Bell matrices, Carleman matrices

- As I have received many e - mails on the subject - actually too many to be answerable individually - and its relationship to tetration, I have decided to produce a summing up. Hope it will be useful !

  This is a *Mathematica* (6.0.1.0) file, a "notebook". The programmed part - all contained in the cyan cell below - can in principle be executed. But, mainly, a text is provided, intented to be simply read by everybody.

  For general results on matrices, I have used mainly F.R. Gantmacher: "The Theory of Matrices", Chelsea, New York, 1990 (2 vols);

  more specific results can be found in R. Aldrovandi: "Special Matrices of *Mathematical* Physics", World Scientific, Singapore, 2001;

  very specific, only concerned with Bell matrices, R. Aldrovandi and L. P. Freitas: "Continuous iteration of dynamical maps", J. Math. Phys. 39, 5324 (1998).

  Of course, the subject is nowadays concentrated in the prize site "http://www.tetration.org/Dynamics/".

  Notice: the name "Carleman matrices" used below is frequently used for its transpose in the literature.

- **We suppose known the lore of Bell matrices $\mathbb{B} = (\mathbb{B}_{Nm}[g])$, with entries $\mathbb{B}_{Nm}[g]$ given by the multinomial theorem**

$$\frac{(g[x])^m}{m!} == \frac{\left(\sum_{j=1}^{\infty} \frac{g_j}{j!} x^j\right)^m}{m!} == \sum_{N=m}^{\infty} \frac{x^N}{N!} \mathbb{B}_{Nm}[g]; \text{ for each value } N,$$

$\mathbb{B}$ will be an N × N matrix which "linearizes" the series if truncated to order N;

examples for N = 3, $\mathbb{B} = \begin{pmatrix} g[1] & 0 & 0 \\ g[2] & g[1]^2 & 0 \\ g[3] & 3\,g[1]\,g[2] & g[1]^3 \end{pmatrix}$; for N = 4,

$$\mathbb{B} = \begin{pmatrix} g[1] & 0 & 0 & 0 \\ g[2] & g[1]^2 & 0 & 0 \\ g[3] & 3\,g[1]\,g[2] & g[1]^3 & 0 \\ g[4] & \frac{1}{2}\left(6\,g[2]^2 + 8\,g[1]\,g[3]\right) & 6\,g[1]^2\,g[2] & g[1]^4 \end{pmatrix}; \text{ notation here: } g_j = g[j];$$

- In *Mathematica*, they can be obtained with the steps

```
g[N_, t_] := Sum[g[i] (t^i) / (i!), {i, 1, N}]
B[N_, n_, m_] := Limit[D[(Sum[g[i] (t^i) / (i!), {i, 1, N}])^m, {t, n}], t -> 0] / (m!)
BELL[N_] := Table[B[N, n, m], {n, N}, {m, N}]
```

- **Example:**

```
MatrixForm[BELL[5]]
```

$$\begin{pmatrix} g[1] & 0 & 0 & 0 & 0 \\ g[2] & g[1]^2 & 0 & 0 & 0 \\ g[3] & 3\,g[1]\,g[2] & g[1]^3 & 0 & 0 \\ g[4] & \frac{1}{2}\left(6\,g[2]^2+8\,g[1]\,g[3]\right) & 6\,g[1]^2\,g[2] & g[1]^4 & 0 \\ g[5] & 5\,(2\,g[2]\,g[3]+g[1]\,g[4]) & 5\,g[1]\left(3\,g[2]^2+2\,g[1]\,g[3]\right) & 10\,g[1]^3\,g[2] & g[1]^5 \end{pmatrix}$$

- **Notice: these matrices are good for series of type g[x] = $\sum_{j=1}^{\infty} \frac{g_j}{j!}\, x^j$, without independent term: g[0] = 0. The series coefficients can be "read" along the first column. It's an lower triangular matrix ($\mathbb{B}_{Nm}$ = 0 for m > N), exhibiting its eigenvalues in the main diagonal. If we are concerned with iteration, these Bell matrices are of interest because they "linearize" function composition: given another series f(x) = $\sum_{j=1}^{\infty} \frac{f_j}{j!}\, x^j$, the Bell matrix of composition g o f is the (right) product of the corresponding Bell matrices: at each order, $\mathbb{B}[g \circ f] = \mathbb{B}[f]\, \mathbb{B}[g]$. Notice the inverse order!**

A matrix $\mathbb{B}$ can be enlarged to a matrix $\tilde{\mathbb{B}}$ with a "zeroth" row and a "zeroth" column, all extra entries being zero except the "0-0" one, which is = 1. For example, for N = 3, $\tilde{\mathbb{B}}$ =

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & g[1] & 0 & 0 \\ 0 & g[2] & g[1]^2 & 0 \\ 0 & g[3] & 3\,g[1]\,g[2] & g[1]^3 \end{pmatrix}$$

. In *Mathematica*, they can be obtained with

```
BELLLARGE[N_] := Table[B[N, n, m], {n, 0, N}, {m, 0, N}]
MatrixForm[BELLLARGE[3]]
```

- **Example:**

```
MatrixForm[BELLLARGE[4]]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & g[1] & 0 & 0 & 0 \\ 0 & g[2] & g[1]^2 & 0 & 0 \\ 0 & g[3] & 3\,g[1]\,g[2] & g[1]^3 & 0 \\ 0 & g[4] & \frac{1}{2}\left(6\,g[2]^2+8\,g[1]\,g[3]\right) & 6\,g[1]^2\,g[2] & g[1]^4 \end{pmatrix}$$

- **We shall call "Carleman matrices" $\mathbb{C}$ the extensions of Bell matrices for series of type G[x] == g_0 + g[x] = $g_0 + \sum_{j=1}^{\infty} \frac{g_j}{j!}\, x^j$, with an independent term (G[0] == g_0). They are defined by the multinomial theorem**

$$\frac{1}{r!}\,(G[x])^r == \sum_{N=0}^{\infty} \frac{x^N}{N!}\, \mathbb{C}_{Nr}[G] == \sum_{N=0}^{\infty} \frac{x^N}{N!} \sum_{m\geq 0} \frac{g_0^{\,r-m}}{(r-m)!}\, \mathbb{B}_{Nm}[g]\,.$$

- **Notice that the triangular condition** $(\mathbb{B}_{Nm} = 0$ **for** $m > N)$ **ensures** $m \le N$, .
  and the factor $\frac{1}{(r-m)!}$ **ensures** $m \le r$

Carleman matrices can be obtained through the *Mathematica* steps

$$\texttt{Carl[N\_, n\_, m\_] :=} \sum_{r=0}^{n} \frac{g[0]^{m-r}}{(m-r)!} \texttt{B[N, n, r]}$$

