Notes on Hyper-operations

Progress Report – NKS Forum III

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These notes refer to a "thread" prepared by K. A. Rubtsov (KAR) and G. F. Romerio (GFR) on 2005-12-04, submitted to the NKS Forum, *News and Announcements* section, on 2005-12-18 and referred to as document *NKS Forum II*. The aim of these notes, mainly originated from an exchange of messages between KAR and GFR (January-July, 2006), is to draw the attention to the elements in which some progress has been achieved concerning ***tetration***. Reference is always made, in this field, to the article prepared by the same Authors and published in the Internet on 2004-09-15, with the title: ***Ackermann's Function and New Arithmetical Operations***. This article is part of the NKS Bibliography and is also cited in the same NKS Forum session, on 2004-03-10, here referred to as *NKS Forum I*. For the full understanding of the present notes, it is always advisable to have read that article. The present notes put also in evidence some interesting characteristics of the hyper-operations with a rank s > 4 (***pentation***, ***hexation***, etc.), to be further analysed. Some extra notes are also included, concerning a possible extension to non integer ranks (***halfation***, ***sesquation, etc.***), for which a connection can be established with the ***arithmetic-geometric mean*** (the Gauss mean). The concept of limit hyper-operation (for a hyper-operation rank $s \to \infty$), to be called ***omegation***, together with its two inverse operations (***omegaroot*** and ***omegalog***), is also introduced.

Recall of Terminology and Symbols

$\boxed{s}$ - general hyper-operator of rank *s*, with $\boxed{1} = +$, $\boxed{2} = \times$, $\boxed{3} = \wedge$ and $\boxed{0} = \circ$ (zeration), $\boxed{4} = \#$ (tetration);

$y = {}^{z}x$ - tetration; *x* # *z*; *x*-tetra-*z* or *x*-tower-*z*, i.e. *x* raised to *x*, *z* times (*z* is the super-exponent);

$y = x \circ z$ - zeration; *x*-zerated-*z* or *x*-ball-*z*; max(x,z)+1 if x≠z, x+2=z+2 if x = z;

$x = y \Delta z$ - deltation; *y*-delta-*z* (inverse function of $y = x \circ z$);

${}^{z}\lceil y^{s}$ - the z^{th} hyper-root of *y*, rank *s*; for s = 4 : ${}^{z}\lceil y^{4} = {}^{z}\sqrt[)]{y}$ (the z^{th} super-root or tetra-root of *y*) and, for s = 3: ${}^{z}\lceil y^{3} = \sqrt[z]{y}$ (the z^{th} root of *y*);

${}_{x}\lfloor y_{s}$ - the hyper-logarithm, base *x*, of *y*, rank *s*, that can also be written as: $\underset{s}{\text{hlog}}_{x}\, y$; for *s* = 4: ${}_{x}\lfloor y_{4} = \text{slog}_x\, y$ (super-log, or tetra-log, base *x*, of *y*) and for *s* = 3: ${}_{x}\lfloor y_{3} = \log_x y$;

$\sqrt[)]{y}$ - the square super-root (tetra-root) of y : $\sqrt[)]{y} = {}^{2}\lceil y^{4}$;

$\text{plog}(x)$ - the ProductLog (Lambert's Function), the inverse function *y* of $x = y \cdot e^{y}$;

$\ln y$ - the natural logarithm of *y*, i.e. : $\log_e y = {}_{e}\lfloor y_{3}$;

$\text{s}\ln y$ - the natural super-logarithm (tetra-logarithm) of *y*, i.e.: $\text{slog}_e\, y = {}_{e}\lfloor y_{4}$.

NB- The present notes don't cover the hyper-operation with rank s=0 (***zeration***, see also [4]), for which a detailed progress report is being prepared. A "poster" also covering this subject will be presented at the International Conference of Mathematicians (ICM-06) to be held in Madrid in August 2006.

Notes on Hyper-operations

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1 - Introduction

It is indispensable to read the article ***Ackermann's Function and New Arithmetical Operations*** [1] available in http://www.rotarysaluzzo.it/filePDF/Iperoperazioni%20(1).pdf as well as the two previous postings in the NKS Forum on 2004-09-15 2005-12-04, before reading the present notes. The authors apologize for having introduced non standard symbols and operators in the presentation of their thoughts in this rather new mathematical field (see also [4]).

Moreover, it would also be advisable to accept some borderline definitions concerning multivalued functions (Knopp 1996), as "functions" that can assume two or more distinct values in their ranges, for at least one point in their domains. For a more complete presentation of this subject, please see: http://mathworld.wolfram.com/MultivaluedFunction.html. Normally, in fact, a "function" $y = f(x)$ must satisfy the property that, for any *x*, there is at-most one unique value of *y*. We can also say that there is a set of values of *x* (the domain of validity of the function) in which *y* can assume one and only one value (the range of the function). The correspondence described by $x \xrightarrow{f} y$ is, therefore, of the type one-to-one or many-to-one. In these notes, we shall call them "one-value (or single-valued) functions". Consequently, we shall call "multi-value functions" all single- or multi-value applications where, given $y = f(x)$, there exists a set of values of *x* (the domain), where *y* can assume one or several different values (the domain). In some cases (e.g.: $y = x^2$ and $x = \sqrt{y}$), the set of the positive *y*'s is chosen as principal value and the inverse is shown as $x = \pm\sqrt{y}$ $(y \geq 0)$. But this cannot generally be done, particularly in all the cases where the function to be inverted is neither periodical nor symmetrical.

Other tools that could be useful for the study of the hyper-operations should be those of the theory of "functional relations", such as $y = f(g(x))$.

2 – The Infinite Tower

We should like to refer to the problem of finding a complete and correct analytical expression for the infinite tower (infinite tetration). The problem has been analyzed by the Euler, who was also able to show that:

(1) $$y = \lim_{z\to\infty} {}^{z}x = {}^{\infty}x \quad \underline{\text{converges if}} \quad e^{-e} \leq x \leq \sqrt[e]{e}\,.$$ [(a)]

The coordinates of the extreme points of the convergence domain are:

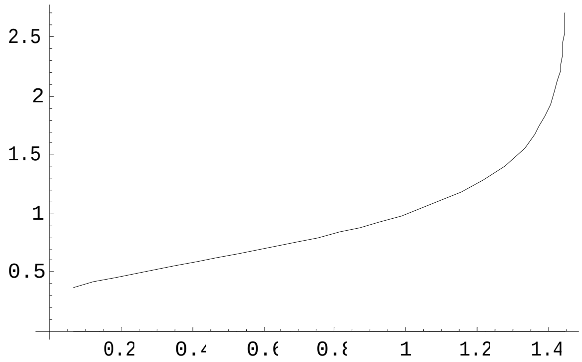
$$\begin{cases} x_1 = e^{-e} = 0.065988036.. \\ y_1 = e^{-1} = 0.367879441.. \end{cases} \qquad \begin{cases} x_2 = \sqrt[e]{e} = 1.444667861.. \\ y_2 = e = 2.7182818284.. \end{cases}$$

It is also well known that the infinite tower can be expressed by using the Lambert's Function (the **ProductLog** function, available within the standard "***Mathematica***" operators). In this case, if we put $x = y \cdot e^{y}$ and we indicate Lambert's function by: $y = w(x) = \text{ProductLog}[x] = \text{plog}(x)$, we get:

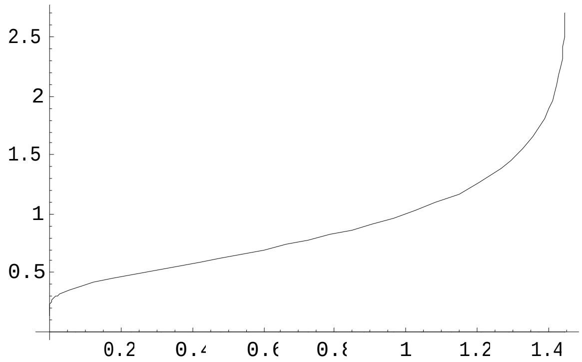
[a] However, there is a problem in the range $1 < x \leq \sqrt[e]{e}$.

