and set functions". I believe this is not a wise example at all!

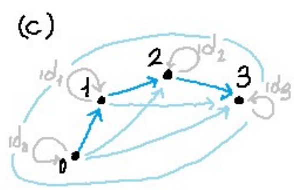
In a category we have a bunch of points (called objects) and a bunch of arrows. Arrows are "things" of which:

1. we can say from where to where they are going;
2. are closed, i.e. if there are two consecutive arrows $x \to y \to z$ then there must exist a third arrow $x \to z$ and we can point to it;
3. for every point (object) there exists a special loop arrow $\underset{\circlearrowleft}{x}$ that when concatenated with the others behaves like an identity.

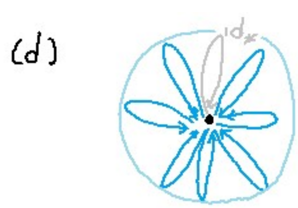
A category is just a bunch of arrows between a bunch of points. So when you think of a category here are some good examples to build an intuition on:



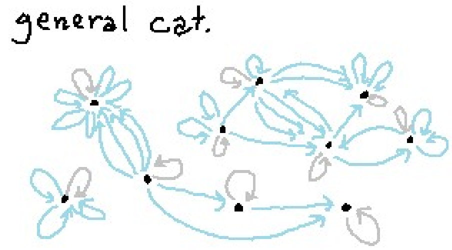
A single point (a) is the simplest category: you can think of it like **pure absolute being**. If you have only points but no arrows except for the identities (b) you just have **pure discreteness**: nothing moves and nothing becomes anything. That is just a bare set where every point is just itself, without any shape or structure.



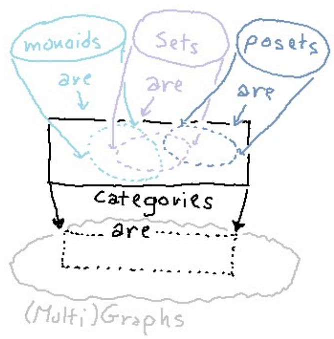
In (c) we have just a finite ordinal number, i.e. a bunch of points and a relation of order between them: the existence of composition express transitivity of the order; the existence of identity express reflexivity.



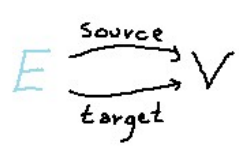
Here things become interesting! When you have only one point and many arrows (d) you have pure "motion"! We have many ways in which the arrows go from the point to itself, think of them like modes of transforming the object in itself, like symmetries. Closure under composition means we obtain a magma, associativity of composition express being a semigroup and existence of identities implies we have a monoid;



As we have seen sets are merely a static kind of categories; monoids (thus groups) are just categories with a single point; and ordered sets are just "slim" categories where there exists at most a single arrow between two points. So in general a category is much like a special (multi)graph, special because edges are closed under concatenation and every vertex has a personal loop-edge.



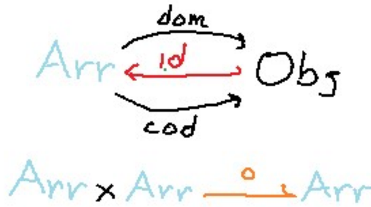
Definiton of multigraph A multigraph is just a set E of edges, a set V of vertexes and two function $s, t : E \to V$ called source and target.



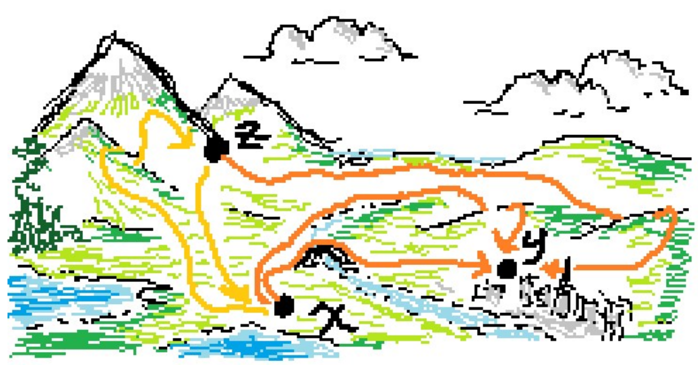
Definiton of category A category is just a special multigraph where we call the edges Arr(-ows), the vertices Ob(-jects) and we have further structure:

- an injection map of objects into arrows $\mathrm{id}_{\_} : \mathrm{Ob} \to \mathrm{Arr}$ called identity;

- a partial binary associative maps on arrows Arr × Arr ⇀ Arr called composition.



So another possibility is to think of categories like partially defined monoids with multiple identity elements. It's evident now that the standard example of a category is the most involved one: we take as objects the class of models of a mathematical theory and as arrows the model-theoretic homomorphisms between them. But the most natural example is our world (space): where every point (position) is an object and an arrow is just a route, i.e. path, from a point to another:



I hope I've provided enough motivation for the claim made at the beginning. One day I could continue with functors and natural transformations.