Migration of flexion points in tetrational functions, for bases $b > \eta$

1) – The natural tetrational function (b = e)

We have already seen that, if we admit $y = {}^{x}e$ smooth, we can put:

$$\frac{d}{dx}\left({}^{x-1}e\right) = \frac{d}{dx}\left(\ln\left({}^{x}e\right)\right)$$

i.e.:

$$\left[{}^{x-1}e\right]' = \frac{1}{{}^{x}e}\cdot\left[{}^{x}e\right]'$$

or:

$$y'(x-1) = \frac{1}{y(x)}\cdot y'(x)$$

In other terms, for $x = 0$, we have $y(0) = {}^{0}e = 1$ and, therefore:

(1) $$\boxed{y'(-1) = y'(0)}$$

The figure built by Andrew Robbins [in: *Solving for the Analytic Piecewise Extension of Tetration and the Super-logarithm*] shows the following plots of $y = e\#x = {}^{x}e$ (e-tetra-x), **red** for ***y***, **green** for ***y'*** and **blue** for ***y''***:

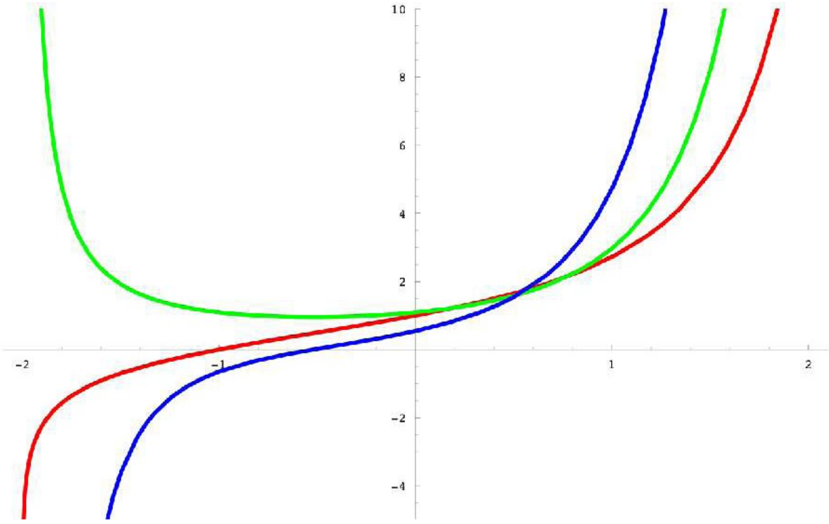

Fig 1 © andydude

A first analysis of the green plot shows that (1) seems to be fully verified [$y'(-1) = y'(0) > 1$]. The same figure also shows that the second derivative is zero at (about?) x = -0.5, where y' has obviously a minimum. The behaviour of y (**red** plot), in $-1 < x < 0$ should be something like this :

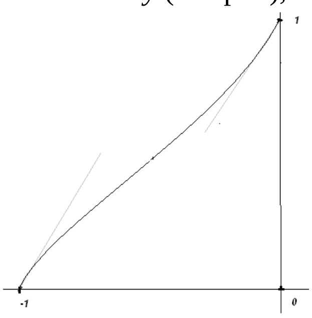

Fig 2 (GFR *gu*-estimation)

In other terms, the minimum of y' (< 1) should appear at about x = -0.5, where y'' = 0. To be checked.
Or, perhaps, the flexion point is at $x = -\sqrt[e]{e} = -1.4446678..$? Also to be checked.

These ideas can be grouped in the following *andydude*'s (annotated) figure:

