Fig. 1 .- Plot of an approximation of tetration, $y = {}^{x}\left(\sqrt{2}\right)$, obtained by using Mathematica and adopting a linear extension in the range 11 < x < 12. The plot shows an expected horizontal asymptote y = 2 corresponding to an attractive fixpoint of the exponential $\left(\sqrt{2}\right)^2 = 2$. Vertical asymptote for x = -2. The result of the repulsive fixpoint of the exponential $\left(\sqrt{2}\right)^4 = 4$ is shown as a horizontal straight-line at y = 4.

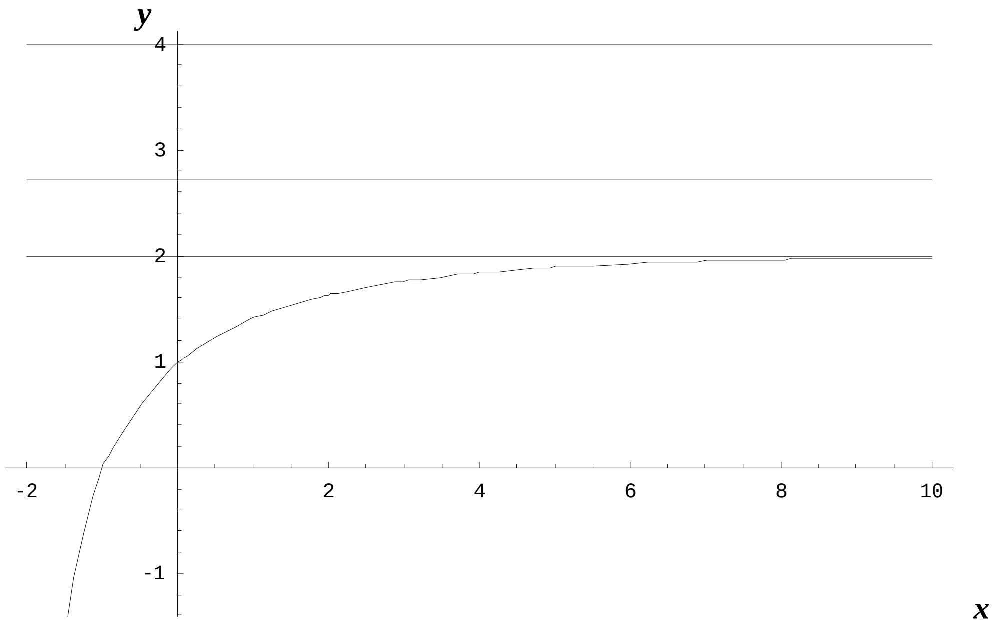
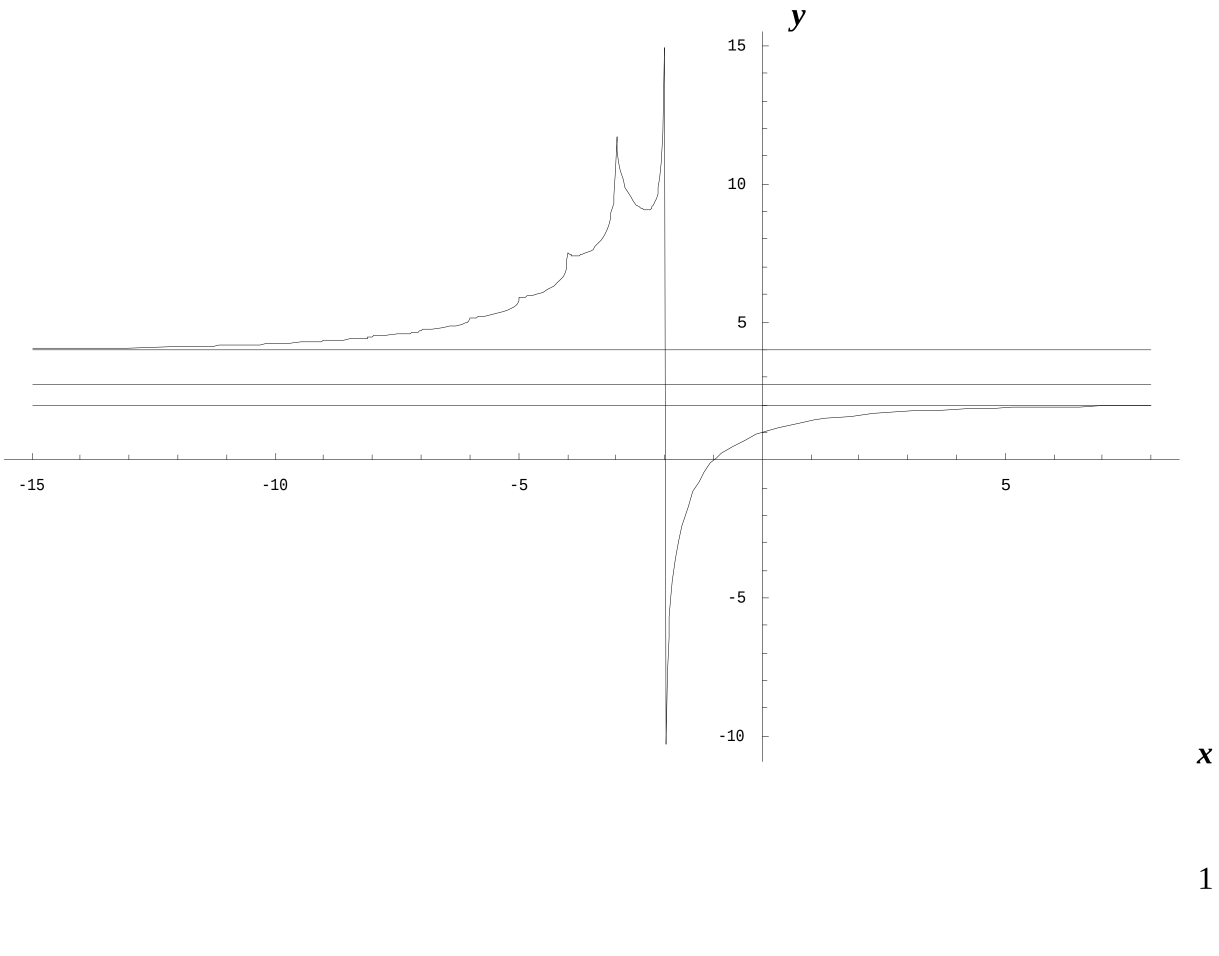
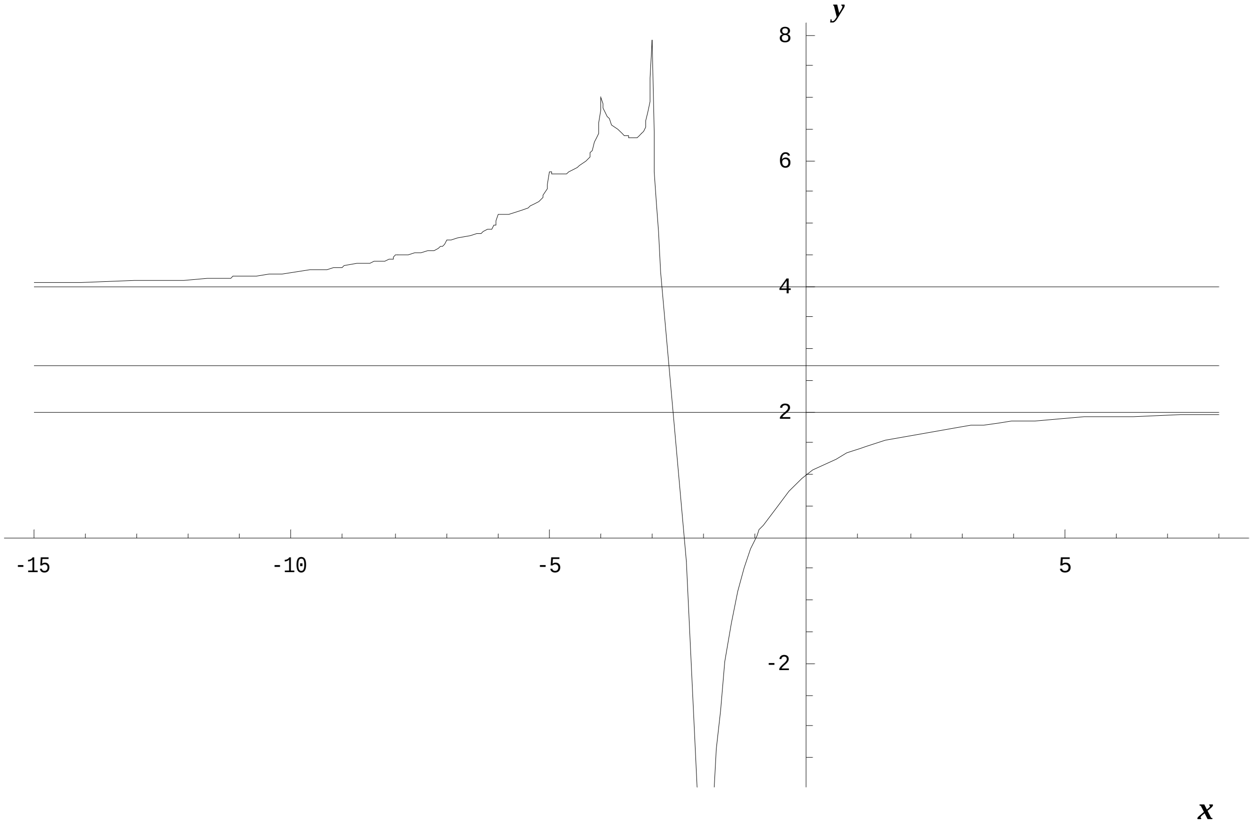