```
CARLEMAN[N_] := Table[Carl[N, n, m], {n, 0, N}, {m, 0, N}]
```

- **Examples: for N = 2 and 3,**

```
MatrixForm[CARLEMAN[2]]
MatrixForm[CARLEMAN[3]]
```

$$\begin{pmatrix} 1 & g[0] & \frac{g[0]^2}{2} \\ 0 & g[1] & g[0]\, g[1] \\ 0 & g[2] & g[1]^2 + g[0]\, g[2] \end{pmatrix}$$

$$\begin{pmatrix} 1 & g[0] & \frac{g[0]^2}{2} & \frac{g[0]^3}{6} \\ 0 & g[1] & g[0]\, g[1] & \frac{1}{2} g[0]^2 g[1] \\ 0 & g[2] & g[1]^2 + g[0]\, g[2] & g[0]\, g[1]^2 + \frac{1}{2} g[0]^2 g[2] \\ 0 & g[3] & 3 g[1]\, g[2] + g[0]\, g[3] & g[1]^3 + 3 g[0]\, g[1]\, g[2] + \frac{1}{2} g[0]^2 g[3] \end{pmatrix}$$

- **It is helpful to the mind to label the first row and column as "zeroth". Thus,** $\mathbb{C}_{02}[G] = \frac{g[0]^2}{2}$.
Using *Mathematica,* you would need a sistematic shift to retain such a convention. Series G(x) can be "read" from column number "1" in this notation, the second column in example above. Unlike Bell matrices, the $\mathbb{C}$ **eigenvalues are non-trivial:**

```
Eigenvalues[CARLEMAN[2]]
```

$$\Big\{1, \frac{1}{2}\left(g[1] + g[1]^2 + g[0]\, g[2] - \sqrt{-4 g[1]^3 + \left(-g[1] - g[1]^2 - g[0]\, g[2]\right)^2}\right),$$
$$\frac{1}{2}\left(g[1] + g[1]^2 + g[0]\, g[2] + \sqrt{-4 g[1]^3 + \left(-g[1] - g[1]^2 - g[0]\, g[2]\right)^2}\right)\Big\}$$

```
Eigenvalues[CARLEMAN[3]]
```

$$\{1, \text{Root}[-2 g[1]^6 + g[1] \left(2 g[1]^2 + 2 g[1]^3 + 2 g[1]^4 + 6 g[0]\, g[1]\, g[2] + 2 g[0]\, g[1]^2 g[2] + 3 g[0]^2 g[2]^2 - g[0]^2 g[1]\, g[3]\right) \#1 + \left(-2 g[1] - 2 g[1]^2 - 2 g[1]^3 - 2 g[0]\, g[2] - 6 g[0]\, g[1]\, g[2] - g[0]^2 g[3]\right) \#1^2 + 2 \#1^3 \&, 1],$$
$$\text{Root}[-2 g[1]^6 + g[1] \left(2 g[1]^2 + 2 g[1]^3 + 2 g[1]^4 + 6 g[0]\, g[1]\, g[2] + 2 g[0]\, g[1]^2 g[2] + 3 g[0]^2 g[2]^2 - g[0]^2 g[1]\, g[3]\right) \#1 + \left(-2 g[1] - 2 g[1]^2 - 2 g[1]^3 - 2 g[0]\, g[2] - 6 g[0]\, g[1]\, g[2] - g[0]^2 g[3]\right) \#1^2 + 2 \#1^3 \&, 2],$$
$$\text{Root}[-2 g[1]^6 + g[1] \left(2 g[1]^2 + 2 g[1]^3 + 2 g[1]^4 + 6 g[0]\, g[1]\, g[2] + 2 g[0]\, g[1]^2 g[2] + 3 g[0]^2 g[2]^2 - g[0]^2 g[1]\, g[3]\right) \#1 + \left(-2 g[1] - 2 g[1]^2 - 2 g[1]^3 - 2 g[0]\, g[2] - 6 g[0]\, g[1]\, g[2] - g[0]^2 g[3]\right) \#1^2 + 2 \#1^3 \&, 3]\}$$

- **Notice that a pattern emerges: for N = 2 and 3,**

$$\mathbb{C}\,[G] == \begin{pmatrix} 1 & g[0] & \frac{g[0]^2}{2} \\ 0 & g[1] & g[0]\,g[1] \\ 0 & g[2] & g[1]^2 + g[0]\,g[2] \end{pmatrix} ==$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & g[1] & 0 \\ 0 & g[2] & g[1]^2 \end{pmatrix} + g[0] \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & g[1] \\ 0 & 0 & g[2] \end{pmatrix} + \frac{g[0]^2}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$\mathbb{C}\,[G] == \begin{pmatrix} 1 & g_0 & \frac{1}{2} g_0^2 & \frac{1}{3!} g_0^3 \\ 0 & g_1 & g_1 g_0 & \frac{1}{2} g_1 g_0^2 \\ 0 & g_2 & g_2 g_0 + g_1^2 & \frac{1}{2} g_2 g_0^2 + g_1^2 g_0 \\ 0 & g_3 & g_3 g_0 + 3 g_1 g_2 & \frac{1}{2} g_3 g_0^2 + 3 g_1 g_2 g_0 + g_1^3 \end{pmatrix} ==$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & g_1 & 0 & 0 \\ 0 & g_2 & g_1^2 & 0 \\ 0 & g_3 & 3 g_1 g_2 & g_1^3 \end{pmatrix} + \begin{pmatrix} 0 & g_0 & \frac{1}{2} g_0^2 & \frac{1}{3!} g_0^3 \\ 0 & 0 & g_1 g_0 & \frac{1}{2} g_1 g_0^2 \\ 0 & 0 & g_2 g_0 & \frac{1}{2} g_2 g_0^2 + g_1^2 g_0 \\ 0 & 0 & g_3 g_0 & \frac{1}{2} g_3 g_0^2 + 3 g_1 g_2 g_0 \end{pmatrix} ==$$

$$\begin{pmatrix} 1 & \begin{matrix} 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} & \mathbb{B} \end{pmatrix} + \begin{pmatrix} 0 & g_0 & \frac{1}{2} g_0^2 & \frac{1}{3!} g_0^3 \\ 0 & 0 & g_1 g_0 & \frac{1}{2} g_1 g_0^2 \\ 0 & 0 & g_2 g_0 & \frac{1}{2} g_2 g_0^2 + g_1^2 g_0 \\ 0 & 0 & g_3 g_0 & \frac{1}{2} g_3 g_0^2 + 3 g_1 g_2 g_0 \end{pmatrix};$$

Terms with $g_0 \neq 0$ are isolated in the second matrix. Isolating progressive powers of g_0, we have

$$g_0 \begin{pmatrix} 0 & 1 & \frac{1}{2} g_0 & \frac{1}{3!} g_0^2 \\ 0 & 0 & g_1 & \frac{1}{2} g_1 g_0^2 \\ 0 & 0 & g_2 & \frac{1}{2} g_2 g_0 + g_1^2 \\ 0 & 0 & g_3 & \frac{1}{2} g_3 g_0 + 3 g_1 g_2 \end{pmatrix} == g_0 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g_1 & 0 \\ 0 & 0 & g_2 & g_1^2 \\ 0 & 0 & g_3 & 3 g_1 g_2 \end{pmatrix} +$$

$$\frac{g_0^2}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & g_1 \\ 0 & 0 & 0 & g_2 \\ 0 & 0 & 0 & g_3 \end{pmatrix} + \frac{g_0^3}{3!} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ==$$

$$g_0 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \mathbb{B}_{11} & 0 \\ 0 & 0 & \mathbb{B}_{21} & \mathbb{B}_{22} \\ 0 & 0 & \mathbb{B}_{31} & \mathbb{B}_{32} \end{pmatrix} + \frac{g_0^2}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathbb{B}_{11} \\ 0 & 0 & 0 & \mathbb{B}_{21} \\ 0 & 0 & 0 & \mathbb{B}_{31} \end{pmatrix} + \frac{g_0^3}{3!} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- **This suggests the introduction of a matrix $\mathbb{M}$ with entries $\mathbb{M}_{rm}[g_0] == \frac{g_0^{m-r}}{(m-r)!}$. It's an upper diagonal matrix with all entries = 1 in the diagonal. Can be obtained as**

