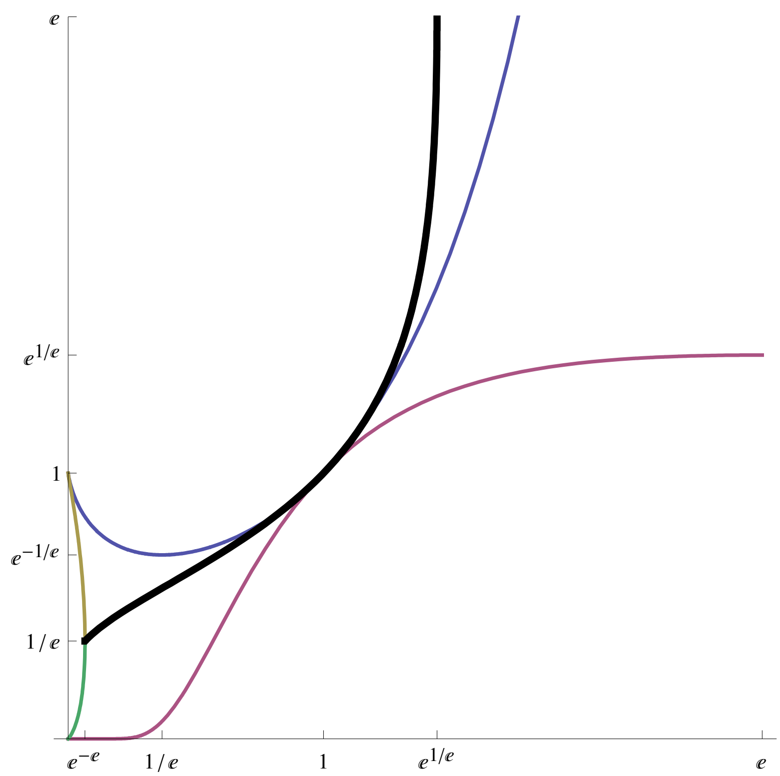
Tetration FAQ

Henryk Trappman
Andrew Robbins

July 10, 2008



Contents

1 Introduction

This FAQ is designed to provide a basic understanding about tetration and its related topics. This FAQ should also prevent researchers from reinventing the wheel and instead allow them to shine their light on undeveloped topics. Revealing dead-end ideas prevents authors from wasting their time and effort. It is hoped that this FAQ makes current tetration research more effective and fruitful. For questions not covered here, there is a **Tetration Reference** (Ref.), available from http://math.eretrandre.org/tetrationforum/.

1.1 Who should read this

This FAQ should be understandable for undergraduates (or advanced high school students) who have completed a few calculus-level math classes. Although higher math is not required, it helps. Basic knowledge of calculus, power series, and linear algebra is assumed.

1.2 Background

The idea of tetration enjoys great popularity among both students and lay mathematicians. Many people have rediscovered tetration in different ways, mostly because there are many problems that tetration solves. When large numbers are too big, tetration is a solution. When the hyper operation sequence just stops, tetration is a solution. However, when tetration is not the solution, but the problem, then the burning central question is:

What is the most "beautiful" extension of tetration beyond integers?

We can see immediately that this question is slightly outside the realm of mathematics and more in the realm of taste. Perhaps this is the reason why professional mathematicians have barely considered this question. A more mathematical task would be to find existence and uniqueness conditions for an extension of tetration, then find it.

What has happened is just the opposite. Dozens of authors have constructed extensions, but their uniqueness conditions remain unknown. So the initial problem of existence and uniqueness has been solved (at least 10 extensions exist), but the problem has now evolved into a search for a *new* problem which *does* admit a unique solution. Needless to say, the new problem must be "beautiful" as well.

This FAQ aims to briefly present some of these extensions of tetration so as to overview the methods that are available. This is partly to provide a sense of a fair competition, and partly to appeal to the taste of the reader, although we do not describe all methods here. For a more complete description of all extensions, see Ref-4.4.

1.3 Acknowledgments

We appreciate any suggestions and contributions for improving and extending it If you contribute a chapter your name will be listed here.