(2) $$y = {}^{\infty}x = \text{inftow}(x) = \frac{\text{plog}(-\ln x)}{-\ln x}$$

with the above-mentioned domain limitations for *x*. The plot of this function (*x* is horizontal) is:

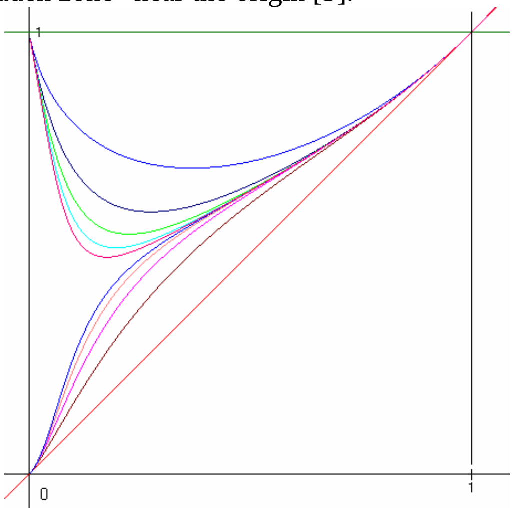


A first problem is that expression (2) is also defined up to *x* = 0, as it is easily shown in the following plot:

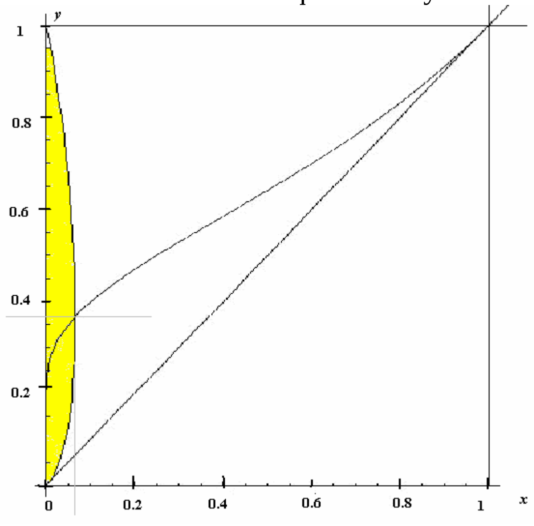


At both the extremities of the full "extension" ($x_0 = 0$ and $x_1 = \sqrt[e]{e}$) , the tangents to the curve are vertical. Near the origin, this can be shown via an appropriate "zooming". Nevertheless, this extension is no more valid for 0 < *x* < $x_1 = e^{-e} = 0.065988036..$, where the infinite tower doesn't have a defined value. A deeper analysis can show that, in reality, in that range, it becomes a multi-value expression (therefore, it is no more an actual standard and … respectable "function").

If we try now to plot the curves of $y = {}^{z}x$, for *z* = *n* (positive integer), we obtain the following plots, which define a kind of "forbidden zone" near the origin [3].



The graphs are obtained for $z = n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$. For n even, the curves cross point (0,1) and, for n odd, point (0,0). The "forbidden zone" can qualitatively be shown as follows (yellow area):



In order to explore the yellow area, we decided to re-examine definition (1) and put it in the form of an implicit functional equation, as follows:

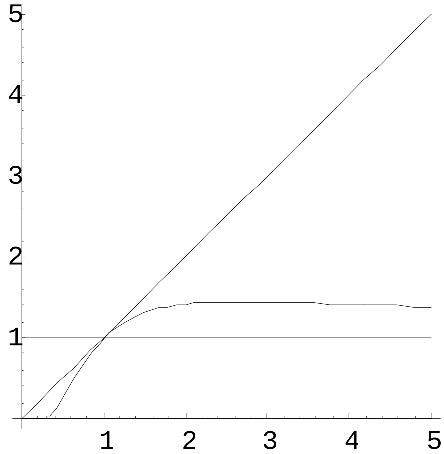
(3) $$y = \lim_{z \to \infty} {}^{z}x = {}^{\infty}x = x^{y}$$

which automatically implies that:

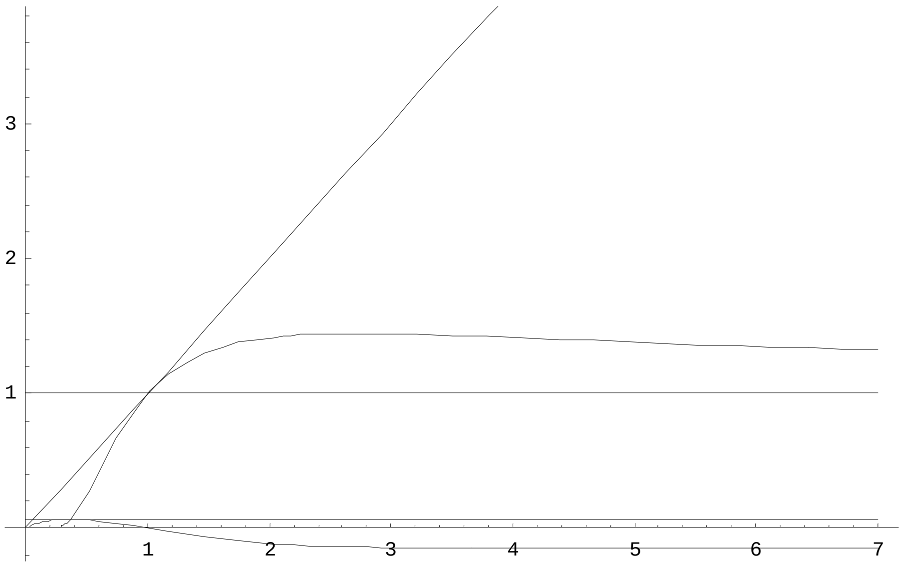
(4) $$x = \sqrt[y]{y}\ .$$

Within the validity of this transformation, we can say that the inverse function of the infinite tower can be expressed via the y^{th} root of y, i.e. by what we might call the "*selfroot*" of y.

Function $x = \sqrt[y]{y}$ is a well-known standard continuous function, defined in the domain $0 < y \leq \infty$. Its plot is as follows (shown together with $y = x$ and $x = 1$):



The plot shows a maximum for $y = e$ ($x = \sqrt[e]{e}$) and a horizontal asymptote for $y \to \infty$ ($x = 1$). Now, y is horizontal and x is vertical. The problem, now, is how to put in evidence the yellow zone, which should in principle also appear in the plot of the inverse of the infinite tower. It should be a continuous function crossing points with (y,x) coordinates in (0,0) and (1,0), with a maximum in: $y_1 = e^{-1} = 0.367879441..$ and $x_1 = e^{-e} = 0.065988036..$.



