Mysteries of Infinite Tetrates

1 – Infinite tetrates

I already said[1] that, according to my point of view, we may assume that:

$$y = b^y = b^{b^y} = b^{b^{b^y}} = ... = b^{b^{\cdot^{\cdot^{y}}}} = \lim_{n\to\infty} {}^n b = {}^\infty b \quad (1)$$

In other words, if: $y = b^y$

then, at the end of the "day": $y = {}^\infty b$ (*y* is the *infinite tetrate* of *b*)

and, from (1), we get $b = \sqrt[y]{y}$ (*b* is the *selfroot* of *y*) (2)

but also, from (1) and (2): $b = {}^\infty\!\!\sqrt{y} = \text{srt}_\infty\, y$ (*b* is the *infinite superroot* of *y*)

We have seen that, with the help of the Lambert Function, we can obtain the inverse of (2), as follows:

$$y = {}^\infty b = \text{plog}(-\ln b)/(-\ln b) \quad (3)$$

The two real branches of $y = \text{plog}_k(z)$ are given by the *Mathematica* operators as:

$$\text{plog}_0(z) = \text{ProductLog}[z]$$

$$\text{plog}_{-1}(z) = \text{ProductLog}[-1, z]$$

For example, considering an infinite tetrate with height *y = 5*, we may find the corresponding base from (2):

$$b = \sqrt[5]{5} = 1.379729661.. \quad (4)$$

which means that: $y = {}^\infty(1.379729661..) = 5$

But this is not the unique value of the infinite tetrate of (4). In fact, from (3), we also have:

$$y = {}^\infty(1.379729661..) = 1.76492..$$

Actually, as another very known example, if we take bases $b = \sqrt{2}$, we have the "famous" result:

$$y = {}^\infty(\sqrt{2}) = \{2, 4\}$$

in fact: $b = \sqrt[2]{2} = \sqrt[4]{4} = 1.414213562..$ (5)

We know that this phenomenon of "double heights" obtained by infinite tetrates of a real base *b* happens in the domain:

$$1 < b < \eta$$

We also know that the maximum real base that we can use for building up "infinite towers" with finite (convergent) heights is $y = \sqrt[e]{e} = \eta = 1.444...$ and that the corresponding height is $y = e$, i.e.:

$$^\infty\eta = e = 2.718281828459045.. \quad (6)$$

For $b > \eta$ the "height" *y* are infinite or/and complex. An accurate study of this domain has been undertaken by the members of the Tetration Forum at the end of 2007. A nice plot was provided by Gottfried, showing the complex values of the infinite tetration height obtained with bases $b > \eta$. For $0 < b < \eta$, all the infinite tetrates of base *b* are not infinite. But this requires, as we know, another type of analysis.

[1] Section 1 is a repetition of what I already posted to the Forum... .

2 – Mysterious tetrates

Indeed, one of the strangest phenomena concerning tetration is the application of (1), in the particular and peculiar case when $y = i$, which gives an infinite tetrate with height i: \[2]

$$i = b^i = b^{b^i} = b^{b^{b^i}} = \ldots = b^{b^{\cdot^{\cdot^{i}}}} = \lim_{n\to\infty} {}^n b = {}^\infty b \qquad (7)$$

The problem is that functional (implicit) equation $i = b^i$ is easily solved, for example, as follows:

$$i = b^i \qquad \to \qquad b = e^{\frac{\pi}{2}} \qquad (8)$$

in fact, we know that: $e^{\frac{\pi}{2}\cdot i} = i$

Actually, *Eulerus dixit* (... again ... !?! ... yes, but in another field of mathematics) that the complete solution of (8) is multiple and that the number of individual solutions is infinite. In fact:

$$e^{\frac{\pi}{2}\cdot(4m+1)\cdot i} = i \qquad m = 0, 1, 2, 3, \ldots..+\infty \qquad (9)$$

Now, let us put : $\boxed{\rho = e^{\frac{\pi}{2}} = 4.810477381..}$ [Greek letter *Rho*] (10)

In this case, the total set of solutions of (8) can be put as follows:

$$b_m = \rho^{(4m+1)} \qquad (11)$$

with: $m = 0, 1, 2, 3, \ldots..+\infty$

According to (11), the base for which the corresponding heights of their infinite tetrates is exactly *i* are given by an infinite set of real numbers, obtained for various values of parameter *m*, as follows:

$$\begin{aligned} b_0 &= 4.810477381.. \\ b_1 &= 2575.970497... \\ b_2 &= 1379410.705... \\ b_3 &= 738662922.3... \end{aligned} \qquad (12)$$

....