1.4 Contact

The main point of communication is the Forum, but if you wish to send us email:

Name	Email (with obvious change)
Andrew Robbins	`and_j_rob(AT)yahoo.com`
Henryk Trappman	`bo198214(AT)eretrandre.org`

2 Iteration

Question 1. *What is iteration?*

Iteration is the process of repeating a function. Given $f(x)$ this is as simple as $f(f(x))$. The problems arise when one begins to ask questions regarding the iteration, like *does it converge?* or *can it be interpolated?*. The first question falls in the subject of dynamical systems, and the second question falls in the subject of interpolation.

Interpolation is the process of taking a list of points $(x, f(x), f(f(x)), f(f(f(x))), \ldots)$ and connecting them with a new function that that passes through each point. For example, the new function will evaluate to $F(1, x) = f(x)$, $F(2, x) = f(f(x))$ and so on.

Question 2. *What is iteration classified under?*

There are two formal subject categories in the American Mathematical Society's (AMS) Mathematics Subject Classification system (at `http://www.ams.org/msc/`) which describe iteration theory, which are **dynamical systems** (37-xx) and **functional equations** (39-xx).

Question 3. *How is iteration written?*

Iteration is written $f^n(x)$, although here we use the notation $f^{\langle n\rangle}(x) = f(f^{\langle n-1\rangle}(x))$ where $f^{\langle 1\rangle}(x) = f(x)$ for clarity. In ASCII, iteration can be written `f^n(x)` or `f<n>(x)`.

Question 4. *What terms are used with iteration?*

In general the expression $f^{\langle n\rangle}(x)$ is just referred to as **iteration**, but if either n or x is constant, then we use slightly different names. When n is constant, we call the function $x \mapsto f^{\langle n\rangle}(x)$ the n-th **iterate** of $f(x)$. When x is constant, we call the function $n \mapsto f^{\langle n\rangle}(x)$ the **iterational** from x of $f(x)$.

2.1 Iteration of functions

Question 5. *It is possible to iterate a function non-integer times?*

Short answer: it depends.

Given a differentiable function with a formal power series, and its power series can be used as a means to study its iterates and interpolate between them, and in some cases, this interpolation also produces an differentiable function as well.

Given a function which is not continuous or differentiable, it is possible to interpolate between iterates of the function, but there are many more possible ways of doing this.

Question 6. *What is the iteration of* $\boldsymbol{e}^x - 1$*?*

The function $\boldsymbol{e}^x - 1$ is one of the simpler applications of continuous iteration. The reason why is because regular iteration requires a fixed point in order to work, and this function has a very simple fixed point, namely zero: $\boldsymbol{e}^0 - 1 = 0$.

2.2 Extension of functions

This part is a discussion about extending the domain of functions beyond their original domain. This usually means extending a function from the integers to the real numbers, although the term *extension* applies to any sets. We will continue the questions later.

2.2.1 Extension of the gamma function

Let us illustrate the problem of extending tetration with the well-known extension of the Gamma function and with the well-known extension of exponentiation.

The factorial function $n \mapsto n!$ which is defined on the natural numbers inductively by

$$1! = 1 \qquad\qquad (n+1)! = n!(n+1)$$

shall be extended to have also values for fractional/real arguments between two consecutive natural numbers, of course not any values but certain nice, smooth, fitting values (whatever that means). We seek for conditions that that narrow the range of possible solutions. The first natural condition is to satisfy the recurrence relation also for real values, i.e. an extension $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ of the factorial function shall satisfy

$$f(1) = 1 \qquad\qquad f(x+1) = (x+1)f(x) \tag{1}$$

for all positive real numbers x. This particularly forces $f(n) = n!$ for natural numbers n, i.e. it implies f being an extension of $n \mapsto n!$. If $f(x)$ is defined on the open interval $(0,1)$ then this condition determines the values of $f(x)$ for $x > 1$ by induction as you can easily verify. However we can arbitrarily define $f(x)$ on $(0,1)$ and so get infinitely many solutions that satisfy (1). The next obvious demand is continuity, differentiability or even analyticity of f. However it is well-known that another criterion suffices:

Proposition 1. *The condition* (1) *together with logarithmic convexity, i.e.*

$$f(\lambda x + (1-\lambda)y) \le f(x)^\lambda f(y)^{1-\lambda} \quad \textit{for all} \quad x, y > 0, 0 < \lambda < 1,$$

uniquely determine f *to be* $f(x) = \Gamma(x+1)$*.*

2.2.2 Extension of exponentiation

Till now let us having defined the exponentiation for natural exponents by

$$b^1 = b \qquad\qquad b^{n+1} = bb^n$$

for real b and natural n. In comparison with the Gamma function we have the additional parameter b. We can anyway try to repeat the considerations for the Gamma function by fixing b in the beginning, i.e. to find a function f_b that satisfies

$$f_b(1) = b \qquad\qquad f_b(n+1) = bf_b(n) \tag{2}$$

Logarithmic convexity here is somewhat self referential as we need the power for real exponents to be defined in the expression

$$f_b(\lambda x + (1-\lambda)y) \le f_b(x)^\lambda f_b(y)^{1-\lambda}$$

So it seems as if this way is not working here (if however someone would like to explore this in a bit more depth I would be happy to hear about the results.)

Now we can easily generalise the original recurrence by induction to $b^{m+n} = b^m b^n$ and by repeated induction to our key relation

$$b^1 = b \qquad\qquad (b^n)^m = b^{nm}. \tag{3}$$

This condition if demanded for real m and n already suffice to uniquely extend exponentiation to fractional exponents. For example take $b^{\frac{1}{2}}$. If it has a value at all then must

$$(b^{\frac{1}{2}})^2 = b^{\frac{1}{2}2} = b^1 = b$$

which means that $b^{\frac{1}{2}}$ is the solution of the equation $x^2 = b$. To make the solution unique we restrict the base domain of our extended exponentiation operation to the positive real numbers. And then we similarly get that $b^{\frac{1}{n}}$ is the solution of $x^n = b$, i.e. $b^{\frac{1}{n}} = p_n^{-1}(x)$ where $p_n(x) := x^n$ is bijective on $\mathbb{R}_+$ and obviously $p_n^{-1}(x) = \sqrt[n]{x}$. Also by equation (3) we get then the general formula for fractional exponents

$$b^{\frac{m}{n}} = p_m(p_n^{-1}(b)).$$

Extension to irrational arguments merely requires the function to be continuous.

Proposition 2. *There is exactly one extension of the exponentiation with natural numbered exponents to positive real exponents such that the function $x \mapsto b^x$ is continuous for each b and which satisfies*

$$(b^x)^n = b^{xn} \tag{4}$$

for each $b > 0$, $x \in \mathbb{R}_+$ and $n \in \mathbb{N}$.

It can be mentioned that we could replace (4) by the stronger demand

$$b^{x+y} = b^x b^y. \tag{5}$$

Because from this follows by induction $b^{xn} = (b^x)^n$. In that case we even could extend our operation to negative exponents. First we notice $b^0 b^x = b^{0+x} = b^x$ and hence $b^0 = 1$ for $b^x \neq 0$. Then $b^x b^{-x} = b^{x-x} = b^0 = 1$ implies $b^{-x} = 1/b^x$.

On the other hand if we need exponentiation for tetration then both arguments of exponentiation must be from the same domain, and the greatest common domain of base and exponent is $\mathbb{R}_+$.

2.2.3 Extension of tetration

By the special bracketing of the tetration the equivalent of neither equation (4) nor (5) hold for all m, n, i.e. for most $m, n \in \mathbb{N}$ we have:

$$ {}^{n+m}x \neq \left({}^{n}x\right)^{{}^{m}x} \tag{6} $$

$$ {}^{nm}x \neq {}^{n}\left({}^{m}x\right) \tag{7} $$

Even

Proposition 3. *There is* no *operation* $*$ *on* $\mathbb{R}_+$ *such that*

$$ {}^{n+m}x = {}^{m}x * {}^{n}x $$

Proof. Suppose there is such an operation, then we gain the contradiction

$$ 65536 = 2^{2^{2^{2}}} = {}^{4}2 = {}^{2+2}2 = {}^{2}2 * {}^{2}2 = 4 * 4 = {}^{2}4 = 4^4 = 256 $$

□

This breaks applying the procedure of extending the exponentiation to extending the tetration. This shall be mentioned with all insistence. Though we have seen that $t_n(x) = {}^{n}x$ is bijective on $\mathbb{R}_{>1}$ for each n, a definition of ${}^{\frac{1}{n}}b$ as $t_n^{-1}(b)$ is as arbitrary as defining it to be $\frac{\pi}{3}$.