Fig. 2. – Extension of Fig. 1 plot to x < - 2 showing the <u>absolute values</u> of the complex magnitude obtained by iteratively applying ${}^{x-1}\left(\sqrt{2}\right) = \log_{\sqrt{2}}\left[{}^{x}\left(\sqrt{2}\right)\right]$, for x < - 2, and its real values, for x > -2. Zero at x = -1, first pole at x = -2. The plot should be completed by the "phase" of the complex variable,, at x < -2.



Unfortunately, the discontinuities of the graph cannot be systematically eliminated, because they pointed out from the iterative application of the "log (base $\sqrt{2}$)" operator to infinite (complex) quantities. Nevertheless, probably, the infinite repetition (the iteration) of these applications would finally give, for $x \to -\infty$, a rather smooth line, as the plot of Fig. 2 seems to suggest. These discontinuities, according to this conjecture, would be "squeezed down" for $x \to -\infty$. This fact could give back again some dignity to the upper value (h = 4) of $h = \text{plog}(-\ln x)/(-\ln x)$, calculated for $\text{plog}(z) = W_{-1}(z)$. The "unreachable" infinite tower would be indeed "reachable" only for $x \to -\infty$.

Fig. 3 – Plot of $\text{Re}\left[{}^{x}\left(\sqrt{2}\right) \right]$, for – 15 < x < 5



The plot shows the real part of the tetrational function (base sqrt 2), together with its two asymptotical values, one of which is a simple conjecture

$$\lim_{x \to +\infty} {}^{x}\left(\sqrt{2}\right) = 2$$

$$\lim_{x \to -\infty} {}^{x}\left(\sqrt{2}\right) = 4 \qquad \text{(conjecture)}$$

The first (lower) asymptotic value (y = 2) is indeed reachable for $x \to +\infty$ and correspond to an attractive fixpoint of the exponential. The second conjectural asymptotic value (y = 4) appears for $x \to -\infty$. It corresponds to a repulsive fixpoint of the exponential.