```
EME[N_, n_, m_] := g0^(m-n)/(m - n) !
MATEME[N_] := Table[EME[N, n, m], {n, 0, N}, {m, 0, N}]
MatrixForm[MATEME[2]]
MatrixForm[MATEME[3]]
```

$$\begin{pmatrix} 1 & g_0 & \frac{g_0^2}{2} \\ 0 & 1 & g_0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & g_0 & \frac{g_0^2}{2} & \frac{g_0^3}{6} \\ 0 & 1 & g_0 & \frac{g_0^2}{2} \\ 0 & 0 & 1 & g_0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- **Then, as it can be show directly, $\mathbb{C} = \mathbb{B}\ \mathbb{M}$. For example,**

```
MatrixForm[BELLLARGE[2].MATEME[2]]
```

$$\begin{pmatrix} 1 & g_0 & \frac{g_0^2}{2} \\ 0 & g[1] & g[1]\, g_0 \\ 0 & g[2] & g[1]^2 + g[2]\, g_0 \end{pmatrix}$$

- **It is possible to define alternatively**

```
CARLEMAN[N_] := BELLLARGE[N].MATEME[N]
MatrixForm[CARLEMAN[3]]
```

$$\begin{pmatrix} 1 & g_0 & \frac{g_0^2}{2} & \frac{g_0^3}{6} \\ 0 & g[1] & g[1]\, g_0 & \frac{1}{2}\, g[1]\, g_0^2 \\ 0 & g[2] & g[1]^2 + g[2]\, g_0 & g[1]^2\, g_0 + \frac{1}{2}\, g[2]\, g_0^2 \\ 0 & g[3] & 3\, g[1]\, g[2] + g[3]\, g_0 & g[1]^3 + 3\, g[1]\, g[2]\, g_0 + \frac{1}{2}\, g[3]\, g_0^2 \end{pmatrix}$$

- But the main interest of this result lies in the fact that, as det $\mathbb{M} = 1$, then det $\mathbb{C}$ = det $\tilde{\mathbb{B}}$ = det $\mathbb{B}$. It follows that $\mathbb{C}$ is invertible whenever $\mathbb{B}$ is invertible, that is, when $g_1 \neq 0$.

- **Now, for composition: if we consider the composition of two series, G[x] == g_0 + g[x] and F[x] == f_0 + f[x], we verify that $\mathbb{C}$[G o F] = $\mathbb{C}$[F] $\mathbb{C}$[G]. This means that $\mathbb{C}$[G o G] = $\mathbb{C}^2$[G]. If we use notation $G^{<n>}$ for the n-th iterate of G, $\mathbb{C}[G^{<n>}] = \mathbb{C}^n$[G], the n-th power of G. Function iterates are translated into powers of matrices, as with Bell matrices. The definition of continuous iterate follows suite: $G^{<t>}$ can be obtained (from the column numbered "1") from $\mathbb{C}^t$[G], provided this arbitrary power of $\mathbb{C}$ can be obtained.**

- Checking whether really $\mathbb{C}$[F o G] = $\mathbb{C}$[G] $\mathbb{C}$[F]:

$$F[G(x)] = f_0 + f_1 g_0 + \frac{1}{2} f_2 g_0^2 + \frac{1}{6} f_3 g_0^3 + x \left(f_1 g_1 + f_2 g_0 g_1 + \frac{1}{2} f_3 g_0^2 g_1\right) + \frac{1}{8} x^4 f_2 g_2^2 + x^2 \left(\frac{1}{2} f_2 g_1^2 + \frac{1}{2} f_3 g_0 g_1^2 + \frac{f_1 g_2}{2} + \frac{1}{2} f_2 g_0 g_2 + \frac{1}{4} f_3 g_0^2 g_2\right) + x^3 \left(\frac{1}{6} f_3 g_1^3 + \frac{1}{2} f_2 g_1 g_2 + \frac{1}{2} f_3 g_0 g_1 g_2 + \frac{f_1 g_3}{6} + \frac{1}{6} f_2 g_0 g_3 + \frac{1}{12} f_3 g_0^2 g_3\right) + \ldots$$

Each coefficient in this composition can be obtained as an entry:

$$\text{Simplify}\left[\begin{pmatrix} 1 & g_0 & \frac{g_0^2}{2} & \frac{g_0^3}{6} \\ 0 & g[1] & g[1]\, g_0 & \frac{1}{2} g[1]\, g_0^2 \\ 0 & g[2] & g[1]^2 + g[2]\, g_0 & g[1]^2 g_0 + \frac{1}{2} g[2]\, g_0^2 \\ 0 & g[3] & 3 g[1]\, g[2] + g[3]\, g_0 & g[1]^3 + 3 g[1]\, g[2]\, g_0 + \frac{1}{2} g[3]\, g_0^2 \end{pmatrix} . \begin{pmatrix} 1 & f_0 & \frac{f_0^2}{2} & \frac{f_0^3}{6} \\ 0 & f[1] & f[1]\, f_0 & \frac{1}{2} f[1]\, f_0^2 \\ 0 & f[2] & f[1]^2 + f[2]\, f_0 & f[1]^2 f_0 + \frac{1}{2} f[2]\, f_0^2 \\ 0 & f[3] & 3 f[1]\, f[2] + f[3]\, f_0 & f[1]^3 + 3 f[1]\, f[2]\, f_0 + \frac{1}{2} f[3]\, f_0^2 \end{pmatrix}\right][[1, 2]]$$

