Tetration FAQ

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Contents

1 Preamble

This FAQ is meant to provide basic understanding about tetration and its related topics. The theme enjoys great popularity among pupils and lay mathematicians. One burning question is: What is the most beautiful/suitable/natural extension of tetration to the real numbers? We see already that this question is slightly outside the realm of mathematics more in the realm of taste. Perhaps this is also the reason why professional mathematicians barely considered this question.

A more mathematical task would be to find (a) uniqueness condition(s) for a real extension of tetration. However there are none known yet. And so this FAQ also aims to briefly present the various extensions already constructed by various authors in a sense of a fair competition (for appeal to the taste of the reader). [Note however that it will take some time until finished.]

The FAQ shall stop researchers from reinventing the wheel and instead focus their light on undeveloped regions. A special request is to also reveal dead-end ideas, to prevent authors from wasting their efforts there. It is hoped that this FAQ makes research about tetration more effective and fruitful.

The FAQ shall be understandable for interested undergraduates. I really appreciate any suggestion and contribution for improving and extending it (and if merely on the level of the English language which is not my native language).

2 Tetration and Ackermann Function

Question 1. *What is tetration?*

Multiplication is repeated addition, exponentiation is repeated multiplication, in similarity to this process one tries to define the next higher operation as repetition of the exponentiation. For better readability we introduce the symbol $\uparrow$ for exponentiation, i.e. $x \uparrow y := x^y$. There is however a difficulty with repeating exponentiation because the operation is not associative (as addition and multiplication are) so we have to chose a certain bracketing scheme. Right bracketing seems to make the most sense (see chapter 6.2 for other bracketing schemes) and so tetration $(x,n) \mapsto {}^n x \colon \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ is defined as

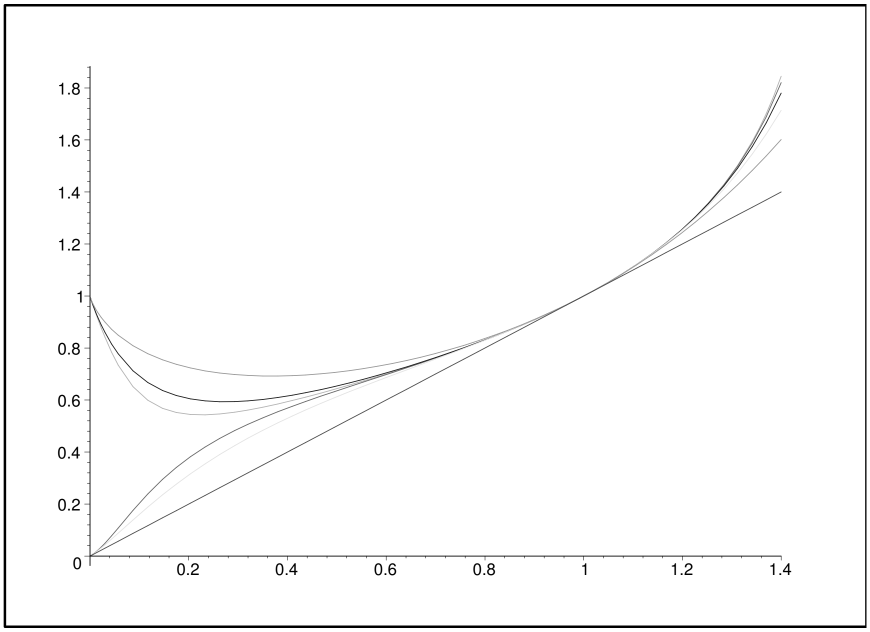
$$ {}^n x := \underbrace{x \uparrow (x \uparrow (\cdots \uparrow x) \cdots))}_{n \times x} $$

or more formally inductively by

$$ {}^1 x := x \tag{1} $$
$$ {}^{n+1} x := x^{{}^n x}. \tag{2} $$

n is called the (tetration) exponent and x is called the (tetration) base.

The functions $x \mapsto {}^n x$ show an interesting behaviour in the interval $(0,1)$. It strikingly resembles the behaviour of the polynomials $x \mapsto x^n$ in $(-\infty, 0)$. Let us have a look at the



graphs. In the area where $x < 1$ the graphs in ascending order are: ${}^1x$, ${}^3x$, ${}^5x$, ${}^6x$, ${}^4x$, ${}^2x$ for $x > 1$ the ascending order is ${}^1x, \dots, {}^6x$.