Our “piecewise”(!?) heuristic analysis starts from the observation that the maximum of $y = {}^{\infty}x$ is obtained for $y = e$ and it gives $x = \sqrt[e]{e}$, and the maximum of the curve defining the yellow zone in the domain $0 < y < 1$ is in $y = e^{-1}$ and it gives $x = e^{-e}$. From one point of maximum we can obtain the second one by exchanging $e \leftrightarrow e^{-1}$. We also found a similar observation in a Web page, where the Author was amazed by this and attributed to Euler some ideas about how we might operate. In fact, after having noticed that:

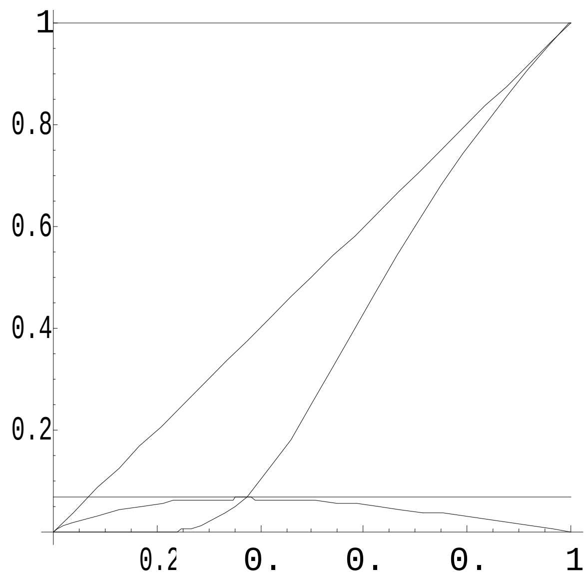
$$\sqrt[1/y]{1/y} = (1/y)^{y} = 1/y^{y} = y^{-y}$$

We tried to use y^{-y} for getting a curve passing through points (0, 0), (1, 0) and with a maximum in point (y_1, x_1). This piecewise curve is shown in the previous figure, in the range $0< y< 1$.

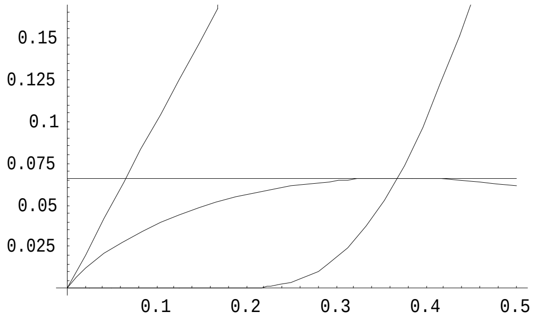
The piecewise formula of that curve is as follows (the part for $y > 1$ has no meaning, … we think):

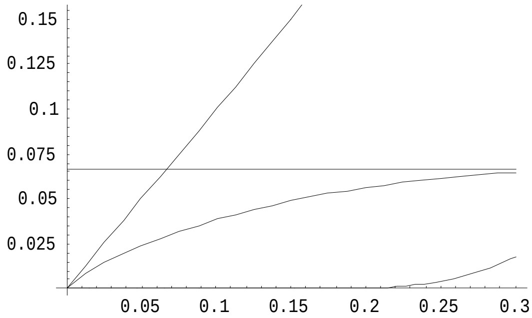
(5) $$x = \kappa \cdot \left(y^{-y} - 1 \right), \text{ i.e.: } y = \sqrt{\frac{1}{1 + x/\kappa}}, \text{ with: } \kappa = \frac{e^{-e}}{e^{1/e} - 1} = 0.148398483..$$

The first zooming of this “attractor” curve for x and y positive < 1 is as follows:



A second and a third zooming give the following plots (*x* vertical, *y* horizontal)





The inversion of such function, i.e. $y = {}^{\infty}x = \mathrm{inftow}\left(x\right)$, is clearly a <u>three-valued function</u> for:

$$0 < x < x_1 = e^{-e} = 0.065988036..$$

3 - <u>**The Square Super-root**</u>

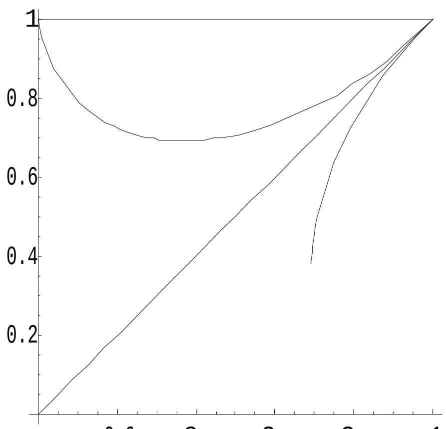
The square super-root (super-squareroot) function is the inverse of the square tower function, i.e.:

(6) $$y = {}^{2}x = x^{x} \quad \leftrightarrow \quad x = \mathrm{ssqrt}\, y = \sqrt[s]{y}$$

Let us recall formula (12, *NKS Forum II*), which stipulates that $1/\sqrt[s]{y} = {}^{\infty}\left(1/y\right)$. Taking also (2) into account, we obtain that the square super-root function can be expressed by using the product-log operator (Lambert's Function) as follows:

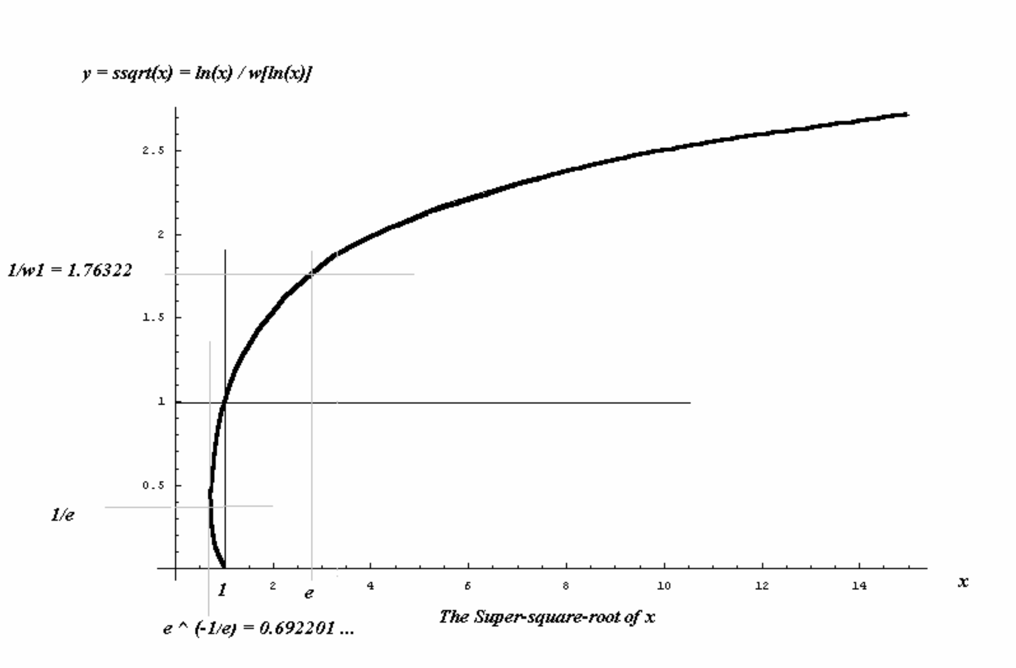
(7) $$\boxed{x = \mathrm{ssqrt}\, y = \sqrt[s]{y} = \frac{\ln y}{\mathrm{plog}\left(\ln y\right)}}$$

where plog(v) = w(v) is the Lambert's Function (in *Mathematica*: ProductLog[v]). The plots of the direct (6) and inverse (7) functions are as follows [shown together with $y = x$ and $y = 1$, after an exchange of variables in (7); x is horizontal]:

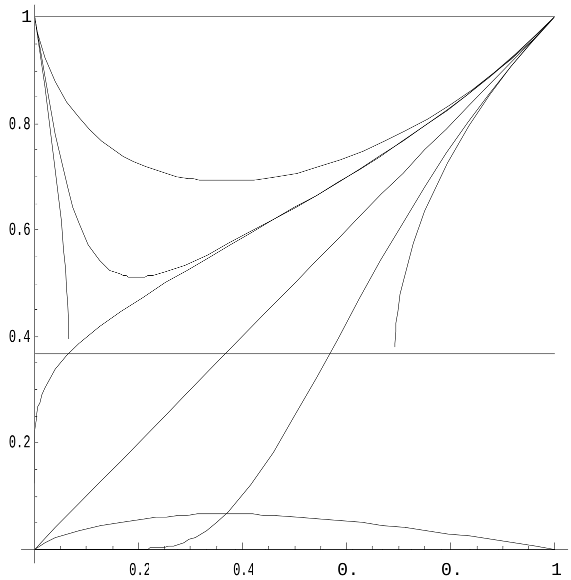


Mathematica doesn't return the complete direct function (the square tower), but it only gives the upper "principal branch" of the inverse (the square super-root), which actually is a two-valued function in the domain $0 < x < 1$. The complete inverse (two-valued) function can be imagined by a graphical inversion of the square tower. Function $y = {}^{2}x$ has a minimum $y = e^{-1/e}$, for $x = e^{-1}$, and the range of validity of $y = \sqrt[s]{x}$ is defined by $x > e^{-1/e}$ = 0,692201.. , where the function's value is $y = e^{-1}$. The same figure clearly shows the problems that we may find when dealing with multi-valued functions.

The following figure gives an idea of the plot of the two-valued (extended) $y = \overline{)x}$ = ssqrt(x) function (figure taken from [1]), obtained by a graphical inversion.



The super-square-root of x for $x < e^{-1/e}$ involves complex solutions. The unique real value of y for $x = e^{-1/e}$ is $1/e$. For $e^{-1/e} < x \leq 1$, the super-square root has two real values. For $x > 1$, it is a continuous increasing function and we have: $\lim_{x\to\infty} ssqrt(x) = \infty$. In particular, with $w_1 = \text{plog}(1) = 0{,}567143...$, the *omega constant*, solution of $w.e^{w} = 1$, for $x = e$, we have: ${}^{2}\overline{)e} = 1/w_1 = 1{,}763220...$. The following figure shows the plots (for x > 0) of some easily identifiable functions (from up/left to down-right)[b]: $y = {}^{2}x$; $y = {}^{4}x$; $y = {}^{\infty}x$ (partial); $y = x$; $y = {}^{\infty}\overline{)x} = \sqrt[x]{x}$ (complete); $y = {}^{2}\overline{)x}$ (partial).



The plots shown in the area where $y > x$ represent tower functions with super-exponents $n > 1$. Those in the area where $y < x$ represent super-root functions (tetra-roots) of n-th order, with $n > 1$.

[b] The plot of the infinite tower is "partial" because of the lack of the lower branch of its "piecewise" complement; the plot of the square super-root doesn't show the lower part of the two-valued "function".

4 – **Infinite Hyperations**

By “*infinite hyperations*” we mean the application of the general hyper-operator $\boxed{s}$ to an infinite operand (- ∞, +∞). This problem, as far as the traditional ranks s=1 (addition), s=2 (multiplication), s=3 (exponentiation) are concerned, is classically dealt with in standard algebra. For instance, we know that $e^{-\infty} = 0$ and $e^{+\infty} = +\infty$. The same problem, applied to tetration (s=4), has been examined in section 2, where we have shown that ${}^{+\infty}a$ may converge or not, according to the value of *a*, while the study of ${}^{-\infty}a$ requires the use of sophisticated mathematical procedures.

For introducing the study of infinite hyperations for ranks s>4, let us begin our research from the study of some properties of the hyper-operations that can be very useful in studying their domain of existence, their ranges and their behaviour. For instance, let us consider formula (6, *NKS Forum II*, see paper [3]):

(10) $$x\boxed{s}(z+1) = x\boxed{s-1}\left(x\boxed{s}z\right)$$

By taking the hyper-log, with rank *s*-1 and to the base *x*, of both members of this expression, we have:

(11) $$x\boxed{s}z = \underset{x\ s-1}{\lfloor\!\underline{\ \ }}\left(x\boxed{s}(z+1)\right)$$

For hyper-base *x = a* (real positive) and hyper-exponent *z = n* (integer positive), we have:

(12) $$\boxed{a\boxed{s}n = \underset{a\ s-1}{\lfloor\!\underline{\ \ }}\left(a\boxed{s}(n+1)\right) = \underset{s-1}{\mathrm{hlog}_a}\left(a\boxed{s}(n+1)\right)}$$

Formula (13) can be implemented, for various hierarchical levels, as follows:

for *s*=2, *multiplication*: $a \times n = \underset{a\ 1}{\lfloor\!\underline{\ \ }}\left(a\times(n+1)\right) = \mathrm{sbt}_a\left(a\times(n+1)\right) = a\cdot n + a - a$ ([c])

for *s*=3, *exponentiation*: $a \wedge n = \underset{a\ 2}{\lfloor\!\underline{\ \ }}\left(a\wedge(n+1)\right) = \mathrm{div}_a\left(a\wedge(n+1)\right) = \dfrac{a^{n+1}}{a}$

for *s*=4, *tetration*: $a\# n = \underset{a\ 3}{\lfloor\!\underline{\ \ }}\left(a\#(n+1)\right) = \log_a\left(a\#(n+1)\right) = \log_a\left({}^{n+1}a\right)$

for *s*=5, *pentation*: $a\S n = \underset{a\ 4}{\lfloor\!\underline{\ \ }}\left(a\S(n+1)\right) = \mathrm{slog}_a\left(a\S(n+1)\right)$

for *s*=6, *hexation*: $a\$n = \underset{a\ 5}{\lfloor\!\underline{\ \ }}\left(a\$(n+1)\right) = \mathrm{sslog}_a\left(a\$(n+1)\right)$

Moreover, we may also say, looking at (10) and (12), that if at least one value $a\boxed{s}n$ of the result of an hyper-operation is known, then an infinite series of other values can be calculated, by iteratively using the following formulas:

(13) $$\boxed{a\boxed{s}(n+1) = a\boxed{s-1}\left(a\boxed{s}n\right), \qquad a\boxed{s}(n-1) = \underset{a\ s-1}{\lfloor\!\underline{\ \ }}\left(a\boxed{s}n\right)}$$

It is also possible to show the use of the logarithm and hyper-log operations for studying the hyper-operations’ behaviour, at levels (for instance) s=4 and s=5. In fact, choosing the hyper-base *a = e*,

for *tetration* (*s*=4), we have: $e\boxed{4}3 = e\#3 = e\wedge(e\wedge e) = 3814279{,}1047602..$

(14) $$e\boxed{4}2 = e\#2 = e\wedge e = 15{,}154262241479..$$

$$e\boxed{4}1 = e\#1 = e = 2{,}718281828455045..$$

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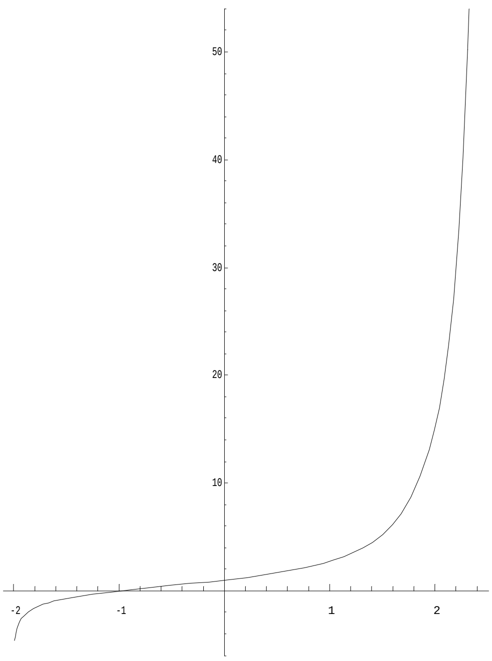
[c] Operators **sbt** $_a$ and **div** $_a$ mean “subtract *a*” and “divide by *a*”, similar to the “**log** $_a$, slog and **sslog** $_a$” operators.