$$f_0 + f[1]\, g_0 + \frac{1}{2} f[2]\, g_0^2 + \frac{1}{6} f[3]\, g_0^3$$

$$\text{Simplify}\left[\begin{pmatrix} 1 & g_0 & \frac{g_0^2}{2} & \frac{g_0^3}{6} \\ 0 & g[1] & g[1]\, g_0 & \frac{1}{2} g[1]\, g_0^2 \\ 0 & g[2] & g[1]^2 + g[2]\, g_0 & g[1]^2 g_0 + \frac{1}{2} g[2]\, g_0^2 \\ 0 & g[3] & 3 g[1]\, g[2] + g[3]\, g_0 & g[1]^3 + 3 g[1]\, g[2]\, g_0 + \frac{1}{2} g[3]\, g_0^2 \end{pmatrix} . \begin{pmatrix} 1 & f_0 & \frac{f_0^2}{2} & \frac{f_0^3}{6} \\ 0 & f[1] & f[1]\, f_0 & \frac{1}{2} f[1]\, f_0^2 \\ 0 & f[2] & f[1]^2 + f[2]\, f_0 & f[1]^2 f_0 + \frac{1}{2} f[2]\, f_0^2 \\ 0 & f[3] & 3 f[1]\, f[2] + f[3]\, f_0 & f[1]^3 + 3 f[1]\, f[2]\, f_0 + \frac{1}{2} f[3]\, f_0^2 \end{pmatrix}\right][[2, 2]]$$

$$f[1]\, g[1] + f[2]\, g[1]\, g_0 + \frac{1}{2} f[3]\, g[1]\, g_0^2$$

- The question is now how to obtain an arbitrary t-th power of $\mathbb{C}$. Let us then address the problem of matrix functions. In the generic case, $\mathbb{C}$ is non-degenerate, so that the simplest approach to the subject will be enough. Given a matrix (N+1) × (N+1) $\mathbb{C}$, it is necessary to

find (i) its (N+1) eigenvalues λ_k and (ii) its (N+1) corresponding "component projectors" $\mathbb{Z}_k$, which are matrices satisfying

$$\mathbb{C}\,\mathbb{Z}_k = \lambda_k\,\mathbb{Z}_k, \quad \mathbb{Z}_k^2 = \mathbb{Z}_k.$$

Then, a function F($\mathbb{C}$), given by the power series $\sum_{j=0}^{\infty} \frac{F_j}{j!}\,\mathbb{C}^j$, can be shown to be the same as

$$F(\mathbb{C}) = \sum_{k=0}^{N} F(\lambda_k)\,\mathbb{Z}_k.$$

To get the $\mathbb{Z}_k$'s, it is necessary to know beforehand (N+1) "seed" functions of $\mathbb{C}$. The simplest are the powers $\mathbb{C}^n$, although any set of functions could be used. Take $\mathbb{C}^n = \sum_{k=1}^{N+1} \lambda_k{}^n\,\mathbb{Z}_k$. Inversion of these last expressions give the $\mathbb{Z}_k$'s as polynomials in $\mathbb{C}$,

$$\mathbb{Z}_i[\mathbb{C}] == \frac{(\mathbb{C}-\lambda_1\,\mathbb{I})\,(\mathbb{C}-\lambda_2\,\mathbb{I})\ldots(\mathbb{C}-\lambda_{i-1}\,\mathbb{I})\,(\mathbb{C}-\lambda_{i+1}\,\mathbb{I})\ldots(\mathbb{C}-\lambda_{N-1}\,\mathbb{I})\,(\mathbb{C}-\lambda_{N+1}\,\mathbb{I})}{(\lambda_i-\lambda_1)\,(\lambda_i-\lambda_2)\ldots(\lambda_i-\lambda_{i-1})\,(\lambda_i-\lambda_{i+1})\ldots(\lambda_i-\lambda_{N-1})\,(\lambda_i-\lambda_{N+1})}.$$

In *Mathematica [b*ecause it reads $\mathbb{C}^n$ not as the n-th power of $\mathbb{C}$, but as that matrix whose entries are $(\mathbb{C}^n{}_{ij})$], it is better to use a more complicated expression involving the characteristic polynomial of $\mathbb{C}$. For an arbitrary matrix M, the steps are

```
L = Eigenvalues[M] // Simplify
Pol[z_, k_] := Cancel[ (Product_{i=1}^{Length[M]} (z - L[[i]])) / (z - L[[k]]) ] // Simplify
R[k_] := Sum_{i=0}^{Length[M]-1} Coefficient[Pol[z, k], z, i] MatrixPower[M, i] // Simplify
Z[k_] := Cancel[R[k] / (Tr[R[k]])] // Simplify
```

- **Thus for M = $\mathbb{C}$ and N = 2, for example, $\mathbb{Z}_1$ is obtained as follows :**

