Progress Report on Hyper-operations (Zeration)

NKS Forum IV

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These notes have been prepared as a *thread* for the NKS Forum, News and Announcement section. It is a kind of progress report concerning some research activities carried out by the authors and described in a document included in the "*A New Kind of Science*" book bibliography. The aim of these notes is to focus on the study of a new arithmetical hyper-operation, called "*zeration*", carried out by one of the authors since 1987, published as a scientific paper in 1989 [1] and, as a monograph, in 1996 [5]. *Zeration* is a new binary operation that might be used for the analytical redefinition of the Dirac function and of the step function, with possible applications in science and engineering [7]. Moreover, zeration can also be used for redefining, in an analytical way, the logical operations of Boolean Algebra. Its inverse operation offers the possibility of defining a new class of numbers, called "*delta numbers*", which can be put in bijection with the logarithms of negative numbers.

Terminology and Symbols Reminder

$\boxed{s}$	- general hyper-operator of rank *s*, with $\boxed{1} = +$, $\boxed{2} = \times$, $\boxed{3} = \wedge$ and $\boxed{0} = \circ$ (zeration), $\boxed{4} = \#$ (tetration);
$y = {}^{z}x$	- tetration; *x* # *z*; *x*-tetra-*z* or *x*-tower-*z*, i.e. *x* raised to *x*, *z* times (*z* is the super-exponent);
$y = x \circ z$	- zeration; *x*-zerated-*z*, max(*x*,*z*)+1 if $x \neq z$, *x*+2 = *z*+2 if *x* = *z*;
$x = y \Delta z$	- deltation; *y*-delta-*z* (inverse function of $y = x \circ z$);
${}^{z}\lceil y^{s}$	- the z^{th} hyper-root of *y*, rank *s*; for s = 4 : ${}^{z}\lceil y^{4} = {}^{z}\rangle y$ (the z^{th} super-root or tetra-root of *y*) and, for s = 3: ${}^{z}\lceil y^{3} = \sqrt[z]{y}$ (the z^{th} root of *y*);
${}_{x}\lfloor y_{s}$	- the hyper-logarithm, base *x*, of *y*, rank *s*, that can also be written as: $\text{hlog}_{s\,x}\, y$; for *s* = 4: ${}_{x}\lfloor y_{4} = \text{slog}_x\, y$ (super-log, or tetra-log, base *x*, of *y*) and for *s* = 3: ${}_{x}\lfloor y_{3} = \log_x y$;
$\rangle y$	- the square super-root (tetra-root) of y : $\rangle y = {}^{2}\lceil y^{4}$;
$\text{plog}(x)$	- the ProductLog (Lambert's Function), the inverse function *y* of $x = y \cdot e^{y}$;
$\ln y$	- the natural logarithm of *y*, i.e. : $\log_e y = {}_{e}\lfloor y_{3}$;
$\text{slog}_a\, y$	- the super-logarithm (tetra-logarithm), to the base *a*, of *y*, i.e.: $\text{slog}_a\, y = {}_{a}\lfloor y_{4}$.
$\text{sln}\, y$	- the natural super-logarithm (natural tetra-logarithm) of *y*, i.e.: $\text{slog}_e\, y = {}_{e}\lfloor y_{4}$.

Progress Report on Hyper-operations (Zeration)

1 – Introduction

While analyzing the two-argument Ackermann's Function $\mathrm{A}(m,n)$, see [6], it appears that, since we know that we must have $\mathrm{A}(0,n)=n+1$, a "level-zero" operation (for *m* = 0) can easily be derived. Nevertheless, in the past, this fact was always considered (as far as we know) as equivalent to pointing to a well known "monary" operation, i.e.: "the successor of *n*", with "*n*" natural. The new approach, suggested by one of us since 1987 (C. A. Rubtsov, see [1], [2], [3]), is to define a new "binary" hyper-operation, which would coincide with "the successor of *n*", in some particular situations. Actually, the "*zeration*" hyper-operation allows this, for example, in the following case:

$x \circ y = y \circ x = x+1$ the successor of *x*, with *x* positive integer, **if:** $y = 0$.

By extending the definition of "successor of *x*" as *x* + 1, with *x* real, "*zeration*" is defined as follows:

$x \circ y = \max(x,y)+1,$ for $x \neq y$

$x \circ y = x+2 = y+2$, for $x = y$.

First of all, we see that zeration is commutative. Then, we could say that zeration gives (in a way) the successor of the greater value of the two operands, when they are different, and the second successor of either of them, if they are equal. This definition allows us to keep the following formulas:

$x \circ x = x+2$ zeration/addition (see [6])
$x + x = x \cdot 2$ addition/multiplication
$x \cdot x = x \wedge 2$ multiplication/exponentiation
$x \wedge x = x \# 2$ exponentiation/tetration (see [6])
and … so on …

In 2002, one of us (G. F. Romerio, see [6]) proposed the English name "zeration" for this new binary hyper-operation, conceived as one of the standard arithmetical operations belonging to an infinite hierarchy (the Grzegorczyk hierarchy). The reasons of the reference to "zero" were that the classical arithmetical operations are three (+, ×, ^), that Rubtsov already called it, in Russian, "the null operation" and that the "superpower" or "tower" (i.e. "tetration") was considered by the majority of researchers as the fourth rank operation (+,×, ^, #). We think that zeration might be used in future for the analytical redefinition of the Dirac function and of the step function, with useful applications in science and engineering. In August 2006, a poster about this subject [7] was presented at the International Congress of Mathematicians (ICM-06, Madrid, 22-30 August 2006), also including a possible use of tetration for the definition of a new "*tetrational format*" to be used for storing very large numbers (the **RRH©** hyper-format). Please see also:
http://forum.wolframscience.com/sho...s=&threadid=579
http://forum.wolframscience.com/sho...s=&threadid=956 .
http://forum.wolframscience.com/showthread.php?s=4c07eaa39f25d09ec5eb69cff78f6808&threadid=1168.
See also: http://answer.google.com/answers/threadview?id=743129

We shall show that zeration, besides being used for the analytical representation of discontinuous functions, can also be used for calculating the absolute and the sign values of a variable, as well as the operators of Boolean Algebra, without the need of any additional "*ad hoc*" definitions.

The commutativity of zeration implies that it has only one inverse, as shown in the next sections. This report is intended to be a short introduction to this new mathematical field.

2 – Definition of Zeration

In paragraphs 8.1 8.2 and 8.3 of [6], we described the basic properties of zeration (hyper-operation of rank s = 0) as shown in the following table (see also [5]):

	$a \circ b = a + 1$	for:	$a > b$
	$a \circ b = b + 1$	for:	$a < b$
(1)	$a \circ b = a + 2 = b + 2$	for:	$a = b$
	$a \circ b = a$	for:	$b = -\infty$
	$a \circ b = b$	for:	$a = -\infty$

We see that a fundamental element of this hyper-operation is "- ∞", which operates as its neutral unit element, i.e. that we have:

(2) $\varepsilon_0 = -\infty$ unit element of zeration

In fact, we started by considering the inverse operations of the "root" type (left-inverse operations) for the hyper-operations of ranks 4, 3, 2, 1 and 0, which can be written as follows:

	s = 4	$z = x \# n$	=>	$x = {}^{n}\!\rangle z$	super-root
(3)	s = 3	$z = x \wedge n$	=>	$x = \sqrt[n]{z}$	root
	s = 2	$z = x \,.\, n$	=>	$x = z / n$	division
	s = 1	$z = x + n$	=>	$x = z - n$	subtraction
	s = 0	$z = x \circ n$	=>	$x = z \Delta n$	"deltation"

where, in the last line, we implicitly defined the inverse operation of *zeration*, called *deltation* and indicated with the "Δ" infixed operator[1]. Then, for showing the properties of *zeration*, we considered the following expression:

$$\lim_{n\to\infty} {}^{n}\!\rangle z = \sqrt[z]{z} = \varepsilon_3 = \text{g}(z) \quad \text{(with g}(z)\text{: unitary function)}$$

(4)
$$\lim_{n\to\infty} \sqrt[n]{z} = z / z = \varepsilon_2 = 1$$

and:
$$\lim_{n\to\infty} z / n = z - z = \varepsilon_1 = 0$$

to which we added:
$$\lim_{n\to\infty} z - n = z \Delta z = \varepsilon_0 = -\infty$$

We then observed that, if we have $x = z \Delta z = -\infty$, we must also have $z \circ (-\infty) = z$, i.e. quantity "-∞" should act as the neutral unit element for zeration, i.e. that we should indeed have:

(5)
$$a \cdot 1 = 1 \cdot a = a$$
$$a + 0 = 0 + a = +a = a$$
$$a \circ (-\infty) = (-\infty) \circ a = \circ a = a \quad \text{(with -}\infty\text{: unit element)}$$

Once the neutral unit element of zeration has been defined, we can also define the first properties of "deltation", as follows:

if: $a \circ b = c$

then: $a = c \Delta b$; $b = c \Delta a$

and if: $a \circ (-\infty) = a$

then: $a \Delta (-\infty) = a$; $(-\infty) \Delta a = \Delta a$

[1] Normally, each hyper-operation has two different inverses, the "hyper-root" and the "hyper-log". In the case of s=0, s=1, s=2, for *a, b* reals, the direct operations are commutative and, therefore, the two inverses are identical (see also section 8).