The very peculiar aspect of this issue is that, obviously, for any real base $b > \eta$, we must also have:

$${}^\infty b = \lim_{n\to+\infty} {}^n b = +\infty \qquad \text{(i.e. the infinite tetrate is infinite)} \qquad (13)$$

To all this we must add that a variant of expression (8) can also be written as:

$$-i = b^{-i} \qquad \to \qquad b = e^{\frac{\pi}{2}} \qquad (14)$$

in fact: $e^{-\frac{\pi}{2}\cdot i} = -i$

This allows us to describe the infinite set of solutions of both (8) and (14) as follows[3]:

$$\boxed{\begin{aligned} b_m &= \rho^{(4m+1)}, \qquad m \in Z \\ \rho &= e^{\frac{\pi}{2}} = 4.810477381.. \end{aligned}} \qquad (15)$$

with also:

$$\begin{aligned} b_{-1} &= 0.008983291... \\ b_{-2} &= 0.000388203.... \\ b_{-3} &= 0.000000031... \end{aligned} \qquad (16)$$

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[2] Number *i* is the "imaginary unit", as we obviously know, defined by: $\sqrt{-1} = \{-i, +i\}$.

[3] If we allow the $-i$ $+i$ alternations, we may also write $b_n = \rho^{(2n+1)}$, with $n = 0, 1, \ldots.. +\infty$.

In conclusion, taking into consideration (13) and (15), we should strangely have:

$$\boxed{\begin{aligned} &{}^{\infty}b_m = \{+\infty, -i, +i\} \\ &\text{with: } b_m = \rho^{(4m+1)}, \qquad m \in Z \\ &\text{and: } \rho = e^{\frac{\pi}{2}} = 4.810477381.. \end{aligned}} \qquad (17)$$

The strangeness of (17) is the appearance, together with the real infinite height (which is ... normal), of two imaginary and conjugate unitary heights – i and + i, normally considered as fixed point, but never thought until now, as far as I know, as ... "imaginary unit" heights.

Gottfried provided us with the following very nice complex plot, showing the complex fixpoints t (from my point of view, equivalent to complex "infinite-tower heights") for real bases b, in:

$$b^t = t \qquad (18)$$

The impressive plot is built with b variable between $b = 2$ and $b = 8192$ (other more complete Gottfried's plots are also available).