If we want to have a unique solution we need conditions that make the solution unique. Till now there aren't known such conditions. However there are several conditions that one certainly want to have satisfied for an (to some real numbers) extended tetration.

1. The recurrence relations (11) and (12) shall be satisfied for arbitrary exponents:

$$ {}^{1}b = x \qquad\qquad {}^{x+1}b = b^{{}^{x}b}. \tag{8} $$

This particularly implies that it is an extension of the tetration for natural exponents.

2. The function $x \mapsto {}^{a}x$ shall be continuous and strictly increasing for each a (so that we can define the inverse: a tetration root). Note that this condition suggests to restrict the base to $\mathbb{R}_{>1}$ (similarly the base of exponentiation is restricted to $\mathbb{R}_+$).

3. The functions $x \mapsto {}^{x}b$ shall be continuous and strictly increasing for each b (so that we can define the inverse: a tetration logarithm).

Definition 1. A tetration extension $(x, y) \mapsto {}^{y}x \colon I \times J \to \mathbb{R}$, where $I, J \subseteq \mathbb{R}$ is called a *real tetration* if it satisfies conditions 1, 2 and 3.

Additional demands could be infinite differentiability or even analyticity.

2.3 Extensions of iteration

TODO

2.3.1 Regular iteration

To understand *regular iteration* one needs to understand power series, so it is best to start with the various type of power series.

2.3.2 Natural iteration

TODO

2.3.3 Abel functional equation

Closely related to continuous iteration is the translation equation

$$F(F(a,x),y) = F(a,x+y)$$

(if we write $F(a,x) = f^{\circ x}(a)$).

Proposition 4. *If $f(x) := F(b,x)$ is continuous and strictly increasing for one b and if F satisfies the translation equation (for all arguments of its domain of definition) then*

$$F(a,x) = f_c(f_c^{-1}(a) + x) \tag{9}$$

for any c, where $f_c(x) := f(x+c)$. Vice versa $G(a,x) := g(g^{-1}(a)+x)$ satisfies the translation equation for each strictly increasing continuous g, then $x \mapsto G(a,x)$ is strictly increasing and continuous for every a.

If we now want to real iterate a function g, i.e. $g(x) = F(x,1)$ and F satisfies the translation equation, $x \mapsto F(a,x)$ is strictly increasing and continuous. Then we merely need to find a strictly increasing continuous function f (which will be equal to $F(a,x)$ for some a) such that $g(x) = F(x,1) = f(f^{-1}(x)+1)$. Or if we put $h := f^{-1}$ such that

$$h(g(x)) = h(x) + 1$$

which is the so called Abel equation.

3 Tetration

Question 7. *What is tetration?*

Tetration is an *iterated exponential*, a function (on $\mathbb{R} \times \mathbb{N} \to \mathbb{R}$) defined as

$${}^n x := \underbrace{x^{x^{\cdot^{\cdot^{x^x}}}}}_{n} \tag{10}$$

where n is called the **height** and x is called the **base**.

Multiplication is repeated addition, exponentiation is repeated multiplication, in similarity to this process one tries to define the next higher operation as repetition of the exponentiation. For better readability we introduce the symbol $\uparrow$ for exponentiation, i.e. $x \uparrow y := x^y$. There is however a difficulty with repeating exponentiation because the operation is not associative (as addition and multiplication are) so we have to chose a certain bracketing scheme. Right bracketing seems to make the most sense (see chapter 4.2 for other bracketing schemes) and so tetration $(x, n) \mapsto {}^n x \colon \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ is defined as

$${}^n x := \underbrace{x \uparrow (x \uparrow (\cdots \uparrow x) \cdots))}_{n \times x}$$

or more formally inductively by

$$
\begin{aligned}
{}^1 x &:= x && (11) \\
{}^{n+1} x &:= x^{{}^n x}. && (12)
\end{aligned}
$$

n is called the (tetration) exponent and x is called the (tetration) base.