Nevertheless, an expression such as "$a\Delta(-\infty) = a$" or "$(-\infty)\Delta a = \Delta a$" cannot be accepted on the basis of definitions valid within the standard theory of numbers like, for instance, expression "$0 - a$" could not be allowed within the theory of positive numbers. We are in fact aware that the introduction of symbol "$-a$" for representing "$0 - a$" with the identity "$0 - a = -a$" has allowed us to introduce what we call the "negative" and the "relative" numbers. We propose that a similar approach be applied for expression "$(-\infty)\Delta a$", by defining a "new number Δa", such that "$\Delta a = (-\infty)\Delta a$". We decided to call Δa a "new number" despite the fact that, as will be shown below, it is closely connected with the logarithm of a negative argument, which has been known about for a long time. We anticipate that this fact justifies the identity "$(-\infty)\Delta a = \Delta a$".

3 – Other properties of zeration

In order to be able to use zeration, deltation and delta numbers in ordinary calculations, we must try to discover the properties of these operations and numbers. It has been proved (see [5]) that, concerning zeration, the following properties are verified:

(6) commutativity (zeration) $a \circ b = b \circ a$

Non-associativity (zeration) $(a \circ b) \circ c \neq a \circ (b \circ c)$ except for: $a = b = c$.

Concerning the use of addition (subtraction) and zeration (deltation) in the same computing environment, the following properties are also verified:

(7) distributivity (addition / zeration): $(a \circ b) + c = (a + c) \circ (b + c)$

similar to: $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$

distributivity (addition / deltation) $(a\Delta b) + c = (a + c)\Delta(b + c)$

similar to: $(a - b) \cdot c = (a \cdot c) - (b \cdot c)$

sign behavior (zeration / deltation): $\Delta c = \Delta a \circ (\Delta b) \leftrightarrow \Delta c = \Delta(a \circ b)$

similar to: $-c = -a + (-b) \leftrightarrow -c = -(a + b)$

Moreover, for delta numbers, the following sign properties are also verified:

(8) $\Delta(\Delta a) = a$; $-(\Delta a) = \Delta(-a)$; $\Delta(a \circ b) = \Delta a \Delta b$

similar to: $-(-a) = a$; $:(-a) = -(:a)$; $-(a + b) = -a - b$

As we can see in the previous formulas, we introduced, for the sake of symmetry, a symbol that is not yet familiar to the reader, the absolute *monary* division sign ":". In fact, the authors propose "$:a$" as a new symbol for indicating the reciprocal of *a*, like $-a$ is the opposite of *a*. This would be similar to the use of the absolute *monary* subtraction sign, the minus sign "-". In this way, it could be used like other signs applied to a number.

Practically, for the following three inverse operations, we shall write:

$$1\,/\,a = \;:a$$

(9) $$0 - a = -a$$

$$-\infty\Delta a = \Delta a$$

The Δ symbol would act similarly to the minus sign, when applied to a number. The relationship between the "$\circ$" and the "+" signs is very clear when zeration is applied to integer operands. In this case expression $c = a \circ b$, to be read "*a*-zerated-*b*", or "*a*-zer-*b*", coincides with the successor of the greater of *a* and *b* (i.e.: *a* + 1 or *b* + 1), if they are different, and with *a* + 2 = *b* + 2, if they are equal.

We must also add the following axiom of order (zeration / deltation):

$$
(10)\qquad \begin{array}{lcl} \Delta b > \Delta a & \leftrightarrow & b < a \\ \Delta b = \Delta a & \leftrightarrow & b = a \\ \Delta b < \Delta a & \leftrightarrow & b > a \end{array} \quad \text{similar to} \quad \begin{array}{lcl} -b > -a & \leftrightarrow & b < a \\ -b = -a & \leftrightarrow & b = a \\ -b < -a & \leftrightarrow & b > a \end{array}
$$

Taking these conclusions into account, we have that:

$$
(11)\qquad \Delta a \circ (\Delta b) = \begin{cases} \Delta b + 1, & \Delta b < \Delta a, \quad a < b \\ \Delta b + 2, & \Delta b = \Delta a, \quad b = a \\ \Delta a + 1, & \Delta a < \Delta b, \quad b < a \end{cases}
$$

Finally, we may observe that, since we have:

$$-\infty < a \quad \forall a \in \mathrm{R}$$

and: $\Delta(-\infty) = -(\Delta\infty) = -\infty$

then, we must have:

$$(12)\qquad \boxed{\Delta a < -\infty < a \quad \forall a \in \mathrm{R}}$$

In conclusion, delta numbers appear to be ranged, in a relation of order, below the limit quantity - ∞ (minus infinity). They can therefore be considered as part of a new set of "*trans-infinite numbers*" (TI), in an extension of the set of real numbers. The denomination "trans-infinite" should not be confused with "transfinite", belonging to the theory of infinite ordinals.

"Transfinite" also belongs to the terminology of "*non-standard analysis*", for indicating "hyper-real numbers", greater (or smaller) than any positive (negative) real number, but limited by the limit positive (negative) infinities, whereas "trans-infinite" numbers are intended to be mathematical objects of a new type, smaller (or greater) than the - ∞ (or + ∞) limit number.

4 – Addition of delta numbers

By using the formula in section 3, one of us (C. A. Rubtsov, [5]) discovered that the "rules of signs" for the delta numbers valid for the addition operation are similar to the "rules of signs" valid for negative numbers in the multiplication operation, i.e. that:

$$
(13)\qquad \begin{array}{ll} (\circ a) + (\circ b) = \circ(a+b) & \qquad (+a)\cdot(+b) = +(a\cdot b) \\ (\circ a) + (\Delta b) = \Delta(a+b) & \qquad (+a)\cdot(-b) = -(a\cdot b) \\ (\Delta a) + (\circ b) = \Delta(a+b) & \qquad (-a)\cdot(+b) = -(a\cdot b) \\ (\Delta a) + (\Delta b) = \circ(a+b) & \qquad (-a)\cdot(-b) = +(a\cdot b) \end{array}
$$

These expressions, can be shortened as follows:

$$
(14)\qquad \begin{array}{ll} a + b = a + b & \qquad a\cdot b = ab \\ a + (\Delta b) = \Delta(a+b) & \qquad a\cdot(-b) = -ab \\ (\Delta a) + b = \Delta(a+b) & \qquad (-a)\cdot b = -ab \\ (\Delta a) + (\Delta b) = a + b & \qquad (-a)\cdot(-b) = ab \end{array}
$$

Symbolically, we could also draw the following "rules of signs" schemes:

(15)

$$\boxed{\begin{array}{l} \circ + \circ = \circ \\ \circ + \Delta = \Delta \\ \Delta + \circ = \Delta \\ \Delta + \Delta = \circ \end{array}} \qquad \text{similar to:} \qquad \boxed{\begin{array}{l} + \cdot + = + \\ + \cdot - = - \\ - \cdot + = - \\ - \cdot - = + \end{array}}$$

With these rules we can start analysing how addition with delta numbers may work. In fact, by putting $a = 0$ and b = n, we have:

(16)

0 + n	= n	
0 + Δn	= Δn	
Δ0 + n	= Δn	!!!
Δ0 + Δn	= n	!!!

