Let $\mathcal{C}$ be a category. $\mathbb{N}$ be the category generated by $\{\cdot \circlearrowleft\}$.

Let $\mathcal{C}^{\mathbb{N}}$ the category of $\mathbb{N}$-Actions, i.e. mono-unary alg. (X, f).

Def: the monoid of abstract/intrinsic iterates of f is defined as

$$\mathcal{E}^1_f := \mathcal{C}^{\mathbb{N}}(f, f)$$

it is the set of morphisms $\alpha : f \Rightarrow f$; i.e. $f\alpha = \alpha f$

Idea: we call them the rank one funtons (relatives to f)

ex: Let $\mathcal{C} = G$ a group. $\mathcal{E}^1_f = C_G(f)$ is the centralizer of f

Let $\mathcal{C} = \mathrm{Set}$ and $f = s : \mathbb{N} \to \mathbb{N}$ the sucessor function

$\mathcal{E}^1_s$ contains all the functions $h : \mathbb{N} \to \mathbb{N}$ s.t. $h(n+1) = h(n) + 1$.

For every $b \in \mathbb{N}$ exists an unique $h \in \mathcal{E}^1_s$ s.t $h(0) = b$

and $h = s^b$, by the rec. thm. Given 2 functions $g, h \in \mathcal{E}^1_s$

by the rec. theorem if $g(0) = h(0)$ then $g = h$.

Thus $\mathbb{N} \simeq \mathcal{E}^1_s$ are in bijection, i.e. $n \longleftrightarrow s^n$

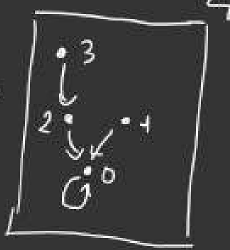
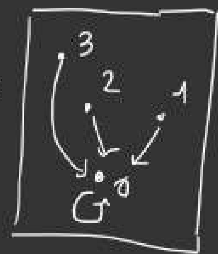
obs: $\mathcal{E}^1_f$ is always a monoid. and $f \in \mathcal{E}^1_f$. Trivially $\langle f \rangle$ is a submonoid.

Def: Define the natural/integer iterates of f as that submonoid

$$\mathbb{N}_f := \langle f \rangle \leq \mathcal{E}^1_f$$

obs $\mathbb{N}_f = \mathrm{im} f^\bullet$ where $f^\bullet : n \longmapsto f^n$ thus $\mathbb{N}/_{\ker f^\bullet} \simeq \mathbb{N}_f$.

<u>ex</u>: let f be the bidempotent function $f : 4 \to 4$

$f =$  $f^2 =$ 

then $\langle f \rangle = \{ id_4, f, f^2 \}$

because $f^{2+n} = f^2$

The monoid $\mathbb{N}_f$ looks like

$\circ$	id	f	f^2
id	id	f	f^2
f	f	f^2	f^2
f^2	f^2	f^2	f^2

so the intrinsic ~~derivatives~~ iterates of f are somekind exotic.

<u>ex</u>: if f is invertible $\mathbb{N}_f$ is an abelian group generated by f. Let $p = ord(f)$ as an element of $Aut(X)$ where $f : X \to X$ is an isomorphism; then

$$\mathbb{N}_f = \mathbb{Z}_p$$

<u>Def</u>: let $\chi : f \Rightarrow g$ be an iso in $\mathcal{C}^{\mathbb{N}}$ define the intrinsic χ iteration of g as conjugation of f-numbers by χ

$$\chi\text{-}exp_g : \mathcal{E}^1_f \longrightarrow \mathcal{C}(X, Y) \quad \text{wher } Y = dom(g)$$

$$\alpha \longmapsto \chi \alpha \chi^{-1}$$

<u>Prop</u> if $\alpha \in \mathcal{E}^1_f$, is an f-iterate, $\chi\text{-}exp_g(\alpha)$ is a g-iterate $(\in \mathcal{E}^f_g)$

proof: $g \chi \alpha \chi^{-1} = \chi f \alpha \chi^{-1} = \chi \alpha \chi^{-1} g$ $\square$

<u>prop</u> $\chi\text{-}\exp_g : \mathcal{E}_f^1 \to \mathcal{E}_g^1$ is a monoid morphism, i.e

$$\chi\text{-}\exp_g(id_X) = id_Y \qquad \chi\text{-}\exp_g(\alpha \circ_{\mathcal{E}_f^1} \beta) = \chi\text{-}\exp_g(\alpha) \circ_{\mathcal{E}_g^1} \chi\text{-}\exp_g(\beta$$