```
M = CARLEMAN[2];
L = Eigenvalues[M] // Simplify;
Pol[z_, k_] := Cancel[ (Product_{i=1}^{Length[M]} (z - L[[i]])) / (z - L[[k]]) ] // Simplify
R[k_] := Sum_{i=0}^{Length[M]-1} Coefficient[Pol[z, k], z, i] MatrixPower[M, i] // Simplify
Z[k_] := Cancel[R[k] / (Tr[R[k]])] // Simplify // MatrixForm
Z[1]
```

$$\begin{pmatrix} 1 & -\frac{g_0\,(-2+2\,g[1]^2+g[2]\,g_0)}{2\,((-1+g[1])^2\,(1+g[1])-g[2]\,g_0)} & \frac{(1+g[1])\,g_0^2}{2\,((-1+g[1])^2\,(1+g[1])-g[2]\,g_0)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- **Once the projectors are obtained, we go back to F($\mathbb{C}$[G]) = $\sum_{k=0}^{N} F(\lambda_k)\,\mathbb{Z}_k$, a matrix whose second column (numbered "1" in our convention) gives the Taylor coefficients of F{G(x}]. Thus,**

$$[F[G(x)]] = \sum_{j=0}^{\infty} \frac{x^j}{j!} \sum_{k=0}^{N} F(\lambda_k)\,[\mathbb{Z}_k]_{j1}.$$

For the arbitrary iteration of G,

$$G^{<t>}(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!} \sum_{k=0}^{N} \lambda_k{}^t\,[\mathbb{Z}_k]_{j1}.$$

Adapted to *Mathematica* index managing, and to order N,

$$G^{<t>}(x) = \sum_{j=0}^{\text{Length}[M]-1} \frac{x^j}{j!} \sum_{k=1}^{\text{Length}[M]} L[[k]]^t Z[k][[j+1, 2]]$$

- **Thus, the complete program to obtain the t-th iterate of G(x) is (we give it already with two tests for N = 2)**

```
g[N_, t_] := Sum[g[i] (t^i) / (i!), {i, 1, N}]
B[N_, n_, m_] := Limit[D[(Sum[g[i] (t^i) / (i!), {i, 1, N}])^m, {t, n}], t → 0] / (m!)
BELL[N_] := Table[B[N, n, m], {n, N}, {m, N}]
BELLLARGE[N_] := Table[B[N, n, m], {n, 0, N}, {m, 0, N}]
Carl[N_, n_, m_] := Sum[g[0]^(m-r)/(m-r)! B[N, n, r], {r, 0, n}]
CARLEMAN[N_] := Table[Carl[N, n, m], {n, 0, N}, {m, 0, N}]
MatrixForm[CARLEMAN[1]];
MatrixForm[CARLEMAN[2]];
MatrixForm[CARLEMAN[3]];

M = CARLEMAN[2] ;
MatrixForm[%]'
L = Eigenvalues[M] // Simplify
Pol[z_, k_] := Cancel[Product[(z - L[[i]]), {i, 1, Length[M]}] / (z - L[[k]])] // Simplify

R[k_] := Sum[Coefficient[Pol[z, k], z, i] MatrixPower[M, i], {i, 0, Length[M]-1}] // Simplify

Z[k_] := Cancel[R[k] / (Tr[R[k]])] // Simplify

Tetr[t_, x_] := Sum[x^j/j! Sum[L[[k]]^t Z[k][[j+1, 2]], {k, 1, Length[M]}], {j, 0, Length[M]-1}]

Tetr[1, x] // Simplify
Tetr[2, x] // Simplify
```

$$\begin{pmatrix} 1 & g[0] & \frac{g[0]^2}{2} \\ 0 & g[1] & g[0]\,g[1] \\ 0 & g[2] & g[1]^2 + g[0]\,g[2] \end{pmatrix}'$$

$$\left\{1, \frac{1}{2}\left(g[1] + g[1]^2 + g[0]\,g[2] - \sqrt{-4\,g[1]^3 + \left(g[1] + g[1]^2 + g[0]\,g[2]\right)^2}\right), \frac{1}{2}\left(g[1] + g[1]^2 + g[0]\,g[2] + \sqrt{-4\,g[1]^3 + \left(g[1] + g[1]^2 + g[0]\,g[2]\right)^2}\right)\right\}$$

$$g[0] + x\,g[1] + \frac{1}{2}x^2\,g[2]$$

$$\frac{1}{2}\left(g[0]^2\,g[2] + x\,g[1]\,(2\,g[1] + x\,(1 + g[1])\,g[2]) + g[0]\left(2 + x^2\,g[2]^2 + 2\,g[1]\,(1 + x\,g[2])\right)\right)$$

- **We give here the solution for G(x) = e^x and N = 2:**

```
g[N_, t_] := Sum[g[i] (t^i) / (i!), {i, 1, N}]
B[N_, n_, m_] := Limit[D[(Sum[g[i] (t^i) / (i!), {i, 1, N}])^m, {t, n}], t → 0] / (m!)
BELL[N_] := Table[B[N, n, m], {n, N}, {m, N}]
BELLLARGE[N_] := Table[B[N, n, m], {n, 0, N}, {m, 0, N}]
```

$$\text{Carl}[N\_, n\_, m\_] := \sum_{r=0}^{n} \frac{g[0]^{m-r}}{(m-r)!} B[N, n, r]$$

```
CARLEMAN[N_] := Table[Carl[N, n, m], {n, 0, N}, {m, 0, N}]
MatrixForm[CARLEMAN[1]];
MatrixForm[CARLEMAN[2]];
MatrixForm[CARLEMAN[3]];

M = CARLEMAN[2] /. {g[0] → 1, g[1] → 1, g[2] → 1, g[3] → 1};
MatrixForm[%]
L = Eigenvalues[M] // Simplify
```

$$\text{Pol}[z\_, k\_] := \text{Cancel}\left[\frac{\prod_{i=1}^{\text{Length}[M]} (z - L[[i]])}{(z - L[[k]])}\right] // \text{Simplify}$$

$$R[k\_] := \sum_{i=0}^{\text{Length}[M]-1} \text{Coefficient}[\text{Pol}[z, k], z, i]\ \text{MatrixPower}[M, i] // \text{Simplify}$$

$$Z[k\_] := \text{Cancel}[R[k] / (\text{Tr}[R[k]])] // \text{Simplify}$$

$$\text{Tetr}[t\_, x\_] := \sum_{j=0}^{\text{Length}[M]-1} \frac{x^j}{j!} \sum_{k=1}^{\text{Length}[M]} L[[k]]^t\, Z[k][[j+1, 2]]$$