In particular, for example, we also have:

0 + Δ1 = Δ1	0 + Δ5 = Δ5	0 - Δ5 = - Δ5
5 + Δ1 = Δ6	Δ6 – 5 = Δ1	5 - Δ2 = Δ3
Δ0 + Δ3 = 3	Δ0 - Δ3 = - 3	Δ5 + Δ2 = 7

Other properties of addition of delta numbers are as follows:

(17) commutativity (sum of delta numbers): $\Delta a + \Delta b = \Delta b + \Delta a$

associativity (sum of delta numbers): $\Delta a + \left(\Delta b + \Delta c\right) = \left(\Delta a + \Delta b\right) + \Delta c$

These formulas suggest the possibility of representing delta numbers on a linear scale, as follows:

Fig. 1

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Δ∞              Δ1  Δ0  Δ(-1)          - ∞                 -1   0  +1              + ∞
_|_______________|___|___|_______________|||||||||_______________|___|___|_______________|_
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From examination of Fig. 1 and of formulas (16), we get the following expressions:

Δ0 + 1 = Δ1	Δ0 - 1 = Δ(-1) = - Δ1
Δ0 + 2 = Δ2	Δ0 - 2 = Δ(-2) = - Δ2
Δ0 + 3 = Δ3	Δ0 - 3 = Δ(-3) = - Δ3
....	....
Δ0 + *n* = Δ*n*	Δ0 - n = Δ(-*n*) = - Δ*n*
....	....
Δ0 + ∞ = Δ∞	Δ0 - ∞= Δ(-∞) = - Δ∞

The last line is supposed to be obtained by letting: $n \to \infty$. The sequence of points representing delta numbers, in the above-mentioned qualitative diagram, indicates (...suggests) that the limit of the Δ0, Δ(-1), Δ(-2) ... sequence should exist and be exactly - ∞ ! In fact, **if** number Δ(-∞) must belong to the delta numbers, **then** that limit must be represented by - ∞. In other words, in this case, we should have:

(18) $\boxed{\Delta\left(-\infty\right) = -\Delta\infty = -\infty}$ "the actual minus infinite" [2]

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[2] The "actual" - ∞ is: $\lim_{x\to\infty}\left(0 - x\right) = -\infty$.

5 – Zeration and Boolean Algebra

In the next sections we shall describe other properties of zeration and its inverse, as well as the usefulness of this tool in mathematical developments. We should like to anticipate here the possibility of using it for the definition of the logical operators of Boolean Algebra (AND, OR, NOT). In fact, we can see that it is possible to describe such Boolean operations with a special and very important use of zeration. For instance, with $a,b,c \in \{0,1\}$, in Boolean Algebra, we have:

(19)

OR $a \vee b = c$			AND $a \wedge b = c$			NOT $\neg b = c$	
0	0	0	0	0	0	0	1
1	0	1	1	0	0	1	0
0	1	1	0	1	0		
1	1	1	1	1	1		

We will see how we can use zeration for calculating $\neg b$ (NOT *b*). First of all, however, concerning the AND operation, for the shown binary ranges fixed for *a*, *b* and *c*, we can write:

(20) $$\boxed{c = a \wedge b = a \cdot b}$$

Then, concerning the NOT and the OR operations, we can write [3]:

(21) $$\boxed{c = \neg b = (b \circ 0) - (b+1)} \qquad = -\,\mathbf{ST}(b;\,0)$$

(22) $$\boxed{c = a \vee b = a + b - ((a+b) \circ 2) + 3} \qquad = \mathbf{ST}(a+b;\,2)$$

Note. See **ST**(…) at page 8 of: http://numbers.newmail.ru/pdf/obj_eng.pdf. With the second formula, it can also be shown that, for *b* = 1, we always have [4]:

$$a \vee 1 = a + 1 - (a + 1 \circ 2) + 3 = 1$$
$$a \vee 0 = a + 0 - (a + 0 \circ 2) + 3 = a - 3 + 3 = a$$

It can be shown that, with the above-mentioned formulas, we have the following "Boolean Algebra".:

(23)
$$\boxed{\begin{array}{l} a \wedge 0 = 0, \qquad a \wedge 1 = a \\ a \vee 0 = a, \qquad a \vee 1 = 1 \\ a \wedge a = a \vee a = a \\ \neg(a \wedge b) = (\neg a) \vee (\neg b) \\ \neg(a \vee b) = (\neg a) \wedge (\neg b) \\ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \\ a \vee (b \vee c) = (a \vee b) \wedge (a \vee c) \end{array}}$$

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[3] See also discontinuous function xsu (*b*) – *b*, Section 10.

[4] Please note that the hyper-operations of higher ranks have an execution priority higher than that of lower rank operations. For example: $a + b \circ c = (a+b) \circ c$, similar to: $a \cdot b + c = (ab) + c$.

6 – Plot of a zeration function

As we know, the properties of zeration ($a \circ b$) are those defined in [5] (formulas 3 .10). The graph of a zeration function, defined as $y(x) = a \circ x$ **c**an be shown, for $a = 2$, in the following diagram ($y = 2 \circ x$). See also [6], paragraph 8.4, page 18.

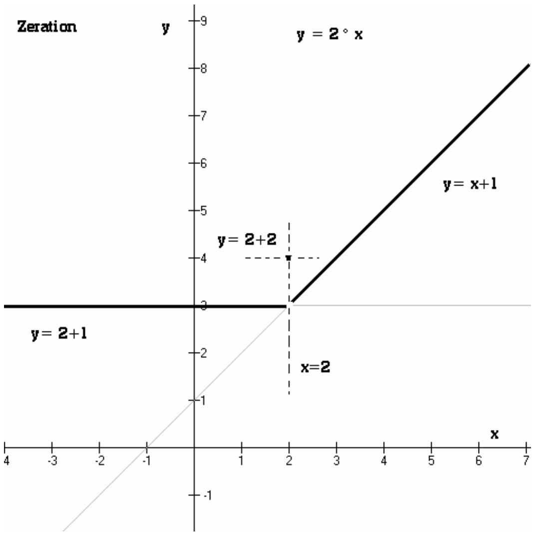

Fig. 2

The plot highlights the main characteristics of zeration: The inverse function of $y = 2 \circ x$ can be obtained in conformity and with the same order used at other hyper-operational levels, such as:

		s=0	s=1	s=2
(24)	*Direct operations*:	$y = 2 \circ x$	$y = 2 + x$	$y = 2 \cdot x$
	Inverse operations	$x = y \Delta 2$	$x = y - 2$	$x = y / 2$

After a change of variables $(x \leftrightarrow y)$, the plot of function $y = x \Delta 2$ is as follows:

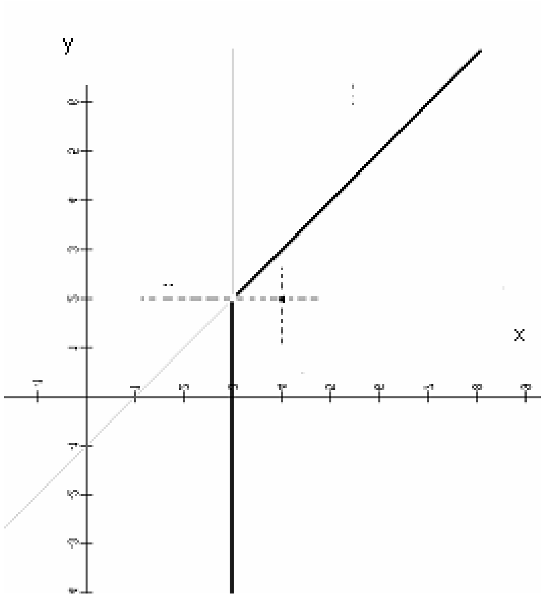

Fig. 3

We may observe that in expression $y = x\Delta 2$, for :

	$x = 2$	$\rightarrow$	$y = -\infty$	
	$x = 3$	$\rightarrow$	$y = \left]-\infty, 2\right[$	(5) infinite range!
(25)	$3 < x < 4$	$\rightarrow$	$y = x - 1$	
	$x = 4$	$\rightarrow$	$y = \{2, 3\}$	two values!
	$x > 4$	$\rightarrow$	$y = x - 1$	

Concerning other possible values for $x < 3$, the function is not only defined for $x = 2$. For a better approach to this question, let us consider again the table of the direct and inverse operations (s = 0, 1, 2). We know that all the three direct operations so defined are commutative and that their inverse are not. Let us therefore add the commuted inverse operations, as follows:

		s=0	s=1	s=2
(26)	*Direct operations*:	$y = 2 \circ x$	$y = 2 + x$	$y = 2 \cdot x$
	Inverse operations	$x = y\Delta 2$	$x = y - 2$	$x = y / 2$
	Commuted inverse:	$\Delta x = 2\Delta y$	$-x = 2 - y$	$: x = 2 / y$

We can, therefore, complete with the $y = x\Delta 2$ function's table with the following values:

	$x = -1$	$\rightarrow$	$y = \Delta 1$	(6)
	$x = 0$	$\rightarrow$	$y = \Delta 0$	
(27)	$x < 1$	$\rightarrow$	Δ numbers	
	$1 < x < 2$	$\rightarrow$	transfinite numbers	(7)
	$x = 2$	$\rightarrow$	$y = -\infty$	
	$2 < x < 3$	$\rightarrow$	transfinite numbers	(8)

Let us now try to calculate $a = c\Delta b$ from the $a \circ b = c$ known values, with $\{a, b\} \in \mathrm{N}$:

	$6\Delta 1 = 5$	because	$5 \circ 1 = 6$		
	$6\Delta 2 = 5$	because	$5 \circ 2 = 6$		
	$6\Delta 3 = 5$	because	$5 \circ 3 = 6$		
	$6\Delta 4 = \{5, 4\}$	because	$5 \circ 4 = 4 \circ 4 = 6$		
	$6\Delta 5 = \left]-\infty, 5\right[$	because	$\left]-\infty, 5\right[\circ 5 = 6$		
(28)	$6\Delta 6 = -\infty$	because	$(-\infty) \circ 6 = 6$		
	$6\Delta 7 = \left]\Delta 6, -\infty\right[$	because	$7\Delta 6 = \left]-\infty, 6\right[$	since:	$\left]-\infty, 6\right[\circ 6 = 7$
	$6\Delta 8 = \{\Delta 6, \Delta 7\}$	because	$8\Delta 6 = \{6, 7\}$	since:	$6 \circ 6 = 7 \circ 6 = 8$
	$6\Delta 9 = \Delta 8$	because	$9\Delta 6 = 8$	since:	$8 \circ 6 = 9$
	$6\Delta 10 = \Delta 9$	because	$10\Delta 6 = 9$	since:	$9 \circ 6 = 10$
	$6\Delta 11 = \Delta 10$	because	$11\Delta 6 = 10$	since:	$10 \circ 6 = 11$

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[5] This relation is satisfied by **any** number *y* such that $-\infty < y < 2$.

[6] In fact, it can be shown that: $y = (-1)\Delta 2 = \Delta(\Delta(-1) \circ 2) = \Delta(2 \circ (\Delta(-1)) = \Delta(2\Delta(-1)) = \Delta(1) = \Delta 1$

[7] Range of numbers < - ∞, but greater than any delta number!

[8] Range of numbers > - ∞, but smaller than any negative number!

In the following table, we shall write the values estimated for $n\Delta 6$ and $n\Delta 2$ (with $n \in \mathbb{Z}$), for trying to find recurrent formulas.

$$1\Delta 6 = \Delta 5 \qquad (-3)\Delta 2 = \Delta 1$$
$$2\Delta 6 = \Delta 5 \qquad (-2)\Delta 2 = \Delta 1$$
$$3\Delta 6 = \Delta 5 \qquad (-1)\Delta 2 = \Delta 1$$

(29)

$$4\Delta 6 = \{\Delta 5, \Delta 4\} \qquad 0\Delta 2 = \{\Delta 1, \Delta 0\}$$
$$5\Delta 6 =]\Delta 5, -\infty[\qquad 1\Delta 2 =]\Delta 1, -\infty[$$
$$6\Delta 6 = -\infty \qquad 2\Delta 2 = -\infty$$
$$7\Delta 6 =]-\infty, 6[\qquad 3\Delta 2 =]-\infty, 2[$$
$$8\Delta 6 = \{6, 7\} \qquad 4\Delta 2 = \{2, 3\}$$

$$9\Delta 6 = 8 \qquad 5\Delta 2 = 4$$
$$10\Delta 6 = 9 \qquad 6\Delta 2 = 5$$
$$11\Delta 6 = 10 \qquad 7\Delta 2 = 6$$

……………………………………………………………………………………

In conclusion, the results of the execution of deltation can be obtained as follows:

(30)

if $c > a + 2$	then:	$c\Delta a = c - 1$
if $c = a + 2$	then:	$c\Delta a = \{c - 1; c - 2\}$
if $a + 1 < c < a + 2$	then:	$c\Delta a = c - 1$
if $c = a + 1$	then:	$c\Delta a = -(\text{transfinite numbers})$
if $a \le c < a + 1$	then:	$c\Delta a = -\infty$
if $c < a$	then:	$c\Delta a = \Delta(a\Delta c)$

These formulas will be integrated into the next version of the KAR-Calc hyper-calculator.

7 – The logarithms of negative numbers

We know that the logarithms of negative numbers have been one of the most controversial subjects since the beginning of modern Mathematics, made particularly so by Euler who proposed his very famous and successful formula for representing number – 1 by an exponential function:

(31) $$-1 = e^{(2n+1)\pi i} \quad \text{with } n = \{0, 1, 2, ...\}$$

From this formula we get:

(32) $$\ln(-1) = (2n+1)\cdot \pi i$$

and: $$\boxed{\ln(-x) = \ln x + (2n+1)\cdot \pi i} \quad \text{with } n = \{0, 1, 2, ...\}$$

which has two extraordinary properties, i.e. to be complex (but this is widely acceptable) and multi-valued (and this can lead to a lot of problems [9]). An application of this formula was the origin of a famous dispute between d'Alembert and Euler in the 18th century.

[9] A way out is to consider only the values obtained by putting $n = 0$ (the principal value). But this is not easily acceptable to everybody.

In fact, we could have:

(33) $\ln(-1)+\ln(-1)=\ln(-1)\cdot 2=\ln\left((-1)^2\right)=\ln(+1)=0$

therefore: $\ln(-1)\neq 0$, while: $\ln(-1)+\ln(-1)=0$

something like: $x+x=0$, only true for $x = 0$ (!!!) [Paradox !]

In order to eliminate this paradox, we shall try to show how to proceed in choosing another complementary way of representing the logarithms of negative numbers, taking into account the similar properties shown by the "new" delta numbers. For this purpose, let us take into consideration the following expression ($a \in \mathrm{R}$; $a > 0$):

(34) $\log_b(1/a)=\log_b(:a)$

to be compared with:

(35) $\log_b(0-a)=\log_b(-a)$.

The first formula is a standard expression used in ordinary algebra and, as a result, it gives $-\log_b a$. The second formula also belongs to algebra, but it creates a lot of problems, due to the presence of a logarithm of a negative quantity. Forgetting for the moment the Euler/d'Alembert dispute, we may simply observe that the two formulas are similar, except for a change in the hyper-operational levels.

We could in fact write:

$$\boxed{\log_b(:a)=-\log_b a=-c}$$ (at: s=2, s=1)

(36)

corresponding to: $$\boxed{\log_b(-a)=\Delta\log_b a=\Delta c}$$ (at: s=1, s=0)

In other words, with *a* real positive, we should have:

(37) $:a=b^{-c}$ which is known

and $-a=b^{\Delta c}$ which is new.

These formulas suggest that delta numbers are identical to the logarithms of negative numbers and can also be summarized as follows:

(38) $$\boxed{\text{if } a=b^c>0, \text{ then } b^{\Delta c}=-\left(b^c\right)}$$

It can be shown that these formulas respect the properties of delta numbers. In fact, from:

(39) $\log_b(-a)=\Delta\log_b a=\Delta c$

for $a = 1$, we get: $\log_b(-1)=\Delta\log_b 1=\Delta 0$

From the known properties of both logarithm operation and the multiplication of delta numbers, we indeed can verify (see next section) that:

(40) $\log_b(-1)\cdot 2=\log_b(-1)^2=\log_b 1=\Delta 0\cdot 2=0$

This justifies the fact that we may have [see (33)]: $\ln(-1)=\Delta 0\neq 0$, while: $\ln(-1)^2=0$.

In fact, in the next session we shall see that multiplication is not always commutative if delta numbers are involved. For the moment, we would like to point out that a delta number can be put in correspondence with the logarithm of a negative number, for instance, as follows:

(41) $$\boxed{\Delta x=\ln\left(-e^x\right)}$$ with $x\in\mathrm{R}$

Recalling formulas (31), (32) and (41), we could write:

(42) $$\Delta x \leftrightarrow x + \{(2n+1)\pi i\}_{n \in N}$$ i.e. with *n* natural (0, 1, 2, …)

This formula shows the link between "delta" numbers and "infinitely-valued complex numbers". The double arrow, instead of the "=" sign, is used because "delta" an "complex" numbers come from two different theories. In the particular case when $x = 0$, we have:

(43) $$\Delta 0 \leftrightarrow \{(2n+1)\pi i\}_{n \in N}$$

Nevertheless, we should not come to the conclusion that $\Delta 0$ is a multi-valued imaginary number, but that formulas (42) and (43) show a bijection between the two representations of the mathematical objects obtained as logarithms of negative numbers.