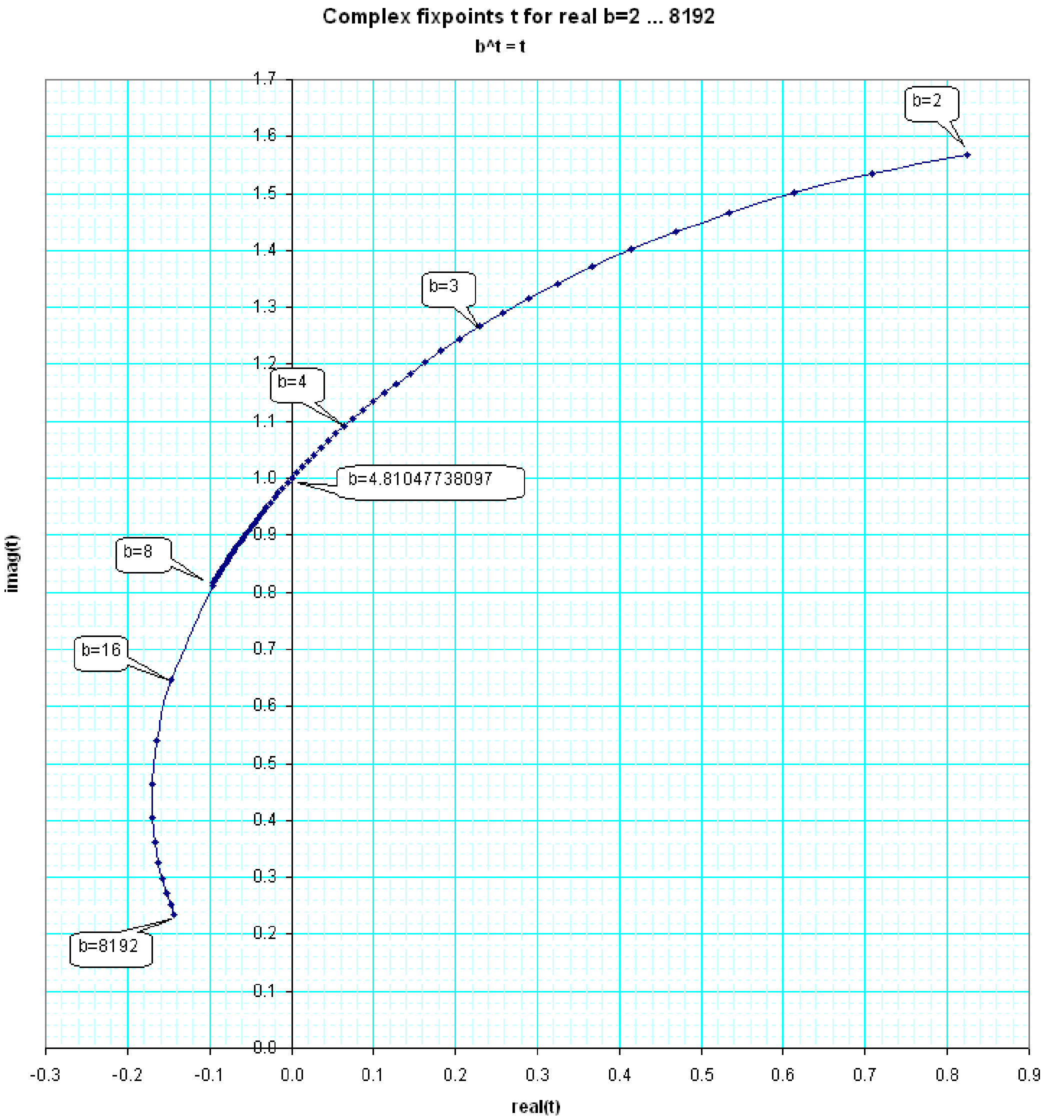

Fig. 1

© Gottfried

First observation: OK, we can see in the plot the key point quoted $b_0 = \rho = 4.810477381..$, but how about the other bases $b_m = \rho^{(4m+1)}$, for which the heights should be {-i,+i}, such as:

$$b_{-3} = 0.000000031...$$
$$b_{-2} = 0.000388203....$$
$$b_{-1} = 0.008983291...$$
$$b_1 = 2575.970497...$$
$$b_2 = 1379410.705...$$
$$b_3 = 738662922.3... \qquad (19)$$

Second observation: In the plot, even point quoted $b_1 = 2575.97..$, where we should have $t = \pm\, i$, is missing. Why? Where are the other points quoted as in (19), if any of them is correct? Is the $b = b(t)$ curve cyclically quoted (if somebody ... understands what I am fuzzily trying to say?).

Third observation: The plot must have a symmetrical part, showing the negative imaginary values (Gottfried provided some). Do we have other real-base branches passing by (0, +*i*) and (0, -*i*)?

Here is the same plot, re-visited by Andrew, showing the two symmetrical parts and the intersections of a family of curves with the real and the imaginary axes (*iy* vertical; *y* horizontal, *b* parameter):

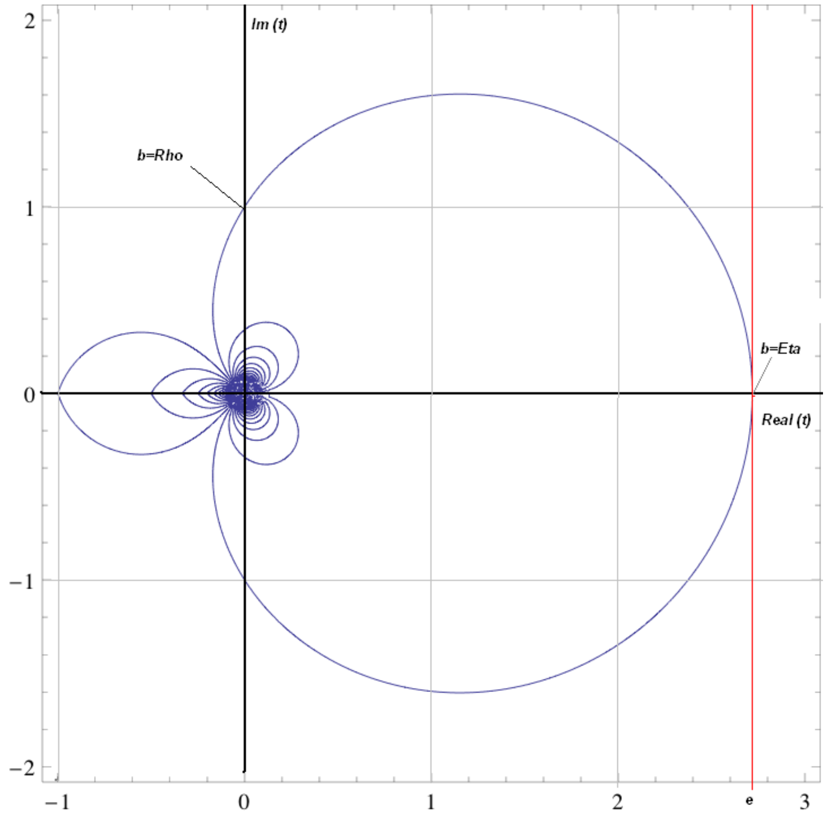

Fig. 2

©Andydude, re-visited by GFR

This figure shows the two {+i, -i} symmetrical imaginary unit heights, both quoted by the same base $b_0 = \rho = 4.810..$. The plot also shows the value of the "tower height" in the real base plot for $t = e = 2.718281828459..$, with Im(t) = 0, obviously quoted by base $b = \eta$. Nice!