```
Tetr[1, x] // Simplify
Tetr[2, x] // Simplify
```

$$\begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\left\{\frac{1}{2}\left(3+\sqrt{5}\right), 1, \frac{1}{2}\left(3-\sqrt{5}\right)\right\}$$

$$1 + x + \frac{x^2}{2}$$

$$\frac{1}{2}\left(5 + 4x + 3x^2\right)$$

- **For arbitrary t,**

```
Tetr[t, x] // Simplify
```

$$\frac{1}{5} 2^{-2-t} \left(5 \left(2^{1+t} - \left(3-\sqrt{5}\right)^t \left(1+\sqrt{5}\right) + \left(-1+\sqrt{5}\right)\left(3+\sqrt{5}\right)^t\right) - 2\left(3-\sqrt{5}\right)^t x \left(-5-\sqrt{5}+\sqrt{5}\,x\right) + 2\left(3+\sqrt{5}\right)^t x \left(5-\sqrt{5}+\sqrt{5}\,x\right)\right)$$

- **This gives, to the order N = 2 of approximation, function G(x) = e^x iterated t times. If we want. by all means, e^e iterated t times, the numerical value is**

```
N[Tetr[t, e], 3] // Simplify
```

$$0.5 - 0.5\, e^{-0.96 t} + 2.7\, e^{0.96 t}$$

- **If we want e^e iterated e times,**

```
N[Tetr[e, e], 3] // Simplify
```

```
37.6
```

- **Of course, the order N necessary to have a good precision depends heavily on the values of t and x (as is the case for the pure exponential function). Clearly the precision of this resuls is very very bad ... For the next values of N, it oscillates a lot, and does not even seems to converge !**

- **For N = 3, profiting to exhibit eigenvalues and projectors:**

```
g[N_, t_] := Sum[g[i] (t^i) / (i!), {i, 1, N}]
B[N_, n_, m_] := Limit[D[(Sum[g[i] (t^i) / (i!), {i, 1, N}])^m, {t, n}], t → 0] / (m!)
BELL[N_] := Table[B[N, n, m], {n, N}, {m, N}]
BELLLARGE[N_] := Table[B[N, n, m], {n, 0, N}, {m, 0, N}]
Carl[N_, n_, m_] := Sum[g[0]^(m-r) / (m - r)! B[N, n, r], {r, 0, n}]
CARLEMAN[N_] := Table[Carl[N, n, m], {n, 0, N}, {m, 0, N}]
MatrixForm[CARLEMAN[1]];
MatrixForm[CARLEMAN[2]];
MatrixForm[CARLEMAN[3]];

M = CARLEMAN[3] /. {g[0] → 1, g[1] → 1, g[2] → 1, g[3] → 1};
MatrixForm[%]
L = N[Eigenvalues[M], 6] // Simplify
Pol[z_, k_] := Cancel[Product[(z - L[[i]]), {i, 1, Length[M]}] / (z - L[[k]])] // Simplify
R[k_] := Sum[Coefficient[Pol[z, k], z, i] MatrixPower[M, i] // Simplify, {i, 0, Length[M]-1}]
Z[k_] := Cancel[R[k] / (Tr[R[k]])] // Simplify

MatrixForm[Simplify[N[Z[1], 3]]]
MatrixForm[Simplify[N[Z[2], 3]]]
MatrixForm[Simplify[N[Z[3], 3]]]
MatrixForm[Simplify[N[Z[4], 3]]]
N[Tetr[e, e], 3] // Simplify
```

$$\begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 1 & 2 & \frac{3}{2} \\ 0 & 1 & 4 & \frac{9}{2} \end{pmatrix}$$

{6.24452, 1.11139, 1.00000, 0.144090}

$$\begin{pmatrix} 0.\times 10^{-8} & 0.0268 & 0.0714 & 0.0691 \\ 0 & 0.0449 & 0.120 & 0.116 \\ 0 & 0.104 & 0.276 & 0.267 \\ 0 & 0.263 & 0.702 & 0.679 \end{pmatrix}$$

$$\begin{pmatrix} 0.\times 10^{-5} & 6.17 & 2.87 & -2.18 \\ 0 & 0.602 & 0.280 & -0.213 \\ 0 & 0.380 & 0.177 & -0.134 \\ 0 & -0.627 & -0.291 & 0.221 \end{pmatrix}$$

$$\begin{pmatrix} 1.00 & -6.00 & -3.17 & 2.17 \\ 0 & 0.\times 10^{-5} & 0.\times 10^{-4} & 0.\times 10^{-5} \\ 0 & 0.\times 10^{-5} & 0.\times 10^{-4} & 0.\times 10^{-4} \\ 0 & 0.\times 10^{-4} & 0.\times 10^{-4} & 0.\times 10^{-4} \end{pmatrix}$$

$$\begin{pmatrix} 0.\times 10^{-5} & -0.201 & 0.227 & -0.0551 \\ 0 & 0.353 & -0.399 & 0.0969 \\ 0 & -0.484 & 0.547 & -0.133 \\ 0 & 0.363 & -0.411 & 0.0997 \end{pmatrix}$$

209.

- **From now on: seems to have difficult convergence. Let us exhibit the results fro N = up to 10. For N = 4:**

```
M = CARLEMAN[4] /. {g[0] → 1, g[1] → 1, g[2] → 1, g[3] → 1, g[4] → 1};
MatrixForm[%]
L = N[Eigenvalues[M], 6] // Simplify

N[Tetr[e, e], 3] // Simplify
```

$$\begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 0 & 1 & 2 & \frac{3}{2} & \frac{2}{3} \\ 0 & 1 & 4 & \frac{9}{2} & \frac{8}{3} \\ 0 & 1 & 8 & \frac{27}{2} & \frac{32}{3} \end{pmatrix}$$

{15.2003, 2.38724, 1.00000, 0.526838, 0.0523089}

981.

- **N = 5:**