We hope that this fact might help in finding an appropriate methodology for handling the logarithms of negative numbers, about which there is no agreement among specialists, so far.

8 – Multiplication of delta numbers

It is universally accepted that multiplication between real or complex numbers is commutative. In other words, if $x, z \in R$ or $x, z \in C$, we have:

(44) $$x \cdot z = z \cdot x.$$

Nevertheless, if x or z are other mathematical objects, for instance "*quaternions*" or "*octonions*" (hyper-complex numbers), the above-mentioned relation is no longer valid. Modern physics also shows other situations in which multiplication is not commutative. We should like to show that multiplication, when at least one of the operands is a delta number, may not be commutative either.

As a matter of fact, in all hyper-operations ranks (except for addition and multiplication of real and complex numbers) the non commutativity property is verified, i.e.:

(45) $$x \boxed{s} z \neq z \boxed{s} x$$

The case of commutativity of s=1 (addition) is intuitively clear ($a + b = b + a$), for all kinds of standard numbers [10]. The case of s=2 (multiplication) seems very clear too, with the above-mentioned exceptions, despite the fact that the philosophical meaning of $x \cdot z$ is different from the meaning of $z \cdot x$. In fact:

(46) $$x \cdot z = x + x + x + ...x \quad (z \text{ times})$$

and: $$z \cdot x = z + z + z + ...z \quad (x \text{ times}).$$

The classical conclusion that, for real and complex numbers, $x \cdot z = z \cdot x$ is one of the most important and beautiful cornerstones of mathematics. Nevertheless, as we have anticipated, this is not always the case in a "delta" environment.

First of all, let us consider how we may calculate the result obtained from the multiplication of a delta number by an integer number. There are two known possibilities for calculating $\Delta a \cdot b$, i.e. the sum of Δa and itself, *b* times (with $b \in N$).

---

[10] Addition is generally commutative for all the investigated numbers. However, the authors already took into consideration numbers generated from the inverse of the hyper-operation with rank s = -1. These numbers will be created with the help of iterative application of the logarithm operator. The numbers thus obtained are already available in the **RRH**© number Hyper-format implemented by the authors (**Snumbers** and **Snumbers2**). See: http://numbers.newmail.ru/My/ . In the framework of these numbers, addition, too, might lose commutativity.

In fact, when, on the one hand, we multiply a Δa number by an <u>even</u> (integer) number, i.e. by $b_e = 2n$, with $n \in \mathrm{N}$ and $a \in \mathrm{R}$, we get, for instance:

(47) $\Delta a \cdot 2 = \Delta a + \Delta a = 2a$

and, generally: $\Delta a \cdot (2n) = 2na$

On the other hand, when we multiply the same Δa by an <u>odd</u> (integer) number, i.e. by: $b_o = 2n+1$ with $n \in \mathrm{N}$ and $a \in \mathrm{R}$, we get, for instance:

(48) $\Delta a \cdot 3 = \Delta a + \Delta a + \Delta a = 2a + \Delta a = \Delta(3a)$

and, generally: $\Delta a \cdot (2n+1) = \Delta((2n+1)a)$

In conclusion, we obtain:

(49) $\Delta a \cdot b = ab$ if *b* is an <u>even</u> integer number

$\Delta a \cdot b = \Delta(ab)$ if *b* is an <u>odd</u> integer number.

However, if *b* <u>is not</u> an integer number, expression $\Delta a \cdot b$ can only be shown, for instance, as:

(50) $\Delta a \cdot b = \ln\left(-e^a\right) \cdot b = \ln\left[\left(-e^a\right)^b\right] = \ln\left[(-1)^b \cdot e^{ab}\right]$

On the contrary, if we consider operation $a \cdot \Delta b$, i.e. the *abbreviation* of the sum of (real) number *a* and itself, Δb times, we appear, at first sight, to be in trouble, because we do not know what exactly "Δb times" may mean. The way out is provided by formula (42), that we can write as follows:

(51) $a \cdot \Delta b = a \cdot \ln\left[-\left(e^b\right)\right]$

Nevertheless, there is a very interesting mathematical problem in connection with the commutativity of addition, leading directly to an apparent paradox. In fact, let us take into consideration the following inequality. It seems peculiar, but it shows a strange fact, considered as trivial by some people and as wrong by … some other people:

(52) $\left(\sqrt[2]{a}\right)^2 \neq \sqrt[2]{a^2}$.

In fact, we have:

(53) $\left(\sqrt[2]{a}\right)^2 = \left(\left\{-\sqrt{a};+\sqrt{a}\right\}\right)^2 = \left\{\left(-\sqrt{a}\right)^2;\left(+\sqrt{a}\right)^2\right\} = +a$

and: $\sqrt[2]{a^2} = \{-a;+a\}$.

We indeed see that, in the first case, we have two values and, in the second, only one. If we put it in exponential form, we get:

(54) $\left(\sqrt[2]{a}\right)^2 = \left(a^{1/2}\right)^2 = a^{\frac{1}{2}\cdot 2} = +a$

$\sqrt[2]{a^2} = \left(a^2\right)^{1/2} = a^{2\cdot\frac{1}{2}} = \{-a;+a\}$

In conclusion, we must accept that, in this case:

(55) $\boxed{\frac{1}{2}\cdot 2 \neq 2\cdot\frac{1}{2}}$ (*<u>non commutativity of multiplication</u> !*) [11]

[11] This seems to show that the quantities involved in (55) **are not** real numbers.

This strange fact may be justified using the formalism of Δ-numbers. In fact, by taking the logarithm to the base 2 of the second formula, we can write:

(56) $$2\cdot\frac{1}{2}=\log_a\{-a;+a\}=\{\log_a(-a);\log_a(+a)\}=\{\Delta 1;+1\}$$

i.e.: $\frac{1}{2}\cdot 2=1$ and $2\cdot\frac{1}{2}=\{\Delta 1;1\}$

In particular, on the one hand, taking into consideration the second case ($\sqrt[2]{a^2}$), if we consider a general formula such as: $\sqrt[c]{a^b}=a^{b\cdot\frac{1}{c}}$, we have that:

- if c is a whole <u>even</u> number: $\sqrt[c]{a^b}=\{-a^{b/c};a^{b/c}\}=\{a^{\Delta(b/c)};a^{b/c}\}$,
- if c is a whole <u>odd</u> number: $\sqrt[c]{a^b}=a^{b/c}$.

On the other hand, if we consider: $\left(\sqrt[c]{a}\right)^b=a^{\frac{1}{c}\cdot b}$, we always have::

- for any c: $\left(\sqrt[c]{a}\right)^b=a^{\frac{1}{c}\cdot b}=a^{b/c}$

This can be explained with the help of formula: "$\ln(-x)=\Delta\ln x$". In fact, whenever we find an exponent (an expression) such as b/c, where c is an <u>even integer number</u>, this implies the generation of a delta number. The above-mentioned situation is perfectly compatible with delta numbers. So, the condition for getting Δ-numbers (in the exponent) is that 1 / *c* must multiply *b*, from the right, and that *c* must be an <u>even</u> integer number.

When 1/*c* has the value of a Δ-number, multiplied from <u>the left</u> by *b*, it always gives Δ-numbers, disregarding the parity of *b*! In other words:

(57) $$b\cdot\Delta(1/c)=\Delta\left(b/c\right).$$

therefore: $a\cdot\Delta b=\Delta(a\cdot b)$.

In conclusion, we have:

- $x\cdot z\neq z\cdot x$ (multiplication is not commutative, if *x* <u>or</u> *z* are Δ-numbers)
- $a\cdot\Delta b=\Delta(a\cdot b)$ (multiplication of a real number by a delta number)

Moreover, since multiplication among delta numbers is not commutative, we must have two inverse operations, i.e. two kinds of divisions (the left and right type division). In other words:

for: $y=x\cdot z$

we shall have $z=y/x=\frac{y}{x}=y:x$ (a) left-side division, "logarithm-type"

(58)

but also: $x=y//z=\frac{y}{\overline{\overline{z}}}=y\therefore z$ (b) right-side division, "root-type" inverse.

Symbol used in (a) has the classical performance of a standard division, it will not create Δ-numbers and can be used as a traditional division operation. Symbol used in (b) is new and it designates a special division, which, if *b* is a <u>whole even number</u>, gives as results multiple values, one of which being possibly a Δ-number.