In the "core of the heart", we can imagine a point where both branches of the heart-like larger plot (like those of all the other shown plots) seems to converge, for $b \to +\infty$. This point has clearly coordinates 0 + *i* 0. i.e. the complex 0, those of the origin of the complex plan. This fact seems to indicate that:

$$\boxed{\lim_{b \to +\infty} {}^{\infty}b = 0} \quad (19)$$

3 – Mysterious tetrations

The application of (3) for $b > \eta$ gives (with *Mathematica*) the following plot with the two coincident real (**black**) and two symmetrical imaginary (**brown** and **blue**) parts of $y = {}^{\infty}b$. We can see point of abscissa $b_0 = \rho = 4.810..$, where $y = \{-i, +i\}$. And we also see that, at that value of *b*, Re(*y*) = 0.

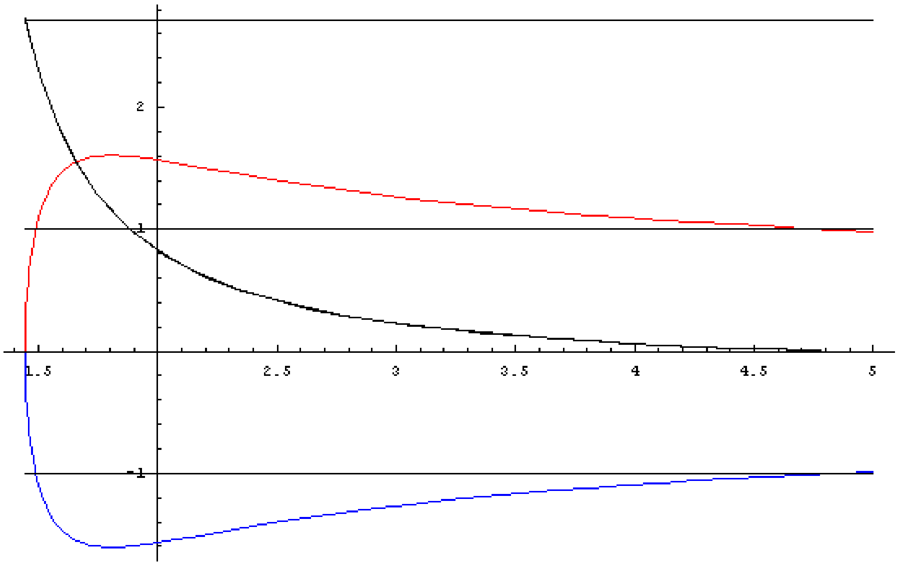

Fig. 3

The plot was obtained by using formula (3), and superposing, vertically, a *iy* imaginary axis to the real ordinates *y*. The colour plots should be better shown on a complex plane, perpendicular to the horizontal *b* axis. In this case, the brown/blue lines would appear on another *iy*-*b* complex plane, perpendicular to the worksheet. The projection of the plots on the *y*-*iy* plane should give the plot of Fig. 2 (rotated of 90°, anti-clockwise). A plot similar to Fig. 3, extended to *b* = 50, is shown in Fig. 4.

