Tetrations with Bases $b < 1$ and Infinite Towers

Hi, Enryk, and good morning to all the Participants!

I am very pleased to have the opportunity of exchanging views with you on this very peculiar subject. Indeed, I think that the idea of trying to find (if existing) a C^{∞} continuity class function for representing $y = {}^{x}b$, conjecturally extended to real x real values is the most appropriate. This is indispensable if we wish to see whether or not tetration (including sexp and its inverses: slog and srt) can be considered a consistent and useful "elementaty" hyperoperation. The ideal solution would be an analytic function (class C^{ω}), probably a complex analytic function. The Robbins' approach seems to me rather good. Perhaps, we are already (partly) around ... the famous corner.

Concerning what happens at $b < 1$, I should like to show you the following thoughts, hoping that you will appreciate my ... artistic approach. These lines are not fully orthodox and need a deep critical assessment. Nevertheless, I think that there might be something useful in them. But, as Enryk correctly said: "*Probably, they are not at the final stage*". I am convinced that this analysis must be done in connection with the study of the "infinite towers" (I like that!).

First of all, let us define the "infinite tower" as follows:

(1) $$\lim_{n\to\infty} {}^{n}b = {}^{\infty}b$$

where the "non-standard" notation, ${}^{\infty}b$, reminds us that the infinite towers are not always infinite, as it had also been stipulated (and demonstrated) by Euler, sometime ago. Euler had also shown that the above-mentioned limit converges in the range $e^{-e} < b < \sqrt[e]{e}$. With the KAR's convention for the "reciprocal of e", noted as "$:e = 1/e$", similar to the traditional representation of the "opposite of e", noted as "$-e = 0 - e$", the "Euler's range" is even more attractive:

$$\boxed{e^{-e} < b < e^{:e}}$$ (the Euler's range)

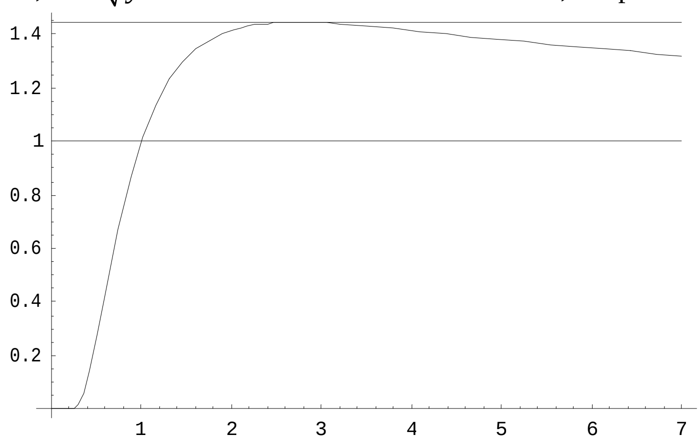
The max and min values of b commute among them, if we exchange $e \rightleftarrows 1/e$. (I don't know what really we can do with that, apart from other funny informal considerations).

Now, let us express (1) by a functional equation (and its inverse), as follows:

(2) $$y = b^{y} = {}^{\infty}b$$ (the infinite tower, base b)

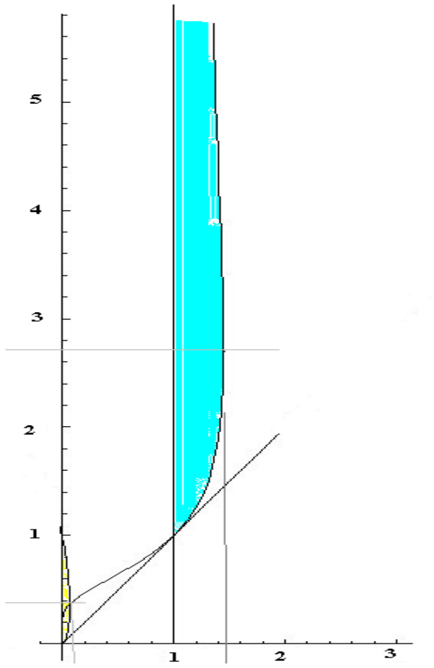
i.e.: $$b = \sqrt[y]{y} = {}^{\infty}\!\!\sqrt{y}$$ (the infinite-order superroot of y)

We know that, for $b > 0$, $b = \sqrt[y]{y}$ is a well known class C^{∞} function, the plot of which is as follows:

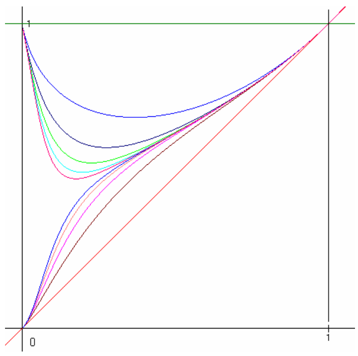


The plot shows a maximum at $y = e$, i.e. $b = \sqrt[e]{e}$ ("jaydfox η"). For any value chosen for the infinite tower $y = b^{y} = {}^{\infty}b$, we can find one value of base b satisfying formula (2).