In conclusion, formula $\sqrt[c]{a^b} = a^{b/c}$ could, more correctly, be written as: $\sqrt[c]{a^b} = a^{b//c}$.
This formula can be analogously obtained by observing that:

(59) $$\Delta a \cdot \Delta b = \Delta\left(\Delta a \cdot b\right)$$

In conclusion multiplication among delta numbers can be summarized as follows :

(60)	$\Delta a \cdot b = ab$	*b*: even integer number
	$\Delta a \cdot b = \Delta\left(ab\right)$	*b*: odd integer number
	$a \cdot \Delta b = \Delta\left(ab\right)$	*b*: any number
	$\Delta a \cdot \Delta b = \Delta\left(\Delta a \cdot b\right)$	*b*: integer number

We can see that, in general, if one of the factor is a delta number, we must have:

(61) $$x \cdot z \neq z \cdot x \,. \qquad \text{(non commutativity of multiplication)}$$

with the exceptions of :

(62) $\Delta a \cdot b = b \cdot \Delta a$ *b*: odd integer number
$\Delta a \cdot \Delta b = \Delta b \cdot \Delta a$ *b*: odd integer number

9 – Mapping Δ-numbers

Non-standard analysis specialists are concentrating their efforts on studying actual infinitely small or infinitely large numbers. They are particularly attracted by the problem of differential entities, like d*y*, as well as by infinite magnitudes, traditionally defined via limit operations. We can see a contact area with non-standard analysis, when studying Δ-numbers, by highlighting an object such as:

(63) $$\boxed{\Delta\left(-\infty\right) = -\Delta\infty = -\infty}$$

which materializes the concept of the negative "actual infinite". Nevertheless, we are also aware that, in the case of delta numbers, research should not be "strangely" limited to the domain of $x < -\infty$, due to the permanent correspondence of each number *x* with its "*opposite*" – *x*.

From a methodological point of view, let us start by saying that zeration and Δ-numbers have been justified as follows:

- zeration is automatically implied by level zero of Ackermann's Function;
- Δ-numbers are numbers generated as a result of the inverse of zeration and are associable with the logarithm of negative real numbers ($\Delta x = \ln\left(-e^x\right)$, $e^{\Delta x} = -e^x$);
- Δ-numbers allow us to build a self-contained and non-contradictory theory, leading to practical applications in science and technology, particularly concerning the analytical representation of discontinuous functions or functions with first derivative discontinuities, as well as the direct analytical implementation of Boolean operators.

The development of the theory and application of Δ-numbers depends on the possibility of representing them either in linear or in multi-dimensional geometrical displays. For analysing this aspect of the problem, let us recall formula (12) valid for defining the relation of order respected by delta numbers:

(64) $$\Delta a < -\infty < a \quad \forall a \in \mathrm{R}$$

This formula allows us to graphically show the display of that relation of order as follows, showing the set of delta numbers on an extension of the real axis, after (below) the $\Delta(-\infty) = -\Delta\infty = -\infty$ value:

Δ∞ Δ1 Δ0 Δ(-1) - ∞ -1 0 +1 + ∞

_|______________|___|___|______________|||||||||______________|___|___|_____________|_

$\Delta\infty = +\Delta\infty = \Delta(+\infty)$

We must admit that a fully acceptable graphical representation of delta numbers is one of the most difficult steps in understanding the theory, due to the very new aspect of these numbers. We are also aware that, as we said, the following formulas plays a very important role in the theory. In fact, given:

(65) $y = \ln x$

for: $x = -a < 0$

we get: $y = \ln x = \ln(-a) = \Delta \ln a$ "*delta*" symbol

i.e.: $\Delta y = \ln(-e^y)$.

Nevertheless, it is always possible to define another function, similar to the logarithm, but with a slightly different form. In fact, given:

(66) $y = -\ln(-x)$

for: $x = a > 0$

we get: $y = -\ln(-x) = -\ln(-a) = \nabla \ln a$ provisional symbol: "*nabla*"

i.e.: $\nabla y = -\ln(-e^y)$

In the second equation system, we introduced the new symbol ∇ (nabla), the meaning of which is self-explanatory. The comparison between the two systems show that we must have:

(67) $\boxed{\nabla y = \Delta(-y) = -\Delta y}$

Next figure shows the two "mapping functions" applied to the same real axis *x*:

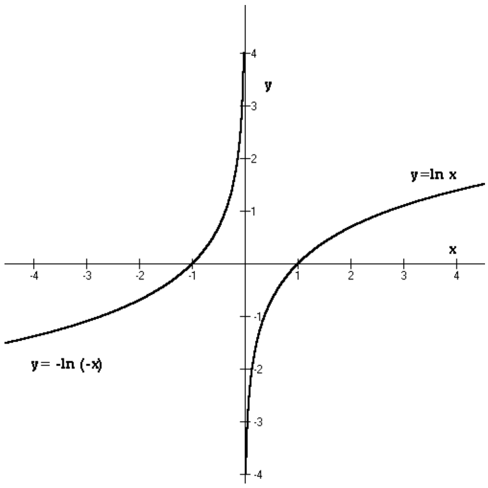


Fig. 4

The new hypothetical "mapping function" $y = -\ln(-x)$ is shown in Fig. 4, left branch. This function would "map" all the negative values of the "*x*" axis to all the values of the "*y*" axis, with a modality similar (but complementary) to that shown for Δ-numbers (See Fig. 5).

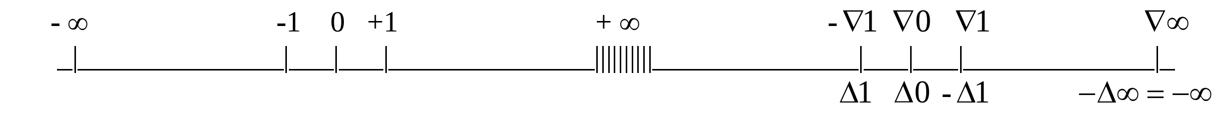


Fig. 5

In fact, we have:

(68) $\Delta a < -\infty < a \quad \forall a \in R$

therefore: $-\nabla a < -\infty < a \quad \forall a \in R$

and then:

(69) $\boxed{a < \infty < \nabla a \quad \forall a \in R}$

However, formula (67) assures us that the hypothetical “nabla” representation is not necessary, because delta numbers cover their entire extension domain.

We should also mention another mapping, alternative to (65), using function $y = 1/\ln(1/x)$, i.e.:

(70) $y = \ :\ln(:x)$

for: $x = a > 0$

we get: $y = \ :\ln(:x) = \ :\ln(:a) = \ \text{-}\ln a$ i.e.: negative numbers

and:: $-y = \ln(:e^y)$

to be compared with $\Delta y = \ln(-e^y)$.

In this case, we shall have the following plot, showing that, for *x* > 1, variable *x* is mapped to the entire set of negative numbers, with no application to + ∞.

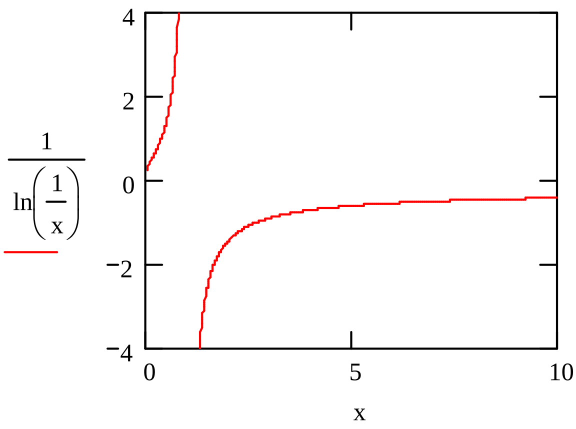


Fig. 6

10 – Classical definitions of the Heaviside and Dirac functions

Mathematical analysis deals function discontinuities, among others, of the following types:

a. poles (infinite values), such as $y = 1/x$;
b. value discontinuities, such as those of square wave signals;
c. tangent discontinuities, such as those of sawtooth wave signals.

Discontinuities of type “a” are normally dealt with within the standard procedures of mathematical analysis. Discontinuities of types “b” and “c” remain objective mathematical problems, requiring case-by-case *ad-hoc* definitions. In science and engineering, for dealing with signals having discontinuities of the three above-mentioned types, the standard procedures include systematic use of the Fourier and/or Laplace transformations.