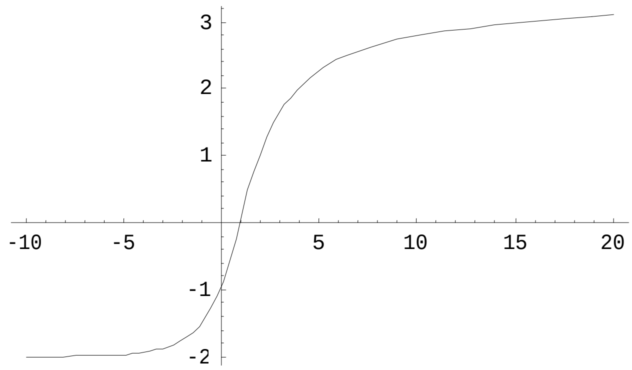
and also:

(15)

$$e\boxed{4}0 = e\#0 = \ln e = 1$$
$$e\boxed{4}(-1) = e\#(-1) = \ln 1 = 0$$
$$e\boxed{4}(-2) = e\#(-2) = \ln 0 = -\infty$$
$$e\boxed{4}(-3) = e\#(-3) = \ln(-\infty) \qquad \text{see [1]}$$
$$e\boxed{4}(-4) = e\#(-4) = \ln(\ln(-\infty))$$
$$\ldots\ldots\ldots$$
$$e\boxed{4}(-\infty) = e\#(-\infty) = \mathbf{\Theta}$$

The characteristics of the tetration function $y = e\boxed{4}n = e\#n$ for *n* integer and *n* < 0 are very peculiar, because the values of *y* are obtained as logarithms of negative numbers (complex multiple numbers), as well as by iterated applications of logarithms to the results. The infinite iteration of this procedure gives, as result, a complex infinite multi-value number, here indicated as "number" **Θ** (Theta)[d]. The study of the $e\boxed{4}n = e\#n$ sequence (for *n* < 0) can be approached via the introduction of the "*Delta Numbers*", also obtained through the inverse of the "*zeration*" operation (hyper-operation of rank s=0). For rank s=5, we have what can be called "*pentation*" or hyper-operation of rank five. Almost no analysis has been made so far concerning this level. We think that the first fundamental tools to be used are based on the implementation of the successful fair approximation obtained for the tetrational function to the base *e* (see [3]). These approximations have been implemented in a prototype hyper-calculator and are shown in the following plots:





On the left: $y = {}^{x}e$ and on the right: $y = \mathrm{slog}_e\, x$.

The tetrational function is the left plot and the super-logarithm is in the right position.. These diagrams have been obtained by supposing that the two functions can be analytically linearly approximated in the domain -1 ≤ *x* ≤ 0 (for the tetrational function) and 0 ≤ *x* ≤ 1 (for the superlog function). These rather fair approximations can also be used for describing some properties of pentation.

In fact, for *s*=5, we have:

(16)

$$e\boxed{5}2 = e\S 2 = e\#e = 2380{,}2121739766.. \quad \text{(approximated value)}$$
$$e\boxed{5}1 = e\S 1 = e = 2.718281828459045..$$
$$e\boxed{5}0 = e\S 0 = \mathrm{sln}\, e = 1$$

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[d] Number **Θ** can be defined as **Θ =** $\ln^{\infty}(0)$, i.e. by applying the "log" operator, infinite times, on number "0".

Moreover, we also have:

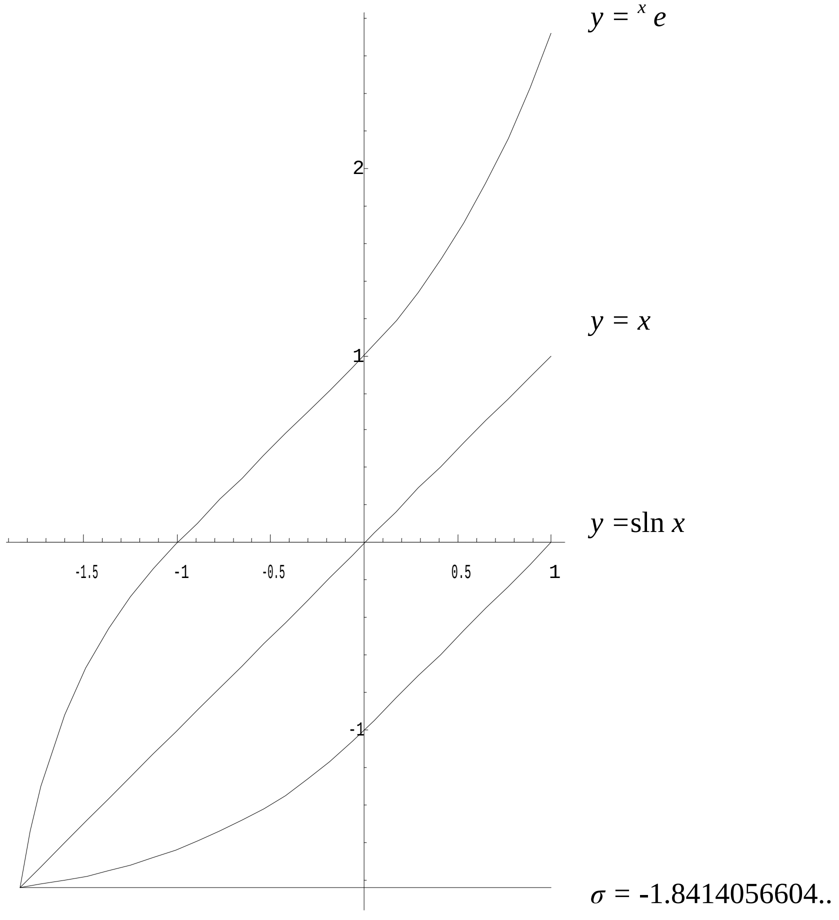
(17)

$$e\boxed{5}(-1) = e\S(-1) = \text{sln}\,1 = 0$$
$$e\boxed{5}(-2) = e\S(-2) = \text{sln}\,0 = -1$$
$$e\boxed{5}(-3) = e\S(-3) = \text{sln}(-1) = -1{,}63212055882856..$$
$$e\boxed{5}(-4) = e\S(-4) = \text{sln}(-1{,}63212..) = -1{,}80448546584741..$$
$$\dots\dots\dots$$
$$e\boxed{5}(-\infty) = e\S(-\infty) = -1{,}84140566043697.. \text{ (asymptotic value } \sigma\text{)}$$

The approximate asymptotic value $\sigma = e\boxed{5}(-\infty)$ has been obtained by using the KAR-Calc hyper-calculator, via iterative calculations. Nevertheless, the asymptotic value $\sigma = e\boxed{5}(-\infty)$ can be also found if we observe that, from the previous formulas, we must have: $\text{sln}\,\sigma = \sigma$ and, therefore:

(18)
$$\boxed{\sigma = \text{sln}\,\sigma = {}^{\sigma}e = -1{,}84140566043697..}$$

determines the position of point (σ,σ) is shown in the graphical presentation of functions $y = x$, $y = \text{sln}\,x$ and $y = {}^{x}e$.



Therefore, number σ appears as a new remarkable number and it is the value of the coordinates of the intersection between the *natural tetration* and the *natural superlog* functions, as well as the asymptotic value of the *minus-infinite pentation* (with hyper-base e). It is also the solution of the implicit functional equation $y = {}^{y}x = x\boxed{5}\infty$, for $x = e$.