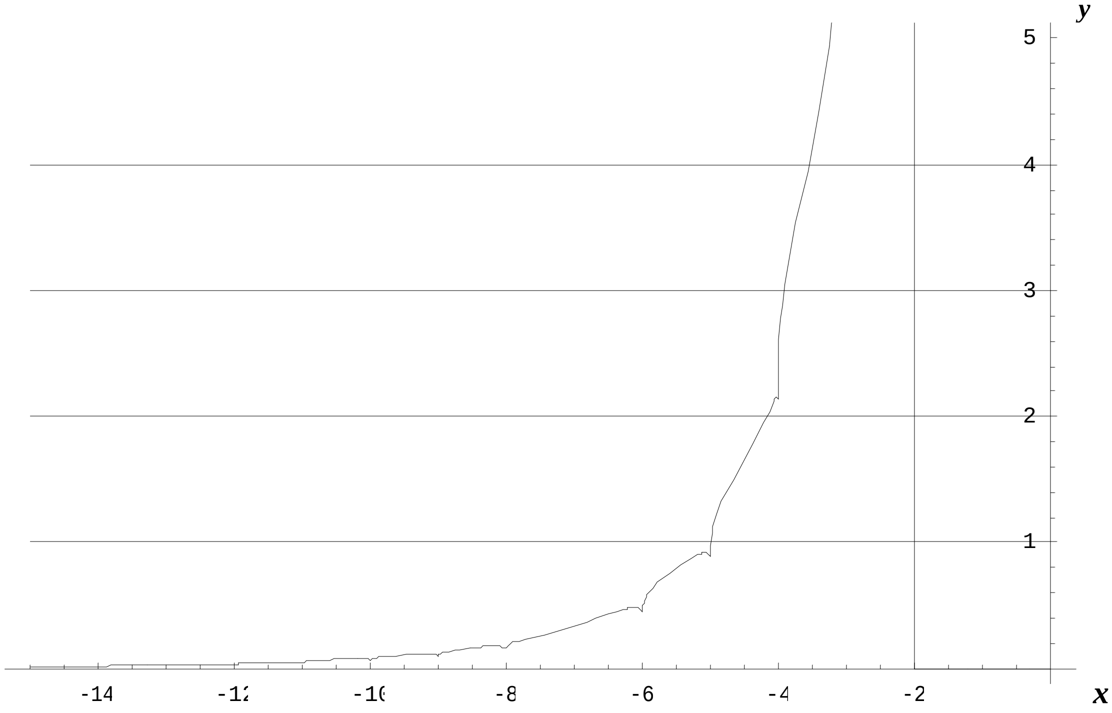
$$y = \left(\sqrt{2}\right)^{x} = x \text{, for } x = 2\text{, attractive fixpoint , } \frac{dy}{dx} < 1$$

$$y = \left(\sqrt{2}\right)^{x} = x \text{, for } x = 4\text{, repulsive fixpoint , } \frac{dy}{dx} > 1$$

The attractive and repulsive fixpoint (of the exponential) will determine, respectively, attractive or repulsive axis for the corresponding tetrational function. The repulsive axis would turn to an attractive axis, for $x \to -\infty$.

<u>Fig 4</u> - Plot of $\operatorname{Im}\left[{}^{x}\left(\sqrt{2}\right) \right]$, for – 15 < x < 0. We can see that the imaginary part "goes to 0" for $x \to -\infty$

This seems to confirm that the value of $\lim_{x\to-\infty} \left[{}^{x}\left(\sqrt{2}\right) \right] = 4$ is a <u>real number</u>.



<u>Provisional conclusions</u> – It is proposed that, probably, in the bases range $1 \le b \le \eta$, $y = {}^{x}b$ is a (one value) function, real for *b* > -2 and complex for *b* < -2, with two real asymptotic values:

$$h_{\text{inf}} = \lim_{x\to+\infty} {}^{x}b \text{ , and}$$

$$h_{\text{sup}} = \lim_{x\to-\infty} {}^{x}b$$

GFR – 2008-05-26