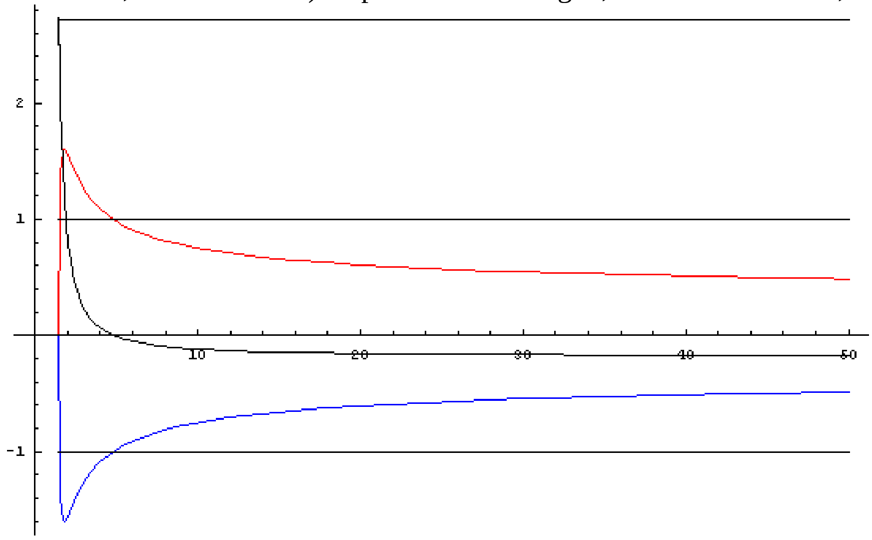

Fig. 4

Again, for $b = \rho$, we have $y = h = \{-i, +i\}$. The real part goes negative, for very high *b* and the modulus of the imaginary parts has a maximum $\text{Im}(y) \approx 1.6$ at $b \approx 1.75$, where $\text{Re}(y) \approx 1.2$, as also indicated by Fig. 2. Fig. 3 and 4 suggest that, in (19), the limit "goes to zero" ... very slowly.

The application of (3) for $b < \beta$ gives the plot of figure 4, in which in **black** we can see the real lower branch of $y = {}^{\infty}b$ and in colour the imaginary (**blue**) and real (**red**) parts of the complex extension of its upper branch. Point of abscissa $b_{-1} = \rho^{-3} = 0.00898...$ is also visible.

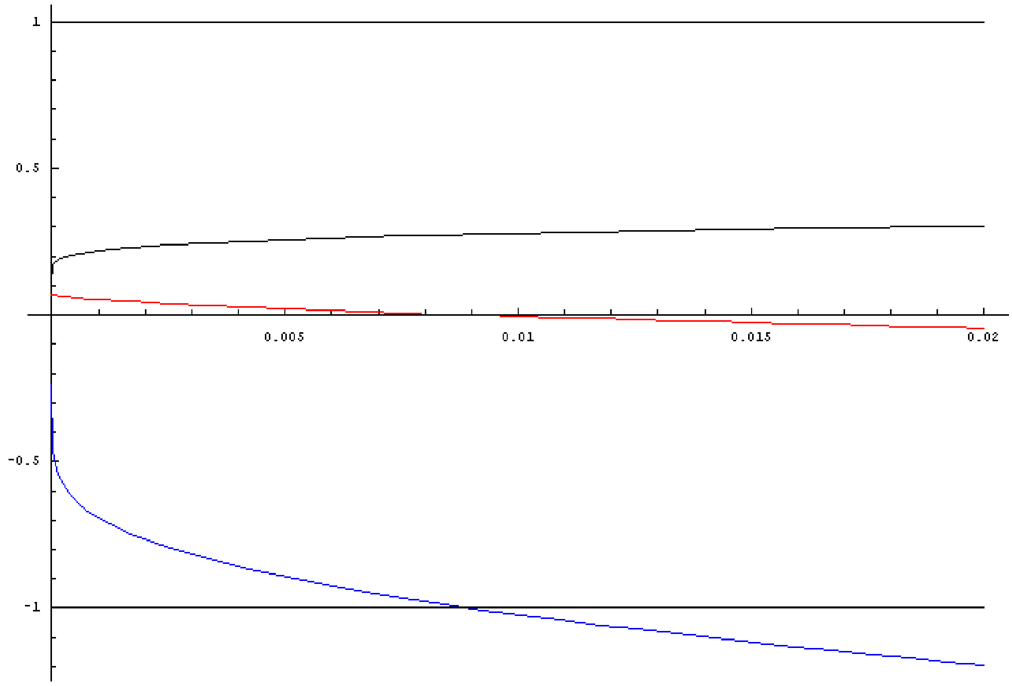

Fig. 4

Fig 4 plot shows another mystery, because point of abscissa $b_{-1} = \rho^{-3} = 0.00898...$ [$\mathrm{Re}(y) = 0,\ \ \mathrm{Im}(y) = -1$] is inside the "yellow zone" ($0 < b < \beta$), with $\beta = 0.065988036..$, where $\lim_{x\to+\infty} y = \lim_{x\to+\infty} {}^{x}b$ is indeterminate, since it is oscillating between two asymptotical $h_{\inf}$ and $h_{\sup}$ values.

Moreover, no evidence is shown of the multiple imaginary unit series shown in (19). Perhaps, Fig. 2 contains cyclical parameterizations. Or, perhaps, formulas (11), (15) and (17) are wrong or not applicable. But I don't really know what this could actually mean.

GFR.
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