Serial Developments of $y = b[4]\infty$

Let us define tetration as follows:

$$\boxed{y = {}^{x}b}$$

and let us put: $h(b) = {}^{\infty}b = \lim_{x \to +\infty} {}^{x}b$

As we know, by using the product-logarithm operator, we may write (sorry, ... my notations):

$$\boxed{y = h(b) = \text{plog}(-\ln b)/(-\ln b)}$$

$\text{plog}(z) = W_0$ $\boxed{\text{or}}$ W_{-1}, with W_i the Lambert Function of rank *i*

Ranks -1 and 0 define the only two real branches of the Lambert Function. We may also see in:
http://mathworld.wolfram.com/PowerTower.html

But, limiting ourselves to the first (principal) branch W_0, we have :

$$h_{\text{inf}}(b) = \text{plog}_0(-\ln b)/(-\ln b) = -\frac{W_0(-\ln b)}{\ln b}$$

with convergence in: $\beta \le b \le \eta$

$\beta = e^{-e} = 0.065988036..$

$\eta = e^{1/e} = 1.444667861..$

(Euler, 1783 ; Eisenstein, 1844 ; Le Lionnais, 1983 ; Wells, 1986).

We also have the following serial expansion of the Lambert function. Please see in:
http://mathworld.wolfram.com/LambertW-Function.html

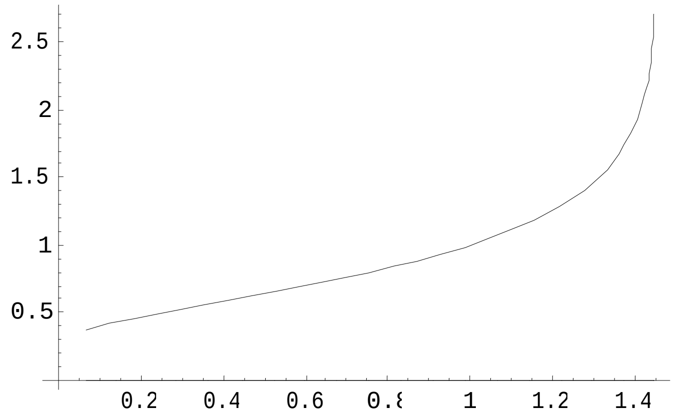
$$W_0(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} \cdot z^n$$

Taking these two expressions into account, Knoebel (1981), for $z = -\ln b$, obtained the following serial development for $h_{\text{inf}}(b)$, the lower branch of *h* :

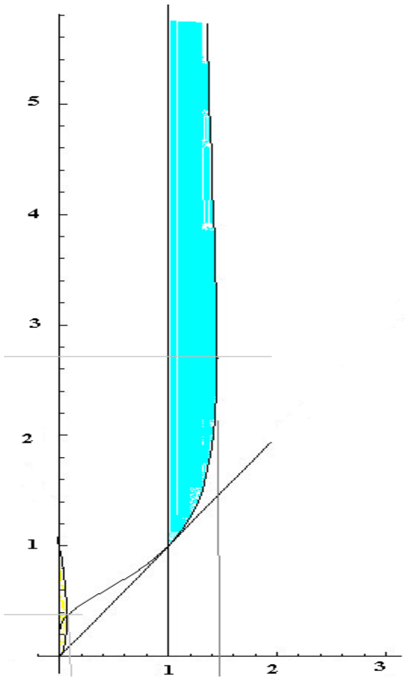
$$\boxed{h_{\text{inf}}(b) = \sum_{n=1}^{+\infty} \frac{(n \cdot \ln b)^{n-1}}{n!}} \qquad n \in N$$

also convergent in $\beta \le b \le \eta$

This formula can be used for finding $h_{\text{inf}}(b)$ for all the *EtaBeta* domain ($\beta \le b \le \eta$). It gives this:



A more accurate and complete analysis allows us to find the following behaviour of the "infinite tetrates", where the serial developments are difficult to obtain for the various "branches":



We may see four areas:

(a) - $0 < b < \beta$ - The "*yellow zone*", where I think that $y = {}^{x}b$ goes to the infinity with probably decreasing but permanent oscillations, assuming there an indeterminate value.

(b) - $\beta \le b < 1$ - First part of the Euler's (EtaBeta) domain, where $y = {}^{x}b$ goes asymptotically to the infinity, where it assumes a real value (h_{inf}).

(c) - $1 < b \le \eta$ - The "blue zone", where $y = {}^{x}b$ goes asymptotically to the infinity, assuming there a real value (h_{inf}). The role of the unreachable upper branch (h_{sup}) should also be determined.

(d) - $b > \eta$ - The super-exponential area.

By observing the fixpoints of the logarithm and exponential "functions" (...I know ...), we see that:

- In areas (a) and (b), we have one real fixpoint; in $\exp_b(x) = x = \log_b(x)$.
- In area (c), equation $\exp_b(x) = x = \log_b(x)$ determines two real fixpoints.
- In area (d), we don't have any real fixpoint in $\exp_b(x) = x = \log_b(x)$.

These facts determine the shape of the $h_{\text{inf}}(b)$ plot, because:

$$\boxed{y = b^y \quad \rightarrow \quad y = {}^{\infty}b = h(b)}$$

NB. The step forward will be to analyze the fixpoints of super-logarithm and tetration Also in those cases, for "pentation", we shall define appropriate "zones", obviously including x = -1 and x = 0, as well as at least one negative hyper-exponent value. Some interesting "pentation" configurations would be detected in the domain $1 < b <\approx 1.5$, where at most two additional tetra-fixpoints, for $x > 0$, will probably be found (due to two additional intersections of the direct and inverse "plateau" plots). Sorry to be ...audio-visual and ... fuzzy. We may discuss of that, if you wish.

GFR -2008/02/08