<u>obs.</u> Notice that $\chi\text{-}\exp_g(f) = g$ and in general $\chi\text{-}\exp_g(f^n) = g^n$
If we denote $\mathcal{E}_f^1$ with additive notation, thinking of it as the abstract number system representing non-integer iterates of f, and we denote f^n by $n_f \in \mathbb{N}_f$ and composition by $\oplus_f$; if we think of $\mathcal{E}_g^1$ as a monoid under function composition we get this translation

$$\chi\text{-}\exp_g(0_f) = id_Y; \quad \chi\text{-}\exp_g(1_f) = g; \quad \chi\text{-}\exp_g(n_f) = g^n$$

$$\chi\text{-}\exp_g(\alpha \oplus_f \beta) = \chi\text{-}\exp_g(\alpha) \circ \chi\text{-}\exp_g(\beta)$$

<u>proof</u>: it is trivial since Υ_χ conjugator is a monoid homomorphism. □

<u>Observation</u>: given a fixed $f : X \to X$ for every pair (Y, g), i.e a discrete dinamicamical system, call it $\mathbb{N}$-system or $\mathbb{N}$ action or $\mathbb{N}$-iteration of $g : Y \to Y$ chosen an invertible $\chi : f \Rightarrow g$ i.e. a solution of $\chi f = g \chi$; we can use χ to extend the $\mathbb{N}$-system $(Y, g) \in \mathcal{C}^{\mathbb{N}}$ to an $\mathcal{E}_f^1$-system $\chi\text{-}\exp_g \in \mathcal{C}^{\mathcal{E}_f^1}$:
given $\alpha, \beta \in \mathcal{E}_f^1$ define for $y \in Y$ $\alpha \cdot y = \chi\text{-}\exp_g(\alpha)(y)$ $\left[1_f \cdot y = g(y) \quad (n+1) \cdot y = g(n \cdot y) \right]$

Def: let grpd($\mathcal{C}^{\mathbb{N}}$) the subcategory of $\mathcal{C}^{\mathbb{N}}$ of isos.

and $f\backslash\mathcal{C}^{\mathbb{N}}$ the (coslice) category of pairs (g,χ)

and morphisms $(g,\chi)\xrightarrow{\omega}(h,\psi)$ are $\omega : g \Rightarrow h$

$$\begin{tikzcd} & g \arrow[d, Rightarrow, "\omega"] \\ f \arrow[ur, Rightarrow, "\chi"] \arrow[r, Rightarrow, "\psi"'] & h \end{tikzcd}$$

i.e. $\omega\chi=\psi$ (χ divides ψ)

Define the functor $\exp : f\backslash\mathcal{C}^{\mathbb{N}} \longrightarrow \mathcal{C}^{\mathcal{E}^1_f}$ where

- exp on objects sends $f \xRightarrow{\chi} g$ in the functor

(g,χ)

$$\boxed{(g,\chi)\longmapsto \chi\text{-}\exp_g}$$

$\chi\text{-}\exp_g : \mathcal{E}^1_f \longrightarrow \mathcal{C}$

$\bullet \longmapsto dom(g)=Y$

$\mathcal{E}^1_f \to \mathcal{C}(Y,Y)$

- exp sends $\omega : g \Rightarrow h$ to the nat tr. $\chi\text{-}\exp_g \overset{\bar{\omega}}{\Downarrow} \chi\text{-}\exp_h$

$$\begin{tikzcd} & g \arrow[d, Rightarrow, "\omega"] \\ f \arrow[ur, Rightarrow, "\chi"] \arrow[r, Rightarrow, "\psi"'] & h \end{tikzcd} \longmapsto \forall \alpha \in \mathcal{E}^1_f \quad \begin{tikzcd} Y \arrow[r, "\chi\text{-}\exp_g(\alpha)"] \arrow[d, "\bar{\omega}"'] & Y \arrow[d, "\bar{\omega}"] \\ Z \arrow[r, "\psi\text{-}\exp_h(\alpha)"'] & Z \end{tikzcd}$$

do $\bar{\omega}\, g^{\chi:\alpha} = h^{\psi:\alpha}\,\bar{\omega}$

if $\bar{\omega} g = h g$ and $\omega\chi=\psi$??

<u>Proof</u> (functoriality) Given a morphism $\omega \in \mathrm{grpd}(f\backslash\mathcal{C}^{\mathbb{N}})((g,\chi)(h,\psi))$

$\forall \alpha$ we have

$$\begin{tikzcd} Y \arrow[r, "\chi\text{-}\exp_g(\alpha)"] \arrow[d, "\omega"'] & Y \arrow[d, "\omega"] \\ Z \arrow[r, "\psi\text{-}\exp_h(\alpha)"'] & Z \end{tikzcd}$$

i.e. $\omega \in \mathcal{C}^{\mathcal{E}_f^1}(g^{\chi:\cdot}, h^{\psi:\cdot})$

$$\begin{aligned} \omega\, \chi\text{-}\exp_g(\alpha) &= \omega \chi \alpha \chi^{-1} \\ &= \psi \alpha \chi^{-1} && (\omega\chi = \psi) \\ &= \psi \alpha \psi^{-1}\omega && (\psi^{-1}\omega = \chi^{-1}) \\ &= \psi\text{-}\exp_h(\alpha)\,\omega \end{aligned}$$