5 – **Omegation and its Inverses (Omegalog and Omegaroot)**

Nothing is against the idea that the hyper-operations' hierarchy is infinite (the Grzegorczyk hierarchy) and that the study of ranks s > 4 will certainly be successfully approached in the future. For the moment, it is interesting to observe that, since the hierarchy is infinite, a limit hyper-operation could exist, for s → ∞, having some common characteristics with the other members of the hierarchy.

For defining the infinite-rank hyper-operation (and of its inverses) we shall use symbol *ω*, similar to the symbol of the first enumerable infinite ordinal. Actually, *omegation* and its inverses (*omegaroot* and *omegalog*) are new mathematical objects, which are not connected at all with the theory of the infinite ordinals. As a matter of fact, let us define:

(19) $y = \lim_{s\to\omega}\left(a\boxed{s}n\right) = a\boxed{\omega}n$ omegation, with $s \in \mathrm{N}$

with: $a = {}^{n}\overset{\omega}{\ulcorner}\, y$ n^{th} omegaroot of *y*

and: $n = {}_{a}\underset{\omega}{\llcorner}\, y$ omegalog, base *a*, of *y*

Moreover, in order to investigate the properties of this new hypothetical operation, the following general properties need also to be taken into consideration.

In fact, since we have: $2 \circ 2 = 2 + 2 = 2 \times 2 = 2 \wedge 2 = 2 \# 2 = \ldots. = 4$

we could also assume that: $2\boxed{0}2 = 2\boxed{1}2 = 2\boxed{2}2 = 2\boxed{3}2 = 2\boxed{4}2 = \ldots\ 2\boxed{n}2 \ldots\ 2\boxed{\omega}2 = 4$

while, since for *s*>1, we have: $a \times 1 = a \wedge 1 = a \# 1 = \ldots = a\boxed{n}1 = \ldots = a$

we can also assume that: $a\boxed{2}1 = a\boxed{3}1 = a\boxed{4}1 = \ldots = a\boxed{n}1 = \ldots\ a\boxed{\omega}1 = a$

and, since for *s*>2, we have: $a \wedge 0 = a \# 0 = \ldots = a\boxed{n}0 = \ldots = 1$

we can also assume that: $a\boxed{3}0 = a\boxed{4}0 = \ldots = a\boxed{n}0 = \ldots = a\boxed{\omega}0 = 1$.

Also, for *s* → ∞, we have: $a\boxed{\omega}n = \underbrace{a\boxed{\omega-1}a\boxed{\omega-1}\ldots\boxed{\omega-1}a}_{n} = \underbrace{a\boxed{\omega}a\boxed{\omega}\ldots\boxed{\omega}a}_{n}$

and we know already that: $a\boxed{\omega}0 = 1, \quad a\boxed{\omega}1 = a$

to which we can now add: $a\boxed{\omega}2 = a\boxed{\omega}a$

and, also: $1\boxed{\omega}2 = 1\boxed{\omega}1 = 1$

which also means that: $1\boxed{\omega}n = 1$.

But we also have that: $2\boxed{\omega}2 = 4$

and, therefore: $2\boxed{\omega}3 = 2\boxed{\omega}\left(2\boxed{\omega}2\right) = 2\boxed{\omega}4$

and, also: $2\boxed{\omega}4 = 2\boxed{\omega}\left(2\boxed{\omega}\left(2\boxed{\omega}2\right)\right) = 2\boxed{\omega}\left(2\boxed{\omega}4\right) = 2\boxed{\omega}5$

then, we must have: $2\boxed{\omega}3 = 2\boxed{\omega}4 = 2\boxed{\omega}5 = \ldots = 2\boxed{\omega}n = \ldots = 2\boxed{\omega}\infty = \infty$

and, therefore: $\boxed{a\boxed{\omega}n = \infty}$ for any $a \geq 2$ and $n \geq 3$.

From (19) of *NKS Forum II* [3] and KAR's notes:

$$\underset{x\;s}{\llcorner} y = \underset{x\;s}{\llcorner}(\underset{x\;s-1}{\llcorner} y)+1$$ hyper-log formula

we have: $$\lim_{s\to\omega} \underset{x\;s}{\llcorner} y = \lim_{s\to\omega} \underset{x\;s}{\llcorner}(\underset{x\;s-1}{\llcorner} y)+1$$

which means: $$\underset{x\;\omega}{\llcorner} y = \underset{x\;\omega}{\llcorner}(\underset{x\;\omega}{\llcorner} y)+1$$ or:

(20) $$\boxed{\underset{x\;\omega}{\llcorner}(\underset{x\;\omega}{\llcorner} y) = \underset{x\;\omega}{\llcorner} y - 1}$$

Then, if we admit that: $$a\boxed{\omega}0 = {}_a\underset{\omega}{\llcorner} a = 1$$ omegalog, base *a* of *a*

we must have: $$a\boxed{\omega}(-1) = {}_a\underset{\omega}{\llcorner} 1 = {}_a\underset{\omega}{\llcorner}\left({}_a\underset{\omega}{\llcorner} a\right) = \left({}_a\underset{\omega}{\llcorner} a\right) - 1 = 1-1 = 0$$

and: $$a\boxed{\omega}(-2) = {}_a\underset{\omega}{\llcorner} 0 = -1$$

and, also: $$a\boxed{\omega}(-3) = {}_a\underset{\omega}{\llcorner}(-1) = -2$$

……………………………

(21)
and, finally, we should have: $$\boxed{a\boxed{\omega}(-n) = {}_a\underset{\omega}{\llcorner}(2-n) = 1-n}$$

From KAR's notes and from (8) of *NKS II*, if we have:

$$y = x\boxed{s}z$$

meaning that: $$x = {}^{z}\overset{s}{\ulcorner} y$$

then $$\lim_{z\to\infty} {}^{z}\overset{s}{\ulcorner} y = {}^{\infty}\overset{s}{\ulcorner} y = {}^{y}\overset{s-1}{\ulcorner} y$$

i.e.: $${}^{\infty}\overset{s}{\ulcorner} y = {}^{y}\overset{s-1}{\ulcorner} y$$

Let us consider now: $$a = x\boxed{\omega}\infty$$ *x*-omega-∞

which gives: $$x = {}^{\infty}\overset{\omega}{\ulcorner} a$$ the infinite omegaroot of *a*

But we also have: $$\lim_{s\to\omega} {}^{\infty}\overset{s}{\ulcorner} a = \lim_{s\to\omega} {}^{a}\overset{s-1}{\ulcorner} a \quad\to\quad {}^{\infty}\overset{\omega}{\ulcorner} a = {}^{a}\overset{\omega}{\ulcorner} a$$

Therefore, we may write:

(22) $$\boxed{{}^{\infty}\overset{\omega}{\ulcorner} a = {}^{a}\overset{\omega}{\ulcorner} a}$$

<u>Conclusion</u>: For any *a*, the *infinite omega-root* of *a* is equal to the a^{th} *omega-root* of *a*.

6 – **Possible Extension to Non-integer Ranks (Sesquation, Halfation)**

In http://personal.lse.ac.uk/williahp/seminar_abstracts.htm, Prof. Paul Williams of the London School of Economics (LSE) made a reflection about the possible existence of an operation "*between Addition and Multiplication and beyond*", saying that such an operation could have practical value if found. He also added that "*… In order to do this we investigate (i) a generalisation of Ackermann's Function, (ii) the solution to the functional equation f f (x) = e^x , (iii) Gauss' Mean which lies between the Arithmetic and Geometric Mean*". He organized a seminar about this problem on 16th March 2006 [6].