```
M = CARLEMAN[5] /. {g[0] → 1, g[1] → 1, g[2] → 1, g[3] → 1, g[4] → 1, g[5] → 1};
MatrixForm[%]
L = N[Eigenvalues[M], 8] // Simplify
N[Tetr[e, e], 3] // Simplify
```

$$\begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} & \frac{1}{120} \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} \\ 0 & 1 & 2 & \frac{3}{2} & \frac{2}{3} & \frac{5}{24} \\ 0 & 1 & 4 & \frac{9}{2} & \frac{8}{3} & \frac{25}{24} \\ 0 & 1 & 8 & \frac{27}{2} & \frac{32}{3} & \frac{125}{24} \\ 0 & 1 & 16 & \frac{81}{2} & \frac{128}{3} & \frac{625}{24} \end{pmatrix}$$

{37.736997, 5.0202687, 1.1933230, 1.00000000, 0.23925720, 0.018487691}

4.08×10^{3}

- **N = 6:**

```
M = CARLEMAN[6] /. {g[0] → 1, g[1] → 1, g[2] → 1, g[3] → 1, g[4] → 1, g[5] → 1, g[6] → 1};
MatrixForm[%]
L = N[Eigenvalues[M], 12] // Simplify

N[Tetr[e, e], 3] // Simplify
```

$$\begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} & \frac{1}{120} & \frac{1}{720} \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} & \frac{1}{120} \\ 0 & 1 & 2 & \frac{3}{2} & \frac{2}{3} & \frac{5}{24} & \frac{1}{20} \\ 0 & 1 & 4 & \frac{9}{2} & \frac{8}{3} & \frac{25}{24} & \frac{3}{10} \\ 0 & 1 & 8 & \frac{27}{2} & \frac{32}{3} & \frac{125}{24} & \frac{9}{5} \\ 0 & 1 & 16 & \frac{81}{2} & \frac{128}{3} & \frac{625}{24} & \frac{54}{5} \\ 0 & 1 & 32 & \frac{243}{2} & \frac{512}{3} & \frac{3125}{24} & \frac{324}{5} \end{pmatrix}$$

```
{95.0154500254, 10.9061640192, 2.35431682929,
 1.00000000000, 0.623528699098, 0.102457710251, 0.00641605007793}
```

1.55×10^4

- **N = 7:**

```
M = CARLEMAN[7] /.
   {g[0] → 1, g[1] → 1, g[2] → 1, g[3] → 1, g[4] → 1, g[5] → 1, g[6] → 1, g[7] → 1};
MatrixForm[%]
L = N[Eigenvalues[M], 20] // Simplify

N[Tetr[e, e], 3] // Simplify
```

$$\begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} & \frac{1}{120} & \frac{1}{720} & \frac{1}{5040} \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} & \frac{1}{120} & \frac{1}{720} \\ 0 & 1 & 2 & \frac{3}{2} & \frac{2}{3} & \frac{5}{24} & \frac{1}{20} & \frac{7}{720} \\ 0 & 1 & 4 & \frac{9}{2} & \frac{8}{3} & \frac{25}{24} & \frac{3}{10} & \frac{49}{720} \\ 0 & 1 & 8 & \frac{27}{2} & \frac{32}{3} & \frac{125}{24} & \frac{9}{5} & \frac{343}{720} \\ 0 & 1 & 16 & \frac{81}{2} & \frac{128}{3} & \frac{625}{24} & \frac{54}{5} & \frac{2401}{720} \\ 0 & 1 & 32 & \frac{243}{2} & \frac{512}{3} & \frac{3125}{24} & \frac{324}{5} & \frac{16807}{720} \\ 0 & 1 & 64 & \frac{729}{2} & \frac{2048}{3} & \frac{15625}{24} & \frac{1944}{5} & \frac{117649}{720} \end{pmatrix}$$

```
{241.71884195402584865, 24.423679761705123493,
 4.6420349211722378955, 1.2685086810034553218, 1.0000000000000000000,
 0.31258935914550811378, 0.041869671065945062737, 0.0021978741041036869726}
```

5.48×10^4

▪ N = 8:

```
M = CARLEMAN[8] /. {g[0] → 1, g[1] → 1, g[2] → 1,
    g[3] → 1, g[4] → 1, g[5] → 1, g[6] → 1, g[7] → 1, g[8] → 1};
MatrixForm[%]
L = N[Eigenvalues[M], 20] // Simplify

N[Tetr[ⅇ, ⅇ], 3] // Simplify
```

$$\begin{pmatrix}
1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} & \frac{1}{120} & \frac{1}{720} & \frac{1}{5040} & \frac{1}{40320} \\
0 & 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} & \frac{1}{120} & \frac{1}{720} & \frac{1}{5040} \\
0 & 1 & 2 & \frac{3}{2} & \frac{2}{3} & \frac{5}{24} & \frac{1}{20} & \frac{7}{720} & \frac{1}{630} \\
0 & 1 & 4 & \frac{9}{2} & \frac{8}{3} & \frac{25}{24} & \frac{3}{10} & \frac{49}{720} & \frac{4}{315} \\
0 & 1 & 8 & \frac{27}{2} & \frac{32}{3} & \frac{125}{24} & \frac{9}{5} & \frac{343}{720} & \frac{32}{315} \\
0 & 1 & 16 & \frac{81}{2} & \frac{128}{3} & \frac{625}{24} & \frac{54}{5} & \frac{2401}{720} & \frac{256}{315} \\
0 & 1 & 32 & \frac{243}{2} & \frac{512}{3} & \frac{3125}{24} & \frac{324}{5} & \frac{16807}{720} & \frac{2048}{315} \\
0 & 1 & 64 & \frac{729}{2} & \frac{2048}{3} & \frac{15625}{24} & \frac{1944}{5} & \frac{117649}{720} & \frac{16384}{315} \\
0 & 1 & 128 & \frac{2187}{2} & \frac{8192}{3} & \frac{78125}{24} & \frac{11664}{5} & \frac{823543}{720} & \frac{131072}{315}
\end{pmatrix}$$

```
{619.79825141573563623, 55.996939866767674624, 9.4678021659188880307,
 2.3812416217435864650, 1.0000000000000000000, 0.70201105026314061543,
 0.14779844368376080673, 0.016519324094288319396, 0.00074563560254871500977}
```

1.84×10^{5}

▪ N = 9 :

```
M = CARLEMAN[9] /. {g[0] → 1, g[1] → 1, g[2] → 1,
    g[3] → 1, g[4] → 1, g[5] → 1, g[6] → 1, g[7] → 1, g[8] → 1, g[9] → 1};
MatrixForm[%]
L = N[Eigenvalues[M], 30] // Simplify