$\square$

<u>obs</u>: We have a functor $\mathrm{grpd}(f\backslash\mathcal{C}^{\mathbb{N}}) \xrightarrow[\exp]{} \mathcal{C}^{\mathcal{E}_f^1}$

<u>obs</u>: restricting $\chi\text{-}\exp_g$ to integers gives $g \in \mathcal{C}^{\mathbb{N}}$, i.e.

$$\begin{tikzcd} \mathrm{grpd}(f\backslash\mathcal{C}^{\mathbb{N}}) \arrow[r, "\exp"] \arrow[dr, "\text{forgetfull}"'] & \mathcal{C}^{\mathcal{E}_f^1} \arrow[d, "f_*"] \\ & \mathcal{C}^{\mathbb{N}} \end{tikzcd}$$

remark: let $\mathcal{C}_{grpd}$ the subcat. of invertible arrows.
~~the functor exp is to be precise~~

$$\begin{tikzcd} f\backslash(\mathcal{C}^{\mathbb{N}})_{grpd} \arrow[r, "-\exp_-"] \arrow[dr, "\text{``forgetful'' } U"'] & (\mathcal{C}^{\mathcal{E}_f^1})_{grpd} \arrow[d, "f_*"] \\ & (\mathcal{C}^{\mathbb{N}})_{grpd} \end{tikzcd}$$

Observation: An "inverse" process to U would give a way to assign f-superfunctions to every $\mathbb{N}$-system..

$$\begin{tikzcd} (\mathcal{C}^{\mathbb{N}})_{grpd} \arrow[r, shift right, "f_{\oplus}"'] & f\backslash(\mathcal{C}^{\mathbb{N}})_{grpd} \arrow[l, shift right, "U"'] \end{tikzcd}$$

An "inverse" to f_* (restriction of iteration to $\mathbb{N}$) would give instead a canonical extension of every (Y,g) to an $\mathcal{E}_f^1$-System $(Y,\exp_g)$.

$$\begin{tikzcd} (\mathcal{C}^{\mathbb{N}})_{grpd} \arrow[r, shift right, "\exp_-"'] & (\mathcal{C}^{\mathcal{E}_f^1})_{grpd} \arrow[l, shift right, "f_*"'] \end{tikzcd}$$

observation: exp can also be thougt as assigning to every $\chi: f \Rightarrow g$ a monoid morphism $\mathcal{E}_f^1 \longrightarrow \mathrm{End}(\mathrm{dom}(g))$. It is natural!!!

<u>theorem</u>: the assignment $\gamma : \mathcal{C}^{\mathbb{N}}(X^{\circlearrowleft f}, Y^{\circlearrowleft g}) \to Mon(\mathcal{E}_f^1, End(Y))$ that assigns to every χ $\mathbb{N}$-system morphism to $X^{\circlearrowleft g}$ an extension of $X^{\circlearrowleft g}$ to an $\mathcal{E}_f^1$-system $(Y, \exp_{g:\chi})$ is natural in g an it's "conjugation"

$$\gamma : \chi \longmapsto \gamma_\chi$$

proof: take arbitrary $g \overset{\omega}{\Rightarrow} h$. We prove in. f. gen. that in Set

$$\boxed{\gamma_{\omega \cdot -} = \gamma_\omega \circ \gamma_-}$$

$$\forall \chi : \gamma(\omega\chi) = \gamma_\omega \gamma_\chi$$

$$\begin{tikzcd} \mathcal{C}^{\mathbb{N}}(f, g) \arrow[r, "\omega \cdot"] \arrow[d, dotted, "\gamma"'] & \mathcal{C}^{\mathbb{N}}(f, h) \arrow[d, dotted, "\gamma"] \\ Mon(\mathcal{E}_f^1, End(Y)) \arrow[r, "\gamma_\omega"'] & Mon(\mathcal{E}_f^1, End(Z)) \end{tikzcd}$$

$$\forall \alpha \in \mathcal{E}_f^1 \quad \gamma_{(\omega\chi)}(\alpha) = (\omega\chi)\alpha(\omega\chi)^{-1} \qquad (\text{def. } \gamma)$$

$$= \omega(\chi\alpha\chi^{-1})\omega^{-1} \qquad (\chi, \omega \text{ invertible})$$

$$= \gamma_\omega \gamma_\chi(\alpha) \qquad (\text{def } \gamma)$$

thus $\gamma_{\omega\chi} = \gamma_\omega \gamma_\chi$ $\square$