In fact, this hypothetical operation, that we might call ***sesquation***, could be labelled as follows (s=3/2):

(23) $y = a\boxed{3/2}n$ *a*-sesqued-*n*

and we probably should have: $y = 2\boxed{3/2}2 = 4$

The word *sesquation* or *sesquition* comes from a list of neologisms found in the following Web site: http://michaelhalm.tripod.com/andre_joyce_s_coined_words.htm, where the term ***sesquition*** is put in connection with new arithmetical operations. The prefix sesqui- means 1,5 or 3/2, like in the term sesqui-oxide (Fe_2O_3).

Another hyper-operation, *between Zeration and Addition*, with rank s=1/2, could be called ***halfation***.

and we should have: $a\boxed{1/2}n$ *a*-halfed-*n*

and we should also have: $2\boxed{1/2}2 = 4$

For establishing the bases for further research in this direction, we have to may recall again the following formulas:

(24) $x\boxed{s}(z+1) = x\boxed{s-1}\left(x\boxed{s}z\right)$ general hyper-operation formula

$\underset{x\ s}{\lfloor} y = \underset{x\ s}{\lfloor}(\underset{x\ s-1}{\lfloor} y)+1$ hyper-log

${}^{\infty}\sqrt[s]{y} = {}^{y}\sqrt[s-1]{y}$ the self-hyperroot

By putting s=3/2, these formulas become:

$$y = x\boxed{3/2}(z+1) = x\boxed{1/2}\left(x\boxed{3/2}z\right)$$

$$\underset{x\ 3/2}{\lfloor} y = \underset{x\ 3/2}{\lfloor}(\underset{x\ 1/2}{\lfloor} y)+1$$

$${}^{\infty}\sqrt[3/2]{y} = {}^{z}\sqrt[1/2]{y}$$

and we should also have:

$a\boxed{3/2}2 = a\boxed{1/2}a$ obviously!

$a\boxed{3/2}1 = a\boxed{1/2}1 = a$ we presume!

$a\boxed{3/2}0 = a\boxed{1/2}0 = 1$ to be verified

Moreover, since we know that for $0 \le s \le 2$ (at least for zeration, addition, multiplication) the direct hyper-operation is commutative, we could assume that also *halfation* and *sesquation* are commutative. In this case, these new operations should also admit only one inverse.

Therefore we should have:

(25) $$\boxed{{}^{3/2}_{\infty}\!\sqrt{y} = {}^{1/2}_{y}\!\sqrt{y} = \underset{y\;1/2}{\lfloor} y = 1}$$ to be verified

and also: $$\boxed{{}^{5/2}_{\infty}\!\sqrt{y} = {}^{3/2}_{y}\!\sqrt{y} = \underset{y\;3/2}{\lfloor} y = 1}$$ to be verified

Taking into account the half ranks of the hyper-operations, the following formula should apply:

$$\boxed{a\boxed{s}a = a\boxed{s+1}2}$$

which can be implemented in the following sequence:

	$a\boxed{0}a = a\boxed{1}2$	$a \circ a = a + 2$	(zeration – addition)
(26)	$a\boxed{1/2}a = a\boxed{3/2}2$		(halfation – sesquation)
	$a\boxed{1}a = a\boxed{2}2$	$a + a = a \times 2$	(addition – multiplication)
	$a\boxed{3/2}a = a\boxed{5/2}2$		(sesquation – *sesteration* ?)
	$a\boxed{2}a = a\boxed{3}2$	$a \times a = a \wedge 2$	(multiplication. – exponentiation)
	………………………………………………………………………		

The word ***Halfation*** obviously comes from English. ***Sesquation*** and ***Sesteration*** come from Latin[e]: The above-mentioned sequence should be justified by means of the Ackermann's Function.

Nevertheless, by applying the theory of functional equations, we might also take into consideration the following formula [6]:

$$y = \mathrm{f} \cdot \mathrm{f}(z) = \mathrm{f}\left[\mathrm{f}(z)\right] = e^z$$ (also called: *iterative root* [7])

In order to find the rank of a possible equivalent hyper-operation, we could put it as follows:

(27) $$y = \mathrm{f}\left[\mathrm{f}(z)\right] = e\boxed{s}\left[e\boxed{s}z\right] = e^z$$

In this case, rank *s* should be in the range 2 < *s* < 3, i. e. between multiplication and exponentiation. In the particular hypothesis that we had exactly s = 5/2 = 2,5, we should also have:

(28) $$e\boxed{5/2}z = {}_{e}\underset{5/2}{\lfloor} e^z = {}_{e}\underset{5/2}{\lfloor} e\boxed{3}z\,.$$

or, if we prefer: $$e\boxed{5/2}z = \underset{2,5}{\mathrm{hlog}_e}\, e^z$$

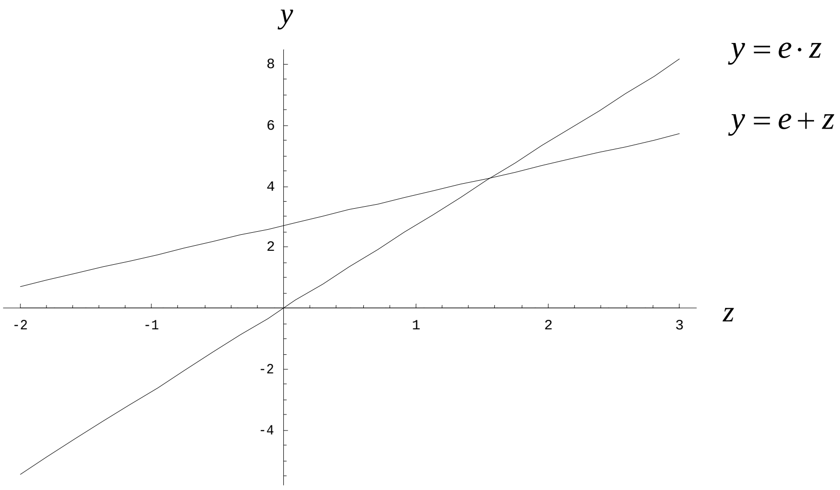
the following plots show the possible domains and ranges of existence of these hypothetical hyper-operations (to be carefully re-analyzed).

---

[e] *Pars prima, semisque altera* : One part and a half of another : 1 + ½ = 3/2
Pars prima, secunda, semisque tertia : One part, a second one and half of a third: 1 + 1 + ½ = 5/2
(Example: the Roman sestertium = 2,5 axes)

Comparison of the possible limit ranges of hyper-operations (hyper-base *e*) with ranks s=1.5, s=2.5 and s=3.5.