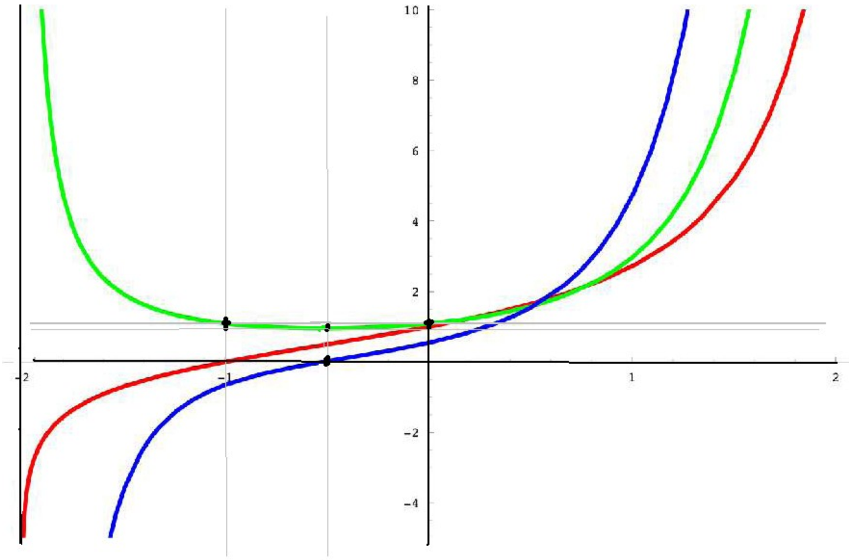

Fig 3

My provisional conclusions seem a little bit too ... quick. Nevertheless, since "natural" $y = {}^{x}e$ is a critical function, the above-mentioned values might also be well know universal constants. Can somebody check this?

2) - Tetrational function with b = 2

For $y = {}^{x}2$, with base $b = 2$, the plots of Fig 1 are of course modified, as shown in the next figure, taken from http://tetration.itgo.com/graph.html (2-tetra-*x*):

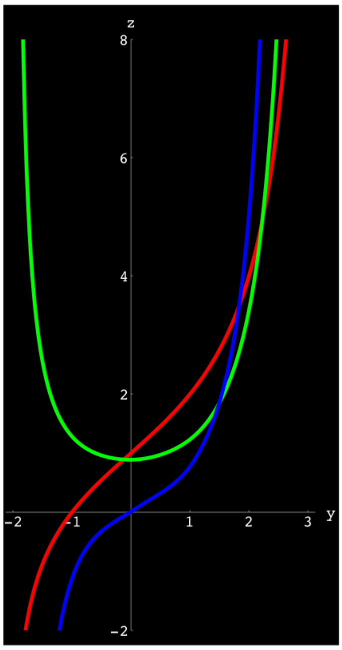

Fig. 4 (© andydude)

The most amazing observation (GFR test of ... experimental mathematics !!!) is that the second derivative (**blue plot**, *andydude* dixit) of $y = {}^{x}2$ seems to be null at *x* = 0 (flexion point in the origin of the *x* axis!).

In this case, formula (1) will change as follows.

$$\frac{d}{dx}\left({}^{x-1}b\right) = \frac{d}{dx}\left(\log_b\left({}^{x}b\right)\right)$$

i.e.: $$\left[{}^{x-1}b\right]' = \frac{1}{{}^{x}b \cdot \ln b} \cdot \left[{}^{x}b\right]'$$

or: $$y'(x-1) = \frac{1}{y(x)\cdot \ln b} \cdot y'(x)$$

Also in this case, for $x = 0$, we have $y(0) = {}^{0}2 = 1$ and, therefore:

(2) $$y'(-1) = \frac{1}{\ln 2} \cdot y'(0)$$

and, for $x = 1$, $y(1) = 2$, we have:

$$y'(0) = \frac{1}{2 \cdot \ln 2} \cdot y'(1)$$

From (2) $y'(1) = 2 \cdot (\ln 2)^2 \cdot y'(-1) = 0.96090628.. \cdot y'(-1)$, therefore:

(3) $$\boxed{y'(1)/y'(-1) = 0.96090628..}$$

As seems to be shown, in Fig. 4, **green** plot. Again, to be carefully checked.

In conclusion, by changing base *b*, from *b* = *e* to *b* = 2, the position of the flexion point moves, from about *x* = - 0.5, to *x* = 0.

3) – Tetrational functions with lower bases *e*^(1/*e*) ≤ *b* < 2

This domain needs to be carefully studied. I should like to show here only two typical cases, which characterize what might happens: the tetrational function for bases *b* = 1.447 and *b* = 1.4446678.. .