Actually, as we know, the Heaviside and Dirac functions are traditionally defined as limits obtained from two generative functions, when letting a "variance" parameter → 0. We shall show here two well known definitions, as normally found in the scientific literature. Practically, we shall take, as "generative" of the step function, the Gaussian cumulative normal distribution used in Statistics and in Physics [8], which can be written as follows:

(71) $$D_\sigma(x-a) = \frac{1}{2}\left(1 + \text{erf}\,\frac{x-a}{\sigma\sqrt{2}}\right)$$ cumulative normal distribution

where erf (z) is the Gauss error function
a: average, mean; σ: variance

From (71), we can get the *step function*, letting $\sigma \to 0$, as follows:

(72) $$\theta(x-a) = \lim_{\sigma\to 0} \frac{1}{2}\left(1 + \text{erf}\,\frac{x-a}{\sigma\sqrt{2}}\right)$$ Heaviside's unitary step function [12]

In a similar way, as "generative" of the Dirac function, we may take the Gaussian standard normal distribution, widely used in statistics and physics and also called the "bell" distribution, that can be written as follows:

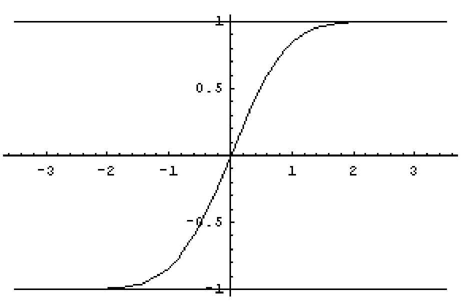
(73) $$P_\sigma(x-a) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-a)^2/2\sigma^2}$$ Gaussian standard normal distribution

a: average, mean; σ: variance

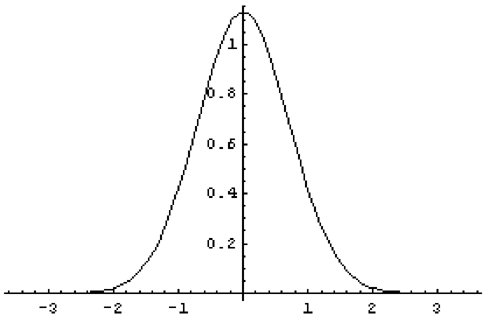
Therefore, as in the case of the step function, the Dirac *infinite impulse function* can be defined as follows:

(74) $$\delta(x-a) = \lim_{\sigma\to 0} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-a)^2/2\sigma^2}$$ Dirac's infinite impulse function

The plots of the Gauss "error function" (component of formula 71) and the Gaussian standard distribution function (73), both in the particular "normalized" situation, in which a = 0 and σ = 1, are as follows:



$y = \text{erf}(x)$



$y = P(x)$

Fig. 7

During the processing of the limit operation, functions (72) and (74) are transformed into the *step* and the *infinite impulse* functions, respectively. It can be shown that, considering formulas (71) and (73), for any "variance" σ, we must have

(75) $$D_\sigma(x-a) = \int_{-\infty}^{x} P_\sigma(x-a)dx \quad \text{and} \quad . \int_{-\infty}^{+\infty} P_\sigma(x-a)dx = 1$$

[12] The Heaviside unitary "step function" is traditionally indicated as $H(x)$, but also as: step(x) or θ (x).

11 – New analytical representation of discontinuous transcendent functions

In [6] (paragraphs 9.1 to 9.4), as well as in section 5 of this report, we have already shown the possibility of using zeration for the analytic definition of the Heaviside and the Dirac functions, as well as of other singular transcendent functions and operators. In this section, we shall try to present the principles for introducing a systematic analytical definition of some elementary discontinuous transcendent functions, exclusively based on the zeration hyper-operation (see also [8]).

First of all, let us show how to use zeration for defining the elementary *unitary discontinuous transcendent function* (**xsu**), for $x \in R$, such that:

(76) $\boxed{\text{xsu}(x-a) = (x \circ a) - (a+1)}$ the elementary **xsu** function

Its properties are:

$\text{xsu}(x-a) = 0$; for: $x < a$

$\text{xsu}(x-a) = 1$; for: $x = a$

$\text{xsu}(x-a) = x - a$; for: $x > a$.

In particular, when $a = 0$, we can also define the *standard unitary discontinuous function*, such that:

(77) $\text{xsu}(x) = (x \circ 0) - 1$ $x \in R$

The meaning of (77) is that, for x real, if $x > 0$ xsu(x) = x, if $x = 0$ xsu(x) = 1 and if $x < 0$ xsu(x) = 0. Function xsu(x), for $x \to 0$, reaches a singular point including both the discontinuities of types “b” and “c”, as shown in the following plot:

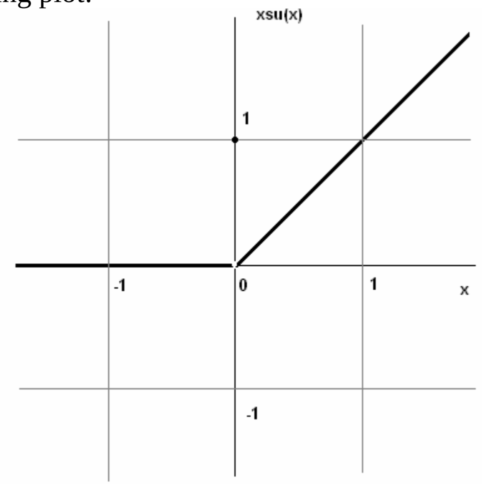


Fig 8

For obtaining the separation of the two discontinuities, we may define functions spot(x) and rmp(x), the unitary single-point [also called spnt(x)] and the unitary ramp functions, respectively, as follows:

(78) $\text{spot}(x) = \text{xsu}(x) \cdot \text{xsu}(-x)$ unitary **single-point** function

with: $\text{spot}(x) = 1$, for $x = 0$

and: $\text{spot}(x) = 0$, for $x \neq 0$

(79) $\text{rmp}(x) = \text{xsu}(x) \cdot [1 - \text{xsu}(-x)]$ unitary **ramp** function

with: $\text{rmp}(x) = 0$, for $-\infty < x < 0$

and: $\text{rmp}(x) = x$, for $x \geq 0$

Function y = spot(x), the "*unitary single-point function*", has the characteristic of always being equal to zero, except for $x = 0$, where its value is 1 and where it describes a single point at the distance "1" from the "x " axis. Function y = rmp(x) is the "*unitary ramp function*", a continuous function, with one tangent (first derivative) discontinuity for $x = 0$ and coincident with $y = x$, for $x \geq 0$. From (77) we can derive another useful unitary discontinuous function, by applying the xsu(- z) function on z = xsu(-x), i.e.by defining functions step(x) and stop(x) = 1 – step(x), as follows:

(80) $\text{step}(x) = \text{xsu}(-\text{xsu}(-x))$ <u>unitary **step** function</u>

with: $\text{step}(x) = 0, \text{ for } x \leq 0$

and: $\text{step}(x) = 1, \text{ for } x > 0$

(81) $\text{stop}(x) = 1 - \text{xsu}(-\text{xsu}(-x))$ <u>unitary **stop** function</u>

with: $\text{stop}(x) = 1, \text{ for } x \leq 0$

and: $\text{stop}(x) = 0, \text{ for } x > 0$

Indeed, we now have the necessary tools for defining the three basic discontinuous transcendent functions widely used, among others, in digital circuit analysis: the generalized unitary <u>ramp</u>, <u>step</u> and <u>infinite impulse</u> functions. Let us see how they can be re-defined, using zeration.