Proposition 1. *The function $x \mapsto {}^nx$ is bijective on $\mathbb{R}_{>1}$ for each $n \in \mathbb{N}$. The function $x \mapsto {}^nx$ is bijective on $\mathbb{R}_+$ if and only if n is uneven and*

$$\lim_{x \downarrow 0} {}^nx = \begin{cases} 0 & \text{for uneven } n \\ 1 & \text{for even } n \end{cases}$$

This gives rise to define the inverse operation.

Question 2. *What different notations and names are in use for tetration? Which one shall I use?*

We propose to write exponentiation and tetration with base x and exponent n as follows.

environment	exponentiation	tetration
generally	x^n	${}^nx$
as symbol	$x \uparrow n$	$x \uparrow\uparrow n$
in ASCII	`x^n`	`x^^n`

The notation ${}^nx$ is probably the most compact way of writing tetration, however we have to be careful about ambiguity as for example in x^nx. With this notation there is also some uncertainty about whether tetration is $\mathbb{R}_+ \times \mathbb{N} \to \mathbb{R}_+$ or $\mathbb{N} \times \mathbb{R}_+ \to \mathbb{R}_+$. The convention coherent with all the other notations is however that tetration is $\mathbb{R}_+ \times \mathbb{N} \to \mathbb{R}_+$.

In some situations it can be quite useful to have an operation symbol instead merely a way of writing, for example in the specification $\uparrow\colon \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ (which however can be synonymously replaced by $(x, n) \mapsto x^n \colon \mathbb{R} \times \mathbb{N} \to \mathbb{R}$) or if we map operations on operations indicated by a decoration, for example $* \mapsto *'$ then we can write $x \uparrow' y$, or if there are difficulties in writing or reading repeated nested levels of exponents.

Alternative names for tetration are hyperpower (often used in professional mathematical context) or superpower. Also the name hyper4 operator is in use. For me hyperpower and superpower sound somewhat unspecific as there is a whole hierarchy of operations which are "hyper" respective the power.

Question 3. *What is the Ackermann function?*

Now we can repeat the process of defining the next higher operation. Given an operation $* \colon X \times X \to X$ we define the right right-bracketing iterator operation $*' \colon X \times \mathbb{N} \to X$ as

$$x *' n := \underbrace{x * (x * (\cdots * x) \cdots)}_{n \times x}$$

Then we have $xn = x +' n$, $x^n = x +'' n$ and $^n x = x +''' n$. Verify however that the left left-bracketing iterator operation $*` \colon \mathbb{N} \times X \to X$

$$n *` x := \underbrace{(\cdots (x * x) * \cdots x) * x}_{n \times x}$$

yields basically the same operations $n +` x = nx$, $n +`` x = x^n$ and $n +``` x = {}^n x$. In generalisation of this method define a sequence of operations $\diamond_n \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ inductively by

$$\begin{aligned} a \diamond_1 b &= a + b \\ a \diamond_{n+1} b &= a \diamond_n' b \end{aligned}$$

Particularly $m \diamond_2 n = mn$, $m \diamond_3 n = m^n$, $m \diamond_4 n = {}^n m$. A similar construction was used by Ackermann 1927 in [1]. He recursively defined a function $\varphi \colon \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ which relates to our construction by $\varphi(a, b, n-1) = a \diamond_n b$ and used it to show that there are recursive but not primitive recursive functions. A function similar to φ is called original Ackermann function, "original" because later Rózsa Péter introduced a simpler function which served the same purpose and which is also usually called Ackermann function.

We can assume $\diamond_1$, $\diamond_2$ and $\diamond_3$ to be extended to $\mathbb{R}_+$. However because $\diamond_4$ (which is tetration) is merely defined for natural numbered exponents (we generally call x the basis and y the exponent in the expression $x \diamond_n y$ for $n \geq 3$) $\diamond_5$ which is the repetition of $\diamond_4$ can merely be defined on $\mathbb{N} \times \mathbb{N}$.

So after extending tetration to real exponents one would aim to extend all the following operations successively to real exponents.

3 Extending Tetration to Fractional Exponents

Question 4. *What is the problem of extending tetration to fractional and real exponents?*

The short answer is: uniqueness. There are no suitable conditions found yet that favour a certain solution over others. Let us illustrate this with the well-known extension of the Gamma function and with the well-known extension of exponentiation.