Sesquation: $y = e\boxed{1.5}z$



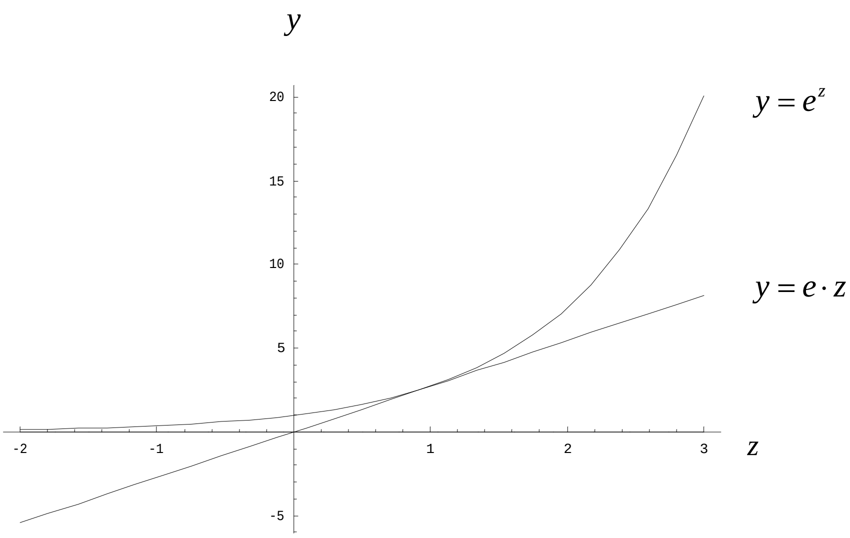
for $z \to -\infty$

$e \cdot z < e\boxed{1.5}z < e + z$

for $z \to +\infty$

$e + z < e\boxed{1.5}z < e \cdot z$

Sesteration: $y = e\boxed{2.5}z$



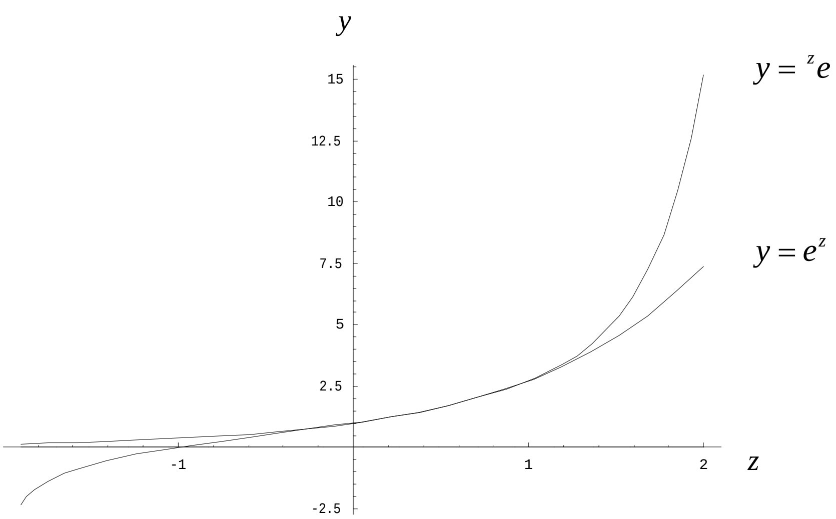
for $z \to -\infty$

$e \cdot z < e\boxed{2.5}z < e^z$

for $z \to +\infty$

$e \cdot z < e\boxed{2.5}z < e^z$

Rank 3.5: $y = e\boxed{3.5}z$



for $z \to -2$ (!)

${}^{z}e < e\boxed{3.5}z < e^z$

for $z \to +\infty$

$e^z < e\boxed{3.5}z < {}^{z}e$

These plots are conjectures concerning the possible behavior of the above-mentioned hyper-operations with fractional ranks that need a deep study and detailed demonstrations.

7 – **Sesquation and the Arithmetic-Geometric Mean**

It is interesting to observe that the problem of finding the properties of hypothetical hyper-operations with non-integer ranks (e.g. s=0.5, s=1.5, s=2.5) is in connection with a study made by Gauss, who defined a "mean" as something between the arithmetic and the geometric mean, called the "*Gauss Mean*" or the "*Arithmetic-Geometric Mean*". This operation is particularly used in advanced mathematical fields and standard operators are available in some known software packages. For example, in ***Mathematica***, operator **ArithmeticGeometricMean [*a*, *b*]** gives the arithmetic-geometric mean **(agm)** of two numbers *a* and *b*. This quantity is used in mathematical developments for computing elliptic integrals and other functions. We should like to remember here the expression of the arithmetic and geometric means (Mean_1, Mean_2), to which we should like to add Mean_3, defined as follows:

(29)

$$\mathrm{Mean}_1(a,b) = (a+b)/2 \qquad \text{arithmetic mean}$$

$$\mathrm{Mean}_2(a,b) = \sqrt{a \cdot b} \qquad \text{geometric mean}$$

$$\mathrm{Mean}_3(a,b) = \big)\overline{a \wedge b} \qquad \textit{exponential mean}^{\,f}$$

The last one, a consequence of the hyper-operations' analysis, can be defined as the square super-root of a^b and it comes from a logical extension of the first two. Due to the non-commutativity of exponentiation, we must have: $\mathrm{Mean}_3(a,b) \neq \mathrm{Mean}_3(b,a)$. A step-by-step procedure can also be envisaged for calculating **agm**, with positive reals *a* and *b*, starting by putting:

$$a_0 = a \qquad b_0 = b$$

and, then, by iterating:

$$a_{n+1} = \frac{a_n + b_n}{2} \qquad \text{[arithmetic mean]}$$

$$b_{n+1} = \sqrt{a_n \cdot b_n}\,, \qquad \text{[geometric mean]}$$

Then, by putting $a_{n+1} \to a_n$ and $b_{n+1} \to b_n$, until $a_n = b_n$ at the acceptable precision.

The **agm** mean was studied by very famous scientists such as J. L. Lagrange (1784-85), C. F. Gauss (1791-99, 1800, 1876) and J. Landen (1771, 1775) and the following formula has been developed and it is available for its calculation. Actually, **agm** is an existing standard operation in ***Mathematica***.
[See also: http://mathworld.wolfram.com/Arithmetic-GeometricMean.html
and: http://functions.wolfram.com/EllipticFunctions/ArithmeticGeometricMean/02/].
:

(30)

$$\boxed{\mathrm{agm}(a,b) = \frac{(a+b)\cdot \pi}{4 \cdot K\left(\dfrac{a-b}{a+b}\right)}} \qquad K: \text{complete elliptic integral of the first kind}$$

The "*Gauss' Mean*" has the following property:

(31)

$$\mathrm{agm}(a,b) = \mathrm{agm}\left(\frac{a+b}{2}, \sqrt{a \cdot b}\right)$$

from which we have:

(32)

$$a\boxed{1}b = a + b = \mathrm{Mean}_1(a,b) \cdot 2 = \mathrm{Mean}_1(a,b)\boxed{2}2$$

$$a\boxed{2}b = a \cdot b = \mathrm{Mean}_2(a,b) \wedge 2 = \mathrm{Mean}_2(a,b)\boxed{3}2$$

$$a\boxed{3}b = a \wedge b = \mathrm{Mean}_3(a,b) \# 2 = \mathrm{Mean}_3(a,b)\boxed{4}2$$

---

[f] The KAR-Calc hyper-calculator is programmed to execute this mean (the square super-root of the results of an exponential operation).

For ***sesquation*** (s=3/2), which should be the operation between addition (s=1) and multiplication (s=2), we should then perhaps adopt the following formula, with the intervention of ***sesteration*** (s=5/2), the operation between multiplication (s=2) and exponentiation (s=3):

(33) $$\boxed{a\boxed{3/2}b = \mathrm{agm}(a,b)\boxed{5/2}2} \quad \text{with: } \mathrm{agm}(a,b) = \frac{(a+b)\cdot\pi}{4\cdot K\left(\dfrac{a-b}{a+b}\right)}$$

A procedure similar to what has been adopted for the definition of **agm**, can probably also be applied for the definition of the *geometric-exponential mean* (rank 2.5), which could perhaps be used in the study of the $\boxed{5/2}$ hyper-operation. As we can see, the problem of extending the hyper-operations to non-integer rank is not fully solved yet! Nevertheless, the possibility of defining new "*mathematical means*" with non-integer ranks gives us the idea that a solution is probably existing. A lot of hope is put in the development of new mathematical procedures for the analytical continuation of operators [8], like "Exp" or "Log", for example.

The need of an operation between addition and multiplication, or of something between the arithmetic and geometric mean has been felt at the scientific and technical level since the years 60's of … last century, in the framework of the study of new methodologies for the assessment of information storage and retrieval systems [9].

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