N[Tetr[ⅇ, ⅇ], 3] // Simplify
```

$$\begin{pmatrix}
1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} & \frac{1}{120} & \frac{1}{720} & \frac{1}{5040} & \frac{1}{40320} & \frac{1}{362880} \\
0 & 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} & \frac{1}{120} & \frac{1}{720} & \frac{1}{5040} & \frac{1}{40320} \\
0 & 1 & 2 & \frac{3}{2} & \frac{2}{3} & \frac{5}{24} & \frac{1}{20} & \frac{7}{720} & \frac{1}{630} & \frac{1}{4480} \\
0 & 1 & 4 & \frac{9}{2} & \frac{8}{3} & \frac{25}{24} & \frac{3}{10} & \frac{49}{720} & \frac{4}{315} & \frac{9}{4480} \\
0 & 1 & 8 & \frac{27}{2} & \frac{32}{3} & \frac{125}{24} & \frac{9}{5} & \frac{343}{720} & \frac{32}{315} & \frac{81}{4480} \\
0 & 1 & 16 & \frac{81}{2} & \frac{128}{3} & \frac{625}{24} & \frac{54}{5} & \frac{2401}{720} & \frac{256}{315} & \frac{729}{4480} \\
0 & 1 & 32 & \frac{243}{2} & \frac{512}{3} & \frac{3125}{24} & \frac{324}{5} & \frac{16807}{720} & \frac{2048}{315} & \frac{6561}{4480} \\
0 & 1 & 64 & \frac{729}{2} & \frac{2048}{3} & \frac{15625}{24} & \frac{1944}{5} & \frac{117649}{720} & \frac{16384}{315} & \frac{59049}{4480} \\
0 & 1 & 128 & \frac{2187}{2} & \frac{8192}{3} & \frac{78125}{24} & \frac{11664}{5} & \frac{823543}{720} & \frac{131072}{315} & \frac{531441}{4480} \\
0 & 1 & 256 & \frac{6561}{2} & \frac{32768}{3} & \frac{390625}{24} & \frac{69984}{5} & \frac{5764801}{720} & \frac{1048576}{315} & \frac{4782969}{4480}
\end{pmatrix}$$

```
{1599.13029757366175826855201733, 130.769366678294388116500168763,
 19.9417673251839302227728579947, 4.50709290760966670357344466012,
 1.34200506706440132573359478068, 1.00000000000000000000000000000,
 0.374791644532846737939466019678, 0.0664011666207600560252502940953,
 0.00634501795095272369881371679787, 0.000251071462248226156767394509161}
```

5.94×10^{5}

N = 10:

```
M = CARLEMAN[10] /. {g[0] → 1, g[1] → 1, g[2] → 1, g[3] → 1,
    g[4] → 1, g[5] → 1, g[6] → 1, g[7] → 1, g[8] → 1, g[9] → 1, g[10] → 1};
MatrixForm[%]
L = N[Eigenvalues[M], 40] // Simplify

N[Tetr[e, e], 3] // Simplify
```

$$\begin{pmatrix}
1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} & \frac{1}{120} & \frac{1}{720} & \frac{1}{5040} & \frac{1}{40320} & \frac{1}{362880} & \frac{1}{3628800} \\
0 & 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} & \frac{1}{120} & \frac{1}{720} & \frac{1}{5040} & \frac{1}{40320} & \frac{1}{362880} \\
0 & 1 & 2 & \frac{3}{2} & \frac{2}{3} & \frac{5}{24} & \frac{1}{20} & \frac{7}{720} & \frac{1}{630} & \frac{1}{4480} & \frac{1}{36288} \\
0 & 1 & 4 & \frac{9}{2} & \frac{8}{3} & \frac{25}{24} & \frac{3}{10} & \frac{49}{720} & \frac{4}{315} & \frac{9}{4480} & \frac{5}{18144} \\
0 & 1 & 8 & \frac{27}{2} & \frac{32}{3} & \frac{125}{24} & \frac{9}{5} & \frac{343}{720} & \frac{32}{315} & \frac{81}{4480} & \frac{25}{9072} \\
0 & 1 & 16 & \frac{81}{2} & \frac{128}{3} & \frac{625}{24} & \frac{54}{5} & \frac{2401}{720} & \frac{256}{315} & \frac{729}{4480} & \frac{125}{4536} \\
0 & 1 & 32 & \frac{243}{2} & \frac{512}{3} & \frac{3125}{24} & \frac{324}{5} & \frac{16807}{720} & \frac{2048}{315} & \frac{6561}{4480} & \frac{625}{2268} \\
0 & 1 & 64 & \frac{729}{2} & \frac{2048}{3} & \frac{15625}{24} & \frac{1944}{5} & \frac{117649}{720} & \frac{16384}{315} & \frac{59049}{4480} & \frac{3125}{1134} \\
0 & 1 & 128 & \frac{2187}{2} & \frac{8192}{3} & \frac{78125}{24} & \frac{11664}{5} & \frac{823543}{720} & \frac{131072}{315} & \frac{531441}{4480} & \frac{15625}{567} \\
0 & 1 & 256 & \frac{6561}{2} & \frac{32768}{3} & \frac{390625}{24} & \frac{69984}{5} & \frac{5764801}{720} & \frac{1048576}{315} & \frac{4782969}{4480} & \frac{156250}{567} \\
0 & 1 & 512 & \frac{19683}{2} & \frac{131072}{3} & \frac{1953125}{24} & \frac{419904}{5} & \frac{40353607}{720} & \frac{8388608}{315} & \frac{43046721}{4480} & \frac{1562500}{567}
\end{pmatrix}$$

```
{4146.591231327360966561979176269791776015,
 309.9205215451875924786347241519135986797, 43.11141724571481530173219031483280598011,
 8.818444911696943786914081643652376950933, 2.435993579436611481789569397243926712159,
 1.000000000000000000000000000000000000000, 0.7720226317249734178554310158722702229112,
 0.1895020548648592927154109727841261548587, 0.02863711929132182501082477596315682467884,
 0.002386396451880993201700066166229189611044,
 0.00008403924005249685119474274975090671363460}
```

1.85×10^{6}

- **Still increasing ... A lot of machine-time seems necessary !**