3.1 Extension of the Gamma Function

The factorial function $n \mapsto n!$ which is defined on the natural numbers inductively by

$$1! = 1 \qquad\qquad (n+1)! = n!(n+1)$$

shall be extended to have also values for fractional/real arguments between two consecutive natural numbers, of course not any values but certain nice, smooth, fitting values (whatever that means). We seek for conditions that that narrow the range of possible solutions. The first natural condition is to satisfy the recurrence relation also for real values, i.e. an extension $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ of the factorial function shall satisfy

$$f(1) = 1 \qquad\qquad f(x+1) = (x+1)f(x) \tag{3}$$

for all positive real numbers x. This particularly forces $f(n) = n!$ for natural numbers n, i.e. it implies f being an extension of $n \mapsto n!$. If $f(x)$ is defined on the open interval $(0,1)$ then this condition determines the values of $f(x)$ for $x > 1$ by induction as you can easily verify. However we can arbitrarily define $f(x)$ on $(0,1)$ and so get infinitely many solutions that satisfy (3). The next obvious demand is continuity, differentiability or even analyticity of f. However it is well-known that another criterion suffices:

Proposition 2. *The condition* (3) *together with logarithmic convexity, i.e.*

$$f(\lambda x + (1-\lambda)y) \le f(x)^\lambda f(y)^{1-\lambda} \quad \text{for all} \quad x, y > 0, 0 < \lambda < 1,$$

uniquely determine f to be $f(x) = \Gamma(x+1)$.

3.2 Extension of the Exponentiation Operation

Till now let us having defined the exponentiation for natural exponents by

$$b^1 = b \qquad\qquad b^{n+1} = bb^n$$

for real b and natural n. In comparison with the Gamma function we have the additional parameter b. We can anyway try to repeat the considerations for the Gamma function by fixing b in the beginning, i.e. to find a function f_b that satisfies

$$f_b(1) = b \qquad\qquad f_b(n+1) = bf_b(n) \tag{4}$$

Logarithmic convexity here is somewhat self referential as we need the power for real exponents to be defined in the expression

$$f_b(\lambda x + (1-\lambda)y) \le f_b(x)^\lambda f_b(y)^{1-\lambda}$$

So it seems as if this way is not working here (if however someone would like to explore this in a bit more depth I would be happy to hear about the results.)

Now we can easily generalise the original recurrence by induction to $b^{m+n} = b^m b^n$ and by repeated induction to our key relation

$$b^1 = b \qquad\qquad (b^n)^m = b^{nm}. \tag{5}$$

This condition if demanded for real m and n already suffice to uniquely extend exponentiation to fractional exponents. For example take $b^{\frac{1}{2}}$. If it has a value at all then must

$$(b^{\frac{1}{2}})^2 = b^{\frac{1}{2}2} = b^1 = b$$

which means that $b^{\frac{1}{2}}$ is the solution of the equation $x^2 = b$. To make the solution unique we restrict the base domain of our extended exponentiation operation to the positive real numbers. And then we similarly get that $b^{\frac{1}{n}}$ is the solution of $x^n = b$, i.e. $b^{\frac{1}{n}} = p_n^{-1}(x)$ where $p_n(x) := x^n$ is bijective on $\mathbb{R}_+$ and obviously $p_n^{-1}(x) = \sqrt[n]{x}$. Also by equation (5) we get then the general formula for fractional exponents

$$b^{\frac{m}{n}} = p_m(p_n^{-1}(b)).$$

Extension to irrational arguments merely requires the function to be continuous.

Proposition 3. *There is exactly one extension of the exponentiation with natural numbered exponents to positive real exponents such that the function $x \mapsto b^x$ is continuous for each b and which satisfies*

$$(b^x)^n = b^{xn} \tag{6}$$

for each $b > 0$, $x \in \mathbb{R}_+$ and $n \in \mathbb{N}$.

It can be mentioned that we could replace (6) by the stronger demand

$$b^{x+y} = b^x b^y. \tag{7}$$

Because from this follows by induction $b^{xn} = (b^x)^n$. In that case we even could extend our operation to negative exponents. First we notice $b^0 b^x = b^{0+x} = b^x$ and hence $b^0 = 1$ for $b^x \neq 0$. Then $b^x b^{-x} = b^{x-x} = b^0 = 1$ implies $b^{-x} = 1/b^x$.