The functions $x \mapsto {}^n x$ show an interesting behaviour in the interval $(0, 1)$. It strikingly resembles the behaviour of the polynomials $x \mapsto x^n$ in $(-\infty, 0)$. Let us have a look at the graphs. This gives rise to define the inverse operation.

Question 8. *How do you make the infinite tetration fractal?*

Question 9. *What is repeated exponentiation?*

Repeated addition is multiplication, repeated multiplication is exponentiation, so is the next one repeated exponentiation? No. The difficulty is that exponentiation is not associative, so we have to chose a bracketing scheme. Right bracketing gives *the* hyper-operations, left bracketing gives lower hyper-operations, balanced bracketing gives balanced hyper-operations, and mixed bracketing gives mixed hyper-operations.

Question 10. *What is iterated exponentiation?*

Exponentiation ($\uparrow$) is a binary function, so there are two ways of iterating it. ($\uparrow c$) is a function from x to x^c, which is called a **power** function. An *iterated power* function gives $(\uparrow c)^{\circ n}(x) = ((x^c)^{\cdots})^c = x^{c^n}$. For the other way, ($b\uparrow$) is a function from x to b^x, which is called an **exponential** function. An *iterated exponential* function gives $(b\uparrow)^{\circ n}(x)$ which is not expressible in any standard closed form.

Question 11. *What is a to the bth power c times?*

It depends on interpretation, since this is unclear. Assuming b is repeated c times:

- If it means: $((a{\uparrow}b){\uparrow}\cdots{\uparrow}b){\uparrow}b = a^{b^c}$ then it is *iterated powers*
- If it means: $((b{\uparrow}b){\uparrow}\cdots{\uparrow}b){\uparrow}a = (b^{b^{c-1}})^a$ then it is a power of *iterated powers*

- If it means: $a\uparrow(b\uparrow\cdots\uparrow(b\uparrow b)) = a^{({}^c b)}$ then it is an exponential of *tetration*
- If it means: $b\uparrow(b\uparrow\cdots\uparrow(b\uparrow a)) = \exp_b^c(a)$ then it is *iterated exponentials*

and as always, it is better to be more clear, since there are so many interpretations.

Question 12. *What is infinite tetration?*

Tetration usually arises in the study of fixed points of exponential functions. Fixed points satisfy $f(x_0) = x_0$, so fixed points of exponential functions would satisfy $\exp_b(c) = b^c = c$. Replacing c in the equation $c = b^c$ gives $c = b^{b^c}$ and repeating this gives $c = \exp_b^n(c)$. To eliminate the c we continue indefinitely to get $c = \exp_b^\infty(x)$ and if c is an attracting fixed point, then it doesn't matter what x is, so this can be written $c = {}^\infty b$. Taking the cth root of the original equation gives $c^{1/c} = b$ meaning **infinite tetration** ${}^\infty x$ (also known as the *infinitely iterated exponential*) is the inverse function of $x^{1/x}$. It is defined as:

$$ {}^\infty x := \lim_{n\to\infty} {}^n x = \frac{W(-\ln(x))}{-\ln(x)} \tag{13} $$

for all $e^{-e} < x < e^{1/e}$ where W is Lambert's W function. For more information, see the Knoebel H-function in the chapter on special functions. [**?**] and [**?**], also [**?**], [**?**], [**?**].

Question 13. *How is tetration written? Which notation shall I use?*

We propose to write exponentiation and tetration with base x and exponent n as follows.

Context	Exponentiation	Tetration
general	x^n	${}^n x$
symbol	$x \uparrow n$	$x \uparrow\uparrow n$
ASCII	`x^n`	`x^^n`

The notation ${}^n x$ is probably the most compact way of writing tetration, however we have to be careful about ambiguity as for example in $x^n x$. With this notation there is also some uncertainty about whether tetration is $\mathbb{R}_+ \times \mathbb{N} \to \mathbb{R}_+$ or $\mathbb{N} \times \mathbb{R}_+ \to \mathbb{R}_+$. The convention coherent with all the other notations is however that tetration is $\mathbb{R}_+ \times \mathbb{N} \to \mathbb{R}_+$.