For *b* = 1.447, we see a kind of "plateau" at $y = e$ and the "characteristic path" for *x* ≈ [30,31], where a flexion point can be detected. There, the first derivative is *y'* ≈ 0, but probably not null. When *b* increases, the "tsunami" coming from the right will transform the $y = {}^{x}b$ plot into a super-exponential. During this process, the flexion point will leave point (≈ 31, *e*) and will be shifted to the left until reaching position (0,-1), passing through the situations described for *b* = 2 and *b* = e. Nevertheless, this plot, without a more precise analysis, can only be a simulation.

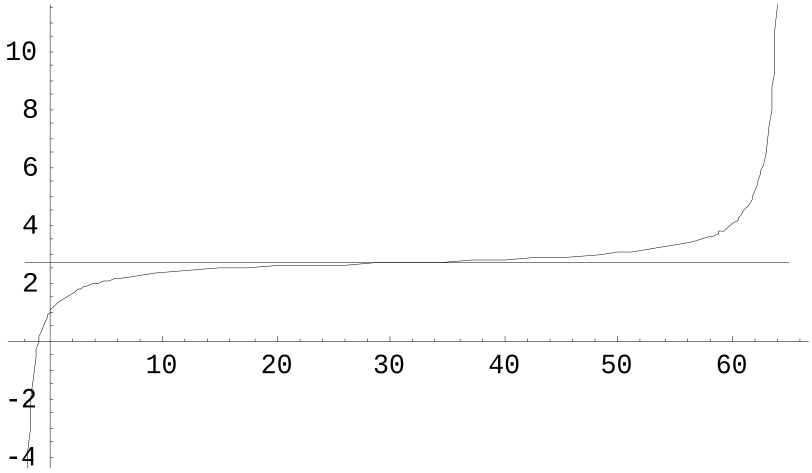

Fig. 5

On the contrary, if the base slightly decreases, reaching $\eta = \sqrt[e]{e} = 1.4446678..$ the precision can be chosen as we wish, by selecting the most appropriate "critical path" (far, ... to the right!). A simple example is as follows:

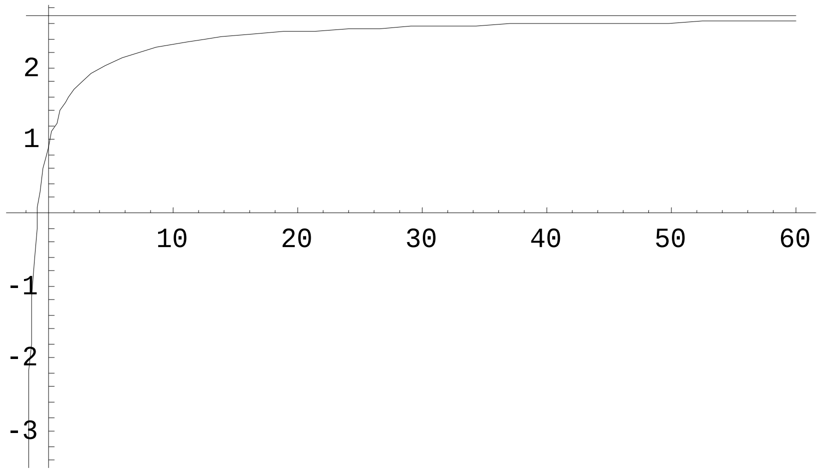

Fig. 6

The plot shows an asymptote at $y = e$ and the abscissa x of the flexion point → + ∞, where we must have: $y' = y'' = 0$.

4) – Conclusions

In conclusion, if we consider a tetrational function $y = {}^{x}b$ ($\eta \le b < +\infty$), we should have:

- If $b = \eta = 1.44466781$, the situation of Fig 6 (a smooth "*knee*", with a horizontal asymptote at $y = e$);
- If $b \approx 1.47$, the situation described by Fig 5 (a "*tsunami*" coming from the right);
- If $b = 2$, the configuration shown in Fig 4 (flexion point in the origin);
- If $b = e$, the configuration described by Fig 1 (flexion point at $x \approx -0.5$ or $x = -\eta$);
- If $b > e$, a fully super-exponential plot (flexion point at $x = -1$?). To be checked, again.

Is this almost right? Can somebody evaluate these general "artistical" statements of mine? Is it possible (of course it is ...) to draw a plot (or to find the law) of the migration of the flexion point, when *b* changes as indicated above? Can we build a Wolfram demonstration project?

Thank you very much in advance.

Gianfranco (GFR) – 14the January 2008