On the other hand if we need exponentiation for tetration then both arguments of exponentiation must be from the same domain, and the greatest common domain of base and exponent is $\mathbb{R}_+$.

3.3 Extension of Tetration

By the special bracketing of the tetration the equivalent of neither equation (6) nor (7) hold for all m, n, i.e. for most $m, n \in \mathbb{N}$ we have:

$$^{n+m}x \neq \left({}^n x\right)^{{}^m x} \tag{8}$$
$$^{nm}x \neq {}^n({}^m x) \tag{9}$$

Even

Proposition 4. *There is* no *operation* $*$ *on* $\mathbb{R}_+$ *such that*

$$^{n+m}x = {}^m x * {}^n x$$

Proof. Suppose there is such an operation, then we gain the contradiction

$$65536 = 2^{2^{2^2}} = {}^4 2 = {}^{2+2}2 = {}^2 2 * {}^2 2 = 4 * 4 = {}^2 4 = 4^4 = 256$$

□

This breaks applying the procedure of extending the exponentiation to extending the tetration. This shall be mentioned with all insistence. Though we have seen that $t_n(x) = {}^n x$ is bijective on $\mathbb{R}_{>1}$ for each n, a definition of ${}^{\frac{1}{n}}b$ as $t_n^{-1}(b)$ is as arbitrary as defining it to be $\frac{\pi}{3}$.

If we want to have a unique solution we need conditions that make the solution unique. Till now there aren't known such conditions. However there are several conditions that one certainly want to have satisfied for an (to some real numbers) extended tetration.

1. The recurrence relations (1) and (2) shall be satisfied for arbitrary exponents:

$$^1 b = x \qquad\qquad {}^{x+1}b = b^{{}^x b}. \tag{10}$$

This particularly implies that it is an extension of the tetration for natural exponents.

2. The function $x \mapsto {}^a x$ shall be continuous and strictly increasing for each a (so that we can define the inverse: a tetration root). Note that this condition suggests to restrict the base to $\mathbb{R}_{>1}$ (similarly the base of exponentiation is restricted to $\mathbb{R}_+$).

3. The functions $x \mapsto {}^x b$ shall be continuous and strictly increasing for each b (so that we can define the inverse: a tetration logarithm).

Definition 1. A tetration extension $(x, y) \mapsto {}^y x \colon I \times J \to \mathbb{R}$, where $I, J \subseteq \mathbb{R}$ is called a *real tetration* if it satisfies conditions 1, 2 and 3.

Additional demands could be infinite differentiability or even analyticity.

4 Fractional Iteration and the Exponential Function

TODO

$$f^{\circ 1}(a) = f(a) \qquad\qquad f^{\circ x+y}(a) = f^{\circ x}(f^{\circ y}(a)) \tag{11}$$

4.1 Translation Equation and Abel Equation

Closely related to continuous iteration is the translation equation

$$F(F(a, x), y) = F(a, x + y)$$

(if we write $F(a, x) = f^{\circ x}(a)$).

Proposition 5. *If $f(x) := F(b, x)$ is continuous and strictly increasing for one b and if F satisfies the translation equation (for all arguments of its domain of definition) then*

$$F(a, x) = f_c(f_c^{-1}(a) + x) \tag{12}$$

for any c, where $f_c(x) := f(x+c)$. Vice versa $G(a, x) := g(g^{-1}(a)+x)$ satisfies the translation equation for each strictly increasing continuous g, then $x \mapsto G(a, x)$ is strictly increasing and continuous for every a.

If we now want to real iterate a function g, i.e. $g(x) = F(x, 1)$ and F satisfies the translation equation, $x \mapsto F(a, x)$ is strictly increasing and continuous. Then we merely need to find a strictly increasing continuous function f (which will be equal to $F(a, x)$ for some a) such that $g(x) = F(x, 1) = f(f^{-1}(x) + 1)$. Or if we put $h := f^{-1}$ such that

$$h(g(x)) = h(x) + 1$$

which is the so called Abel equation.

4.2 Tetration versus Iterated Exp

TODO

If we have a real iteration of the function then we can construct a real tetration from it, illustrated by the example:

$$\exp_b^{\circ 2}(x) = b^{b^x}$$
$$\exp_b^{\circ 2}(b) = b^{b^b} = {}^3b$$

Proposition 6. *For each real iteration $\exp_b^{\circ x}$ of the function $\exp_b$ the operation defined by*

$${}^ab = \exp_b^{\circ a-1}(b) \tag{13}$$

is a real tetration.