In some situations it can be quite useful to have an operation symbol instead merely a way of writing, for example in the specification $\uparrow\colon \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ (which however can be synonymously replaced by $(x, n) \mapsto x^n\colon \mathbb{R} \times \mathbb{N} \to \mathbb{R}$) or if we map operations on operations indicated by a decoration, for example $* \mapsto *'$ then we can write $x \uparrow' y$, or if there are difficulties in writing or reading repeated nested levels of exponents.

Alternative names for tetration are hyperpower (often used in professional mathematical context) or superpower. Also the name hyper4 operator is in use. For me hyperpower and superpower sound somewhat unspecific as there is a whole hierarchy of operations which are "hyper" respective the power.

Question 14. *What terms used with tetration?*

Question 15. *What is the Ackermann function?*

Now we can repeat the process of defining the next higher operation. Given an operation $*\colon X \times X \to X$ we define the right right-bracketing iterator operation $*'\colon X \times \mathbb{N} \to X$ as

$$x *' n := \underbrace{x * (x * (\cdots * x) \cdots)}_{n \times x}$$

Then we have $xn = x +' n$, $x^n = x +'' n$ and ${}^n x = x +''' n$. Verify however that the left left-bracketing iterator operation $*^{`}\colon \mathbb{N} \times X \to X$

$$n *^{`} x := \underbrace{(\cdots (x * x) * \cdots x) * x}_{n \times x}$$

yields basically the same operations $n +^{`} x = nx$, $n +^{``} x = x^n$ and $n +^{```} x = {}^n x$. In generalisation of this method define a sequence of operations $\diamond_n\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ inductively by

$$\begin{aligned} a \diamond_1 b &= a + b \\ a \diamond_{n+1} b &= a \diamond_n' b \end{aligned}$$

Particularly $m \diamond_2 n = mn$, $m \diamond_3 n = m^n$, $m \diamond_4 n = {}^n m$. A similar construction was used by Ackermann 1927 in [**?**]. He recursively defined a function $\varphi\colon \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ which relates to our construction by $\varphi(a, b, n-1) = a \diamond_n b$ and used it to show that there are recursive but not primitive recursive functions. A function similar to φ is called original Ackermann function, "original" because later Rózsa Péter introduced a simpler function which served the same purpose and which is also usually called Ackermann function.

We can assume $\diamond_1$, $\diamond_2$ and $\diamond_3$ to be extended to $\mathbb{R}_+$. However because $\diamond_4$ (which is tetration) is merely defined for natural numbered exponents (we generally call x the basis and y the exponent in the expression $x \diamond_n y$ for $n \geq 3$) $\diamond_5$ which is the repetition of $\diamond_4$ can merely be defined on $\mathbb{N} \times \mathbb{N}$.

So after extending tetration to real exponents one would aim to extend all the following operations successively to real exponents.

3.1 Tetration as an iterated exponential

TODO

$$f^{\circ 1}(a) = f(a) \qquad\qquad f^{\circ x+y}(a) = f^{\circ x}(f^{\circ y}(a)) \tag{14}$$

TODO

If we have a real iteration of the function then we can construct a real tetration from it, illustrated by the example:

$$\begin{aligned} \exp_b^{\circ 2}(x) &= b^{b^x} \\ \exp_b^{\circ 2}(b) &= b^{b^b} = {}^3 b \end{aligned}$$

Proposition 5. *For each real iteration* $\exp_b^{\circ x}$ *of the function* $\exp_b$ *the operation defined by*

$$ {}^{a}b = \exp_b^{\circ a-1}(b) \tag{15} $$

is a real tetration.