First of all, the *generalized unitary ramp function*, applied to variable $x – a$, can be written as follows:

(82) $\rho(x-a) = \text{rmp}(x-a)$

i.e.: $\boxed{\rho(x-a) = \text{xsu}(x-a) \cdot [1 - \text{xsu}(a-x)]}$ <u>gen. unitary ramp function</u>

$\rho(x-a)$, generalized unitary ramp function, shifted at $x = a$

the plot of which is as follows:

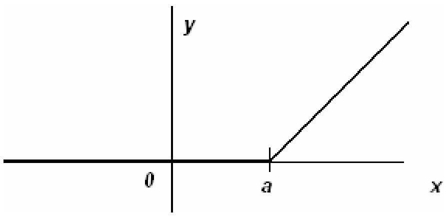


Fig. 9

Moreover, the Heaviside *generalized unitary step function*, according to our definitions based on zeration applied to variable $x – a$, is coincident with function "$\text{step}(x-a)$", which is as follows:

(83) $\theta(x-a) = \text{step}(x-a)$

and, therefore $\boxed{\theta(x-a) = \text{xsu}(-\text{xsu}(a-x))}$ <u>gen. unitary step function</u>

$\theta(x-a)$, the Heaviside unitary step function, shifted at $x = a$

with the following qualitative plot:

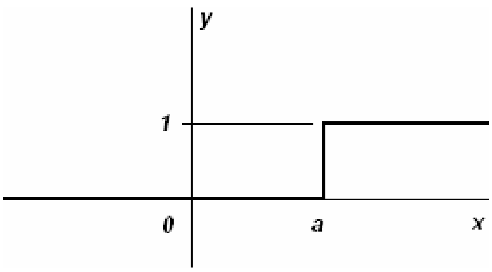


Fig. 10

As far as the definition of the Dirac infinite impulse function is concerned, based on the properties of zeration, we can write:

(84) $\boxed{\delta(x-a) = -\text{spot}(x-a)\cdot(a\Delta x)}$ <u>unitary infinite impulse function</u>

$\delta(x-a)$, the Dirac <u>unitary infinite impulse</u> function, shifted at $x = a$ which has the following qualitative plot:

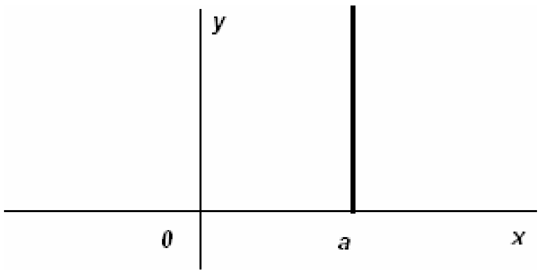


Fig 11

Function $\delta(x-a)$ is called "*infinite impulse*" because, for $x = a$, its value is $+\infty$ and "*unitary*", because we must always have:

(85) $\int_{-\infty}^{x} \delta(x-a) = 1$, for $x \geq a$ and $\int_{-\infty}^{x} \delta(x-a) = 0$, for $x < a$.

i.e.: $\int_{-\infty}^{x} \delta(x-a) = \theta(x-a)$

It should also be pointed out that, also with the direct representation using the above-mentioned discontinuous functions, we may always write the following cumulative formula [see also (75)], which must be valid in both in the "zeration" and the classical "generative-limit" representation types:

(86) $$\boxed{\delta(x-a) = \frac{d}{dx}\theta(x-a) = \frac{d^2}{dx^2}\rho(x-a)}$$

We also have the possibility of defining, with the same methodology, two other important functions: the "*absolute value function*" $y = \text{abs}(x)$, a continuous function with a tangent discontinuity for $x = 0$, and the "*sign function*", $y = \text{sgn}(x)$, a typical discontinuous function for $x = 0$, as follows:

(85) $|x| = \text{abs}(x) = \text{rmp}(x) + \text{rmp}(-x)$

or: $\boxed{|x| = \text{xsu}(x) + \text{xsu}(-x) - 2\cdot\text{xsu}(x)\cdot\text{xsu}(-x)}$ <u>absolute value function</u>

with: $|x| = \text{abs}(x) = 0,\ \text{for } x = 0$

and: $|x| = \text{abs}(x) = x,\ \text{for } x > 0$

and also: $|x| = \text{abs}(x) = -x,\ \text{for } x < 0$

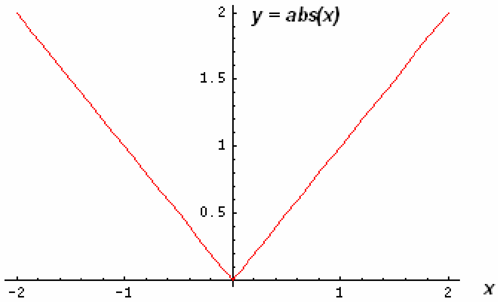


Fig. 12

Under the same assumptions, the “sign of *x*” function, sgn(*x*), can be represented as:

(86) $$\langle x \rangle = \mathrm{sgn}(x) = \mathrm{step}(x) - \mathrm{step}(-x)$$ sign function

i.e.: $$\boxed{\langle x \rangle = \mathrm{sgn}(x) = \mathrm{xsu}\left[-\mathrm{xsu}(-x)\right] - \mathrm{xsu}\left[-\mathrm{xsu}(x)\right]}$$

with: $$\langle x \rangle = \mathrm{sgn}(x) = 0, \text{ for } x = 0$$

and: $$\langle x \rangle = \mathrm{sgn}(x) = 1, \text{ for } x > 0$$

and also: $$\langle x \rangle = \mathrm{sgn}(x) = -1, \text{ for } x < 0$$

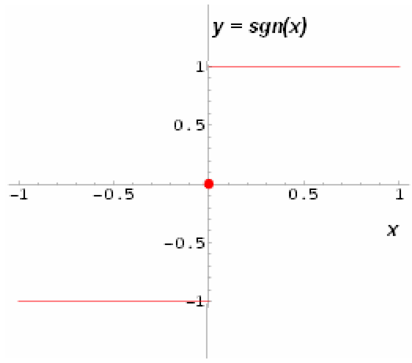


Fig. 13

It is interesting to observe that the last two functions are linked by the following formulas: [13]

(87) $$x = \langle x \rangle \cdot |x| = \mathrm{sgn}(x) \cdot \mathrm{abs}(x) \quad \text{and} \quad \langle x \rangle = \frac{d|x|}{dx}$$

We can therefore conclude that the basic components of the elementary discontinuous transcendent functions, i.e. the Heaviside and the Dirac functions, could systematically be represented, in a very precise way, using zeration and the (83) and (84) formulas, with the help of the **xsu** *unitary discontinuous transcendent function*, analytically defined by (76) and (77). In particular, formulas (85) and (86) can be explicitly shown, only using zeration, addition, subtraction and multiplication, as follows:

(88) $$|x| = \left(1 - (-x \circ 0)\right) \cdot \left((x \circ 0) - 2\right) + \left((x \circ 0) - 1\right) \cdot \left(2 - (-x \circ 0)\right)$$

$$\langle x \rangle = \left(2 - (-x \circ 0)\right) \cdot \left(2 - \left(1 - (x \circ 0) \circ 0\right)\right) - \left((x \circ 0) - 2\right) \cdot \left(\left(\left(1 - (-x \circ 0)\right) \circ 0\right) - 2\right)$$

Other interesting relationships among elementary discontinuous functions are as follows:

(89)
$$\boxed{\begin{aligned}
&\mathrm{xsu}(x) + \mathrm{xsu}(-x) = \mathrm{abs}(x) + 2 \cdot \mathrm{spot}(x) \\
&\mathrm{xsu}(x) - \mathrm{xsu}(-x) = x \\
&\mathrm{xsu}(x) \cdot \mathrm{xsu}(-x) = \mathrm{spot}(x) \\
&\mathrm{step}(x) + \mathrm{stop}(x) = 1 \\
&\mathrm{step}(x) \cdot \mathrm{stop}(x) = 0 \\
&\mathrm{step}(x) - \mathrm{step}(-x) = \mathrm{sgn}(x) \\
&\mathrm{stop}(x) \cdot \mathrm{stop}(-x) = \mathrm{spot}(x) \\
&\mathrm{step}(x) \cdot \mathrm{step}(-x) = 0
\end{aligned}}$$

[13] We propose to indicate the “sign of x” function by $\langle x \rangle = \mathrm{sgn}(x)$, in analogy with $|x| = \mathrm{abs}(x)$.

12 – Provisional conclusions and possible further developments

In conclusion, we can say that rank s=0 hyperoperation (zeration) generates a new set of numbers that can be put in bijection with the logarithms of negative numbers (the delta numbers) and can help in defining a set of fundamental discontinuous functions to be used for digital signal analysis. All that is needed for applying this methodology in science and engineering [7] is an arithmetical operational routine, implementing the "zeration" operation.

Among the developments expected in the (near) future, we should like to mention the use of zeration for the following implementations:

- Listing all useful discontinuous functions that can be generated using zeration and its inverse, implementing all the possible logical Boolean and conditional operators, by using single strip instructions;
- Finalizing the rules of the algebra of delta numbers, for an alternative representation of multi-valued complex numbers, obtained as logarithms of negative numbers.
- Use of a new mathematical notation of transients and of special functions for the extension of the possibilities of creating mathematical models of physical processes, treated until now with nonlinear algorithms and with the help of logical conditions and branchings;
- Analyze the possible connection of "trans-infinite numbers" with numbers $(x+a)\Delta\, x$, where $a \in \left]0;1\right[$ (*the parentheses are not necessary, due to the priority of addition on deltation*).

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