Proof.

$${}^1b = \exp_b^{\circ 0}(b) = \mathrm{id}(b) = b$$
$${}^{a+1}b = \exp_b^{\circ a}(b) = b^{\exp_b^{\circ a-1}(b)} = b^{{}^ab}$$

□

Vice versa if we have defined a real tetration we can derive a real iteration of $\exp_b$. Though we have to reach out some more before. Whenever we have a real tetration then the function $x \mapsto {}^xb$ maps $(0, \infty)$ to $(1, \infty)$ and is injective. Hence we can define the tetration logarithm by

$${}^{\mathrm{tlog}_b(x)}b := x \tag{14}$$

And this enables us then to define real iterations of $\exp_b$ by the reasoning

$$\begin{aligned}\exp_b^{\circ x}(a) &= \exp_b^{\circ x}({}^{\operatorname{tlog}_b(a)}b) = \exp_b^{\circ x}(\exp_b^{\circ \operatorname{tlog}_b(a)}(1)) = \exp_b^{\circ x+\operatorname{tlog}_b(a)}(1)\\ &= {}^{x+\operatorname{tlog}_b(a)}b\end{aligned}$$

Proposition 7. *For each real tetration $(x, y) \mapsto {}^y x$ the operation defined by*

$$\exp_b^{\circ x}(a) = {}^{x+\operatorname{tlog}_b(a)}b$$

is a real iteration of $\exp_b$.

Proof.

$$\begin{aligned}\exp_b^{\circ 1}(a) &= {}^{1+\operatorname{tlog}_b(a)}b = b^{{}^{\operatorname{tlog}_b(a)}b} = b^a = \exp_b(a)\\ \exp_b^{\circ x+y}(a) &= {}^{x+y+\operatorname{tlog}_b(a)}b = {}^{x+\operatorname{tlog}_b({}^{y+\operatorname{tlog}_b(z)}a)}b = \exp_b^{\circ x}(\exp_b^{\circ y}(a))\end{aligned}$$

□

These both translations (iteration to tetration and tetration to iteration) are at least formally inverse. To see what this means let us generalise and simplify the notation.

Given a function h with $h(1) = b$ (in our case was $h = \exp_b$). Let $\mathfrak{A}$ be the set of all functions $f(x)$ that are strictly increasing, continuous and that satisfy

$$f(1) = b \qquad\qquad f(x+1) = f(h(x))$$

for all x. Let $\mathfrak{B}$ be the set of all operations $F(a, x)$ such that $x \mapsto F(b, x)$ is strictly increasing and continuous, and such that

$$F(a, 1) = h(a) \qquad\qquad F(a, x+y) = F(F(a, x), y)$$

for all x, y, a.

Then we define in the sense of our previous considerations the mappings $f \mapsto f^{\uparrow} : \mathfrak{A} \to \mathfrak{B}$ and $F \mapsto F^{\downarrow} : \mathfrak{B} \to \mathfrak{A}$ by

$$f^{\uparrow}(a, x) = f(f^{-1}(a) + x) \tag{15}$$

$$F^{\downarrow}(x) = F(b, x-1). \tag{16}$$

We can see that they are formally inverse to each other in the sense that $(f^{\uparrow})^{\downarrow} = f$ and $(F^{\downarrow})^{\uparrow} = F$ for each $f \in \mathfrak{A}$ and $F \in \mathfrak{B}$.

$$\begin{aligned}(F^{\downarrow})^{\uparrow}(a, x) &= F^{\downarrow}(F^{\downarrow^{-1}}(a) + x) = F(b, F^{\downarrow^{-1}}(a) + x - 1)\\ &= F(F(b, F^{\downarrow^{-1}}(a) - 1), x) = F(F^{\downarrow}(F^{\downarrow^{-1}}(a)), x)\\ &= F(a, x)\\ (f^{\uparrow})^{\downarrow}(x) &= f^{\uparrow}(b, x-1) = f(f^{-1}(b) + x - 1) = f(1 + x - 1) = f(x)\end{aligned}$$