Proof.

$$ \begin{aligned} {}^{1}b &= \exp_b^{\circ 0}(b) = \mathrm{id}(b) = b \\ {}^{a+1}b &= \exp_b^{\circ a}(b) = b^{\exp_b^{\circ a-1}(b)} = b^{{}^{a}b} \end{aligned} $$

□

Vice versa if we have defined a real tetration we can derive a real iteration of $\exp_b$. Though we have to reach out some more before. Whenever we have a real tetration then the function $x \mapsto {}^{x}b$ maps $(0,\infty)$ to $(1,\infty)$ and is injective. Hence we can define the tetration logarithm by

$$ {}^{\mathrm{tlog}_b(x)}b := x \tag{16} $$

And this enables us then to define real iterations of $\exp_b$ by the reasoning

$$ \begin{aligned} \exp_b^{\circ x}(a) &= \exp_b^{\circ x}({}^{\mathrm{tlog}_b(a)}b) = \exp_b^{\circ x}(\exp_b^{\circ\, \mathrm{tlog}_b(a)}(1)) = \exp_b^{\circ x+\mathrm{tlog}_b(a)}(1) \\ &= {}^{x+\mathrm{tlog}_b(a)}b \end{aligned} $$

Proposition 6. *For each real tetration* $(x,y) \mapsto {}^{y}x$ *the operation defined by*

$$ \exp_b^{\circ x}(a) = {}^{x+\mathrm{tlog}_b(a)}b $$

is a real iteration of $\exp_b$.

Proof.

$$ \begin{aligned} \exp_b^{\circ 1}(a) &= {}^{1+\mathrm{tlog}_b(a)}b = b^{{}^{\mathrm{tlog}_b(a)}b} = b^{a} = \exp_b(a) \\ \exp_b^{\circ x+y}(a) &= {}^{x+y+\mathrm{tlog}_b(a)}b = {}^{x+\mathrm{tlog}_b({}^{y+\mathrm{tlog}_b(z)}a)}b = \exp_b^{\circ x}(\exp_b^{\circ y}(a)) \end{aligned} $$

□

These both translations (iteration to tetration and tetration to iteration) are at least formally inverse. To see what this means let us generalise and simplify the notation.

Given a function h with $h(1) = b$ (in our case was $h = \exp_b$). Let $\mathfrak{A}$ be the set of all functions $f(x)$ that are strictly increasing, continuous and that satisfy

$$ f(1) = b \qquad\qquad f(x+1) = f(h(x)) $$

for all x. Let $\mathfrak{B}$ be the set of all operations $F(a,x)$ such that $x \mapsto F(b,x)$ is strictly increasing and continuous, and such that

$$F(a,1) = h(a) \qquad\qquad F(a,x+y) = F(F(a,x),y)$$

for all x, y, a.

Then we define in the sense of our previous considerations the mappings $f \mapsto f^{\uparrow}\colon \mathfrak{A} \to \mathfrak{B}$ and $F \mapsto F^{\downarrow}\colon \mathfrak{B} \to \mathfrak{A}$ by

$$f^{\uparrow}(a,x) = f(f^{-1}(a)+x) \tag{17}$$

$$F^{\downarrow}(x) = F(b,x-1). \tag{18}$$

We can see that they are formally inverse to each other in the sense that $(f^{\uparrow})^{\downarrow} = f$ and $(F^{\downarrow})^{\uparrow} = F$ for each $f \in \mathfrak{A}$ and $F \in \mathfrak{B}$.

$$\begin{aligned}
(F^{\downarrow})^{\uparrow}(a,x) &= F^{\downarrow}(F^{\downarrow^{-1}}(a)+x) = F(b, F^{\downarrow^{-1}}(a)+x-1) \\
&= F(F(b,F^{\downarrow^{-1}}(a)-1),x) = F(F^{\downarrow}(F^{\downarrow^{-1}}(a)),x) \\
&= F(a,x) \\
(f^{\uparrow})^{\downarrow}(x) &= f^{\uparrow}(b,x-1) = f(f^{-1}(b)+x-1) = f(1+x-1) = f(x)
\end{aligned}$$