But beware! We did not consider the domains of definitions yet. As we can already see for natural iteration exponents, the domain $D \subseteq \mathbb{R}\times\mathbb{R}$ of an F may not be rectangular, but be dependent on the second parameter:

$$\begin{aligned}
\exp_b^{\circ 0} = \mathrm{id}\colon & (-\infty,\infty) \leftrightarrow (-\infty,\infty)\\
\exp_b^{\circ 1} = \exp_b\colon & (-\infty,\infty) \leftrightarrow (0,\infty)\\
\exp_b^{\circ n}\colon & (-\infty,\infty) \leftrightarrow (\exp_b^{\circ n-1}(0),\infty)\\
\exp_b^{\circ -1} = \log_b\colon & (0,\infty) \leftrightarrow (-\infty,\infty)\\
\exp_b^{\circ -n} = \log_b^{\circ n}\colon & (\exp_b^{\circ n-1}(0),\infty) \leftrightarrow (-\infty,\infty)
\end{aligned}$$

So for a given boundary function $d\colon \mathbb{R}\to\mathbb{R}$ define $D_d = \{(x,t)\colon x\in(d(t),\infty), t\in\mathbb{R}\}$. So let $F\colon D_d\to\mathbb{R}$ then $F^{\downarrow}\colon \mathbb{R}\to(\sup_{x\in\mathbb{R}} d(x),\infty)$ because by (15) $F^{\downarrow^{-1}}$ must be defined on $(d(x),\infty)$ for each $x\in\mathbb{R}$.

5 Particular Extensions of Tetration

5.1 Ioannis Galidakis' solution

TODO
See [5].

5.2 Robert Munafo's solution

TODO
See [8].

5.3 Andrew Robbin's solution

TODO
See [9]

5.4 Gottfried Helms's solution

TODO
Posting in sci.math.research

6 Adaption of the Problem

6.1 Making the Exponentiation Associative and Commutative

TODO

$$x \bigtriangleup_1 y = x + y$$
$$x \bigtriangleup_{n+1} y = \exp(\log(x) \bigtriangleup_n \log(y))$$

6.2 Alternative Bracketings

6.2.1 Left Bracketing

TODO

$$\underbrace{((x^x)^{\cdots})^x}_{n\times x} = x^{x^{n-1}}$$

Uniqueness of the solution $(x, y) \mapsto x^{x^{y-1}}$?

6.2.2 Balanced Bracketing

TODO
${}^1x = x$, ${}^2x = x^x$, ${}^4x = (x^x)^{(x^x)}$, et cetera. Generally we define *balanced tetration* as

$$^{2^0}x = x$$
$$^{2^{n+1}}x = \left({}^{2^n}x\right)^{\left({}^{2^n}x\right)}$$

This reduces extending tetration to extending the iteration of $F(x) := x^x$ because ${}^{2^n}x = F^{\circ n}(x)$. So if we found a continuous iteration of F then we have also found a continuous iteration of the balanced tetration by

$$^{y}x = F^{\circ \log_2(y)}(x)$$

We can simplify the treatment of $F(x)$ by introducing $G(x) := xe^x$ (which is the inverse of the Lambert's W function) and noticing that $F^{\circ n} = exp \circ G^{\circ n} \circ \ln$ because $F := \exp \circ G \circ \ln$. So continuous iteration is reduced to continuous iteration of G. Now G has a fixed point at 0 (corresponding to the fixed point of F at 1) and can be developed at 0 into the powerseries

$$G(x) = \sum_{j=1}^{\infty} j \frac{x^j}{j!}$$

Formal powerseries of the form $x + \dots$ have a unique continuous iterate. So if it converges (which seems quite so) then G has a unique continuous iterate and then F has a unique continuous iterate and then there is a unique analytic balanced tetration.

6.3 Chose a number system that reflects non-associativity

TODO
Arborescent Numbers and Higher Arithmetic Operations (article)
Tree fraction calculator (web application)

7 Infinite Power Towers

TODO
$\lim_{n\to\infty} x \uparrow\uparrow n = \frac{W(-\ln(x))}{-\ln(x)}$ for $e^{-e} < x < e^{1/e}$ where W is Lambert's W function, see [4] and [6], also [2], [7], [3].

8 Coming in Contact

The main point of communication is the Tetration Forum. You can also reach the author via e-mail at `bo198214:at:eretrandre.org` where you as usual have to replace the ":at:" by "@".

8.1 Credits

If you contribute a chapter your name will be listed here.

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