But beware! We did not consider the domains of definitions yet. As we can already see for natural iteration exponents, the domain $D \subseteq \mathbb{R}\times\mathbb{R}$ of an F may not be rectangular, but be dependent on the second parameter:

$$\begin{aligned}
\exp_b^{\circ 0} = \mathrm{id}&\colon (-\infty,\infty) \leftrightarrow (-\infty,\infty) \\
\exp_b^{\circ 1} = \exp_b&\colon (-\infty,\infty) \leftrightarrow (0,\infty) \\
\exp_b^{\circ n}&\colon (-\infty,\infty) \leftrightarrow (\exp_b^{\circ n-1}(0),\infty) \\
\exp_b^{\circ -1} = \log_b&\colon (0,\infty) \leftrightarrow (-\infty,\infty) \\
\exp_b^{\circ -n} = \log_b^{\circ n}&\colon (\exp_b^{\circ n-1}(0),\infty) \leftrightarrow (-\infty,\infty)
\end{aligned}$$

So for a given boundary function $d\colon \mathbb{R} \to \mathbb{R}$ define $D_d = \{(x,t)\colon x \in (d(t),\infty), t \in \mathbb{R}\}$. So let $F\colon D_d \to \mathbb{R}$ then $F^{\downarrow}\colon \mathbb{R} \to (\sup_{x\in\mathbb{R}} d(x),\infty)$ because by (17) $F^{\downarrow^{-1}}$ must be defined on $(d(x),\infty)$ for each $x \in \mathbb{R}$.

3.2 Applications of tetration

TODO

3.2.1 Binary representation

TODO

3.2.2 Generating functions

TODO

3.2.3 Large numbers

TODO

3.3 Extensions of tetration

Question 16. *What is the problem of extending tetration to non-integer heights?*

The short answer is: uniqueness. There are no suitable conditions found yet that favour a certain solution over others.

3.3.1 Ioannis Galidakis' bump method

TODO
See [**?**].

3.3.2 Robert Munafo's tower method

TODO
See [**?**].

3.3.3 Jay D. Fox's linear method

3.3.4 Gottfried Helms's solution

TODO
Posting in sci.math.research

4 Related problems

4.1 Alternative definitions of exponentiation

Making the Exponentiation Associative and Commutative

$$x \bigtriangleup_1 y = x + y$$
$$x \bigtriangleup_{n+1} y = \exp(\log(x) \bigtriangleup_n \log(y))$$

4.2 Alternative bracketing of exponentiation

4.2.1 Left Bracketing

TODO

$$\underbrace{((x^x)^{\cdots})^x}_{n\times x} = x^{x^{n-1}}$$

Uniqueness of the solution $(x,y) \mapsto x^{x^{y-1}}$?

4.2.2 Balanced Bracketing

TODO
${}^1x = x$, ${}^2x = x^x$, ${}^4x = (x^x)^{(x^x)}$, et cetera. Generally we define *balanced tetration* as

$$\begin{aligned} {}^{2^0}x &= x \\ {}^{2^{n+1}}x &= \left({}^{2^n}x\right)^{\left({}^{2^n}x\right)} \end{aligned}$$

This reduces extending tetration to extending the iteration of $F(x) := x^x$ because ${}^{2^n}x = F^{\circ n}(x)$. So if we found a continuous iteration of F then we have also found a continuous iteration of the balanced tetration by

$${}^yx = F^{\circ \log_2(y)}(x)$$

We can simplify the treatment of $F(x)$ by introducing $G(x) := xe^x$ (which is the inverse of the Lambert's W function) and noticing that $F^{\circ n} = exp \circ G^{\circ n} \circ \ln$ because $F := \exp \circ G \circ \ln$. So continuous iteration is reduced to continuous iteration of G. Now G has a fixed point at 0 (corresponding to the fixed point of F at 1) and can be developed at 0 into the powerseries

$$G(x) = \sum_{j=1}^{\infty} j \frac{x^j}{j!}$$

Formal powerseries of the form $x + \dots$ have a unique continuous iterate. So if it converges (which seems quite so) then G has a unique continuous iterate and then F has a unique continuous iterate and then there is a unique analytic balanced tetration.

4.3 Choose a number system that reflects non-associativity

TODO
Arborescent Numbers and Higher Arithmetic Operations (article)
Tree fraction calculator (web application)