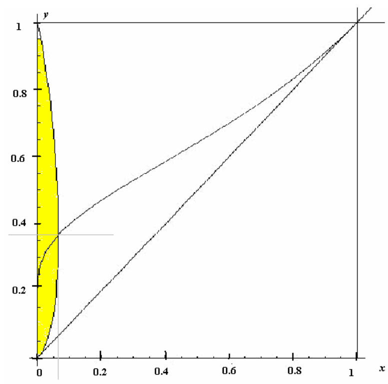
The problem is the inverse procedure, involving a non-bijective function. Nevertheless, a “brutal” graphical inversion would give the following $y = f(b)$ plot:



What seems to happen is that, in the $1 < b < \sqrt[e]{e}$ range, we have a two-valued “function”, giving two distinct values for the corresponding infinite tower (coincident values, for $b = \sqrt[e]{e}$). For the range $0 < b < 1$ (in b = 0 and b = 1 we have singular points), the plot shows (yellow zone!) a $0 < b \le e^{-e}$ sub-range (not shown by the graphic inversion, but artificially added by me), outside the Euler’s convergence area, the possible interpretation of which might be discussed later. In the $0 < b < 1$ range, the available plots of functions $y = {}^{n}b$ ($n \in N$) are as follows:



The graphs are obtained for n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. For n even, the curves cross point (0,1) and, for n odd, point (0,0). The “yellow area” can qualitatively be shown as follows:



The provisional conclusion that I draw is that, in the $0 < b < 1$ range, the tetrational function ${}^{x}b$, for x real, (if existing and smooth) should have an oscillatory behaviour.

It is also important to observe that an analytic "inversion" of $b = \sqrt[y]{y}$ can be obtained through the Lambert's function ($y = \mathrm{W}(z) = \mathrm{plog}(z)$, where $z = y \cdot e^{y}$), i.e.:

(3) $$b = \sqrt[y]{y} = {}^{\infty}\big)y \qquad \leftrightarrow \qquad y = \mathrm{plog}(-\ln b)/(-\ln b)$$

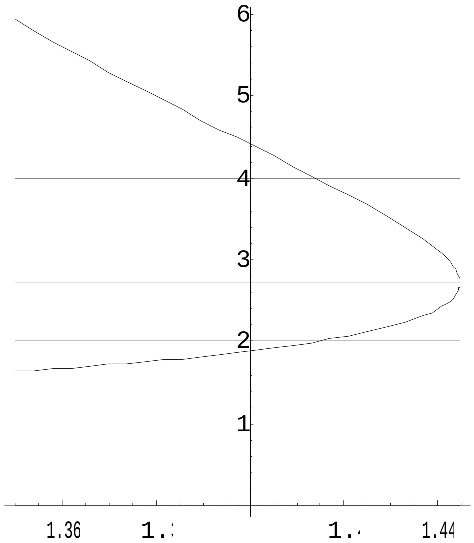
It is again important to say that the Lambert's function (the product logarithm) is a <u>complex function</u>, showing only two real branches, normally called W_{-1} and W_{0} (the "ordinary" ProductLog). In the framework of my ignorance of the theory of "molti-valued functions", I thought to simply indicate by "plog" the logical union of the two real "branches" of the product logarithm. In other words, by using the Mathematica operators, we can define these two real branches as:

(4) $\mathrm{W}_{0}(z)$ = **ProductLog[z]** $\qquad$ $\mathrm{W}_{-1}(z)$ = **ProductLog[-1, z]**

and their logical union by:

(5) $$\mathrm{plog}(z) = \mathrm{ProductLog}[z] \cup \mathrm{ProductLog}[-1, z]$$

In the following (zoomed) figure, we can see the two branches of $y = \mathrm{plog}(-\ln b)/(-\ln b)$, jointly plotted by using Mathematica. The lower (well known) branch is obtained via $\mathrm{W}_{0}(z)$ and the upper branch by using $\mathrm{W}_{-1}(z)$:



The same plot also shows the two values of the infinite towers, for base $b = \sqrt{2}$, which are 2 and 4, in the lower and upper branch, respectively. This is correct, since:

(6) $${}^{\infty}\left(\sqrt{2}\right) = \{2, 4\} \qquad \text{because} \qquad b = \sqrt[2]{2} = \sqrt[4]{4} \qquad \text{from (3), } b = \sqrt[y]{y}$$

Moreover, around the $b = 1$ singular point, we must have:

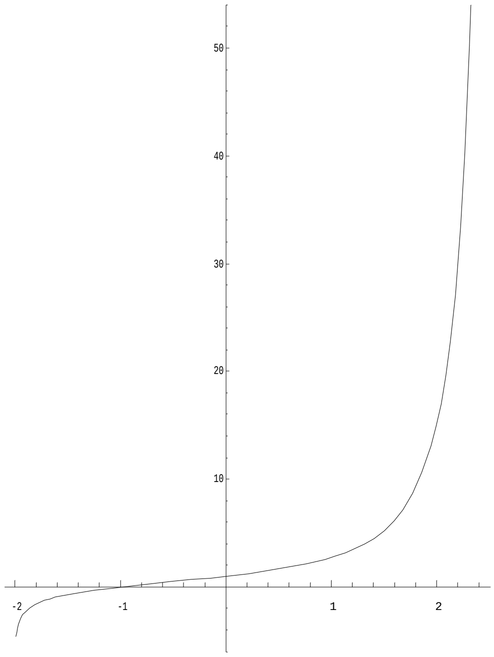
(7) $\boxed{\lim_{b \to 1-} {}^{\infty}b = 1 \text{ and } {}^{\infty}1 = 1}$ $\qquad$ continuity up to b = 1, from the left

$\boxed{\lim_{b \to 1+} {}^{\infty}b = \{1, +\infty\}}$ $\qquad$ semi-continuity (!!!), at b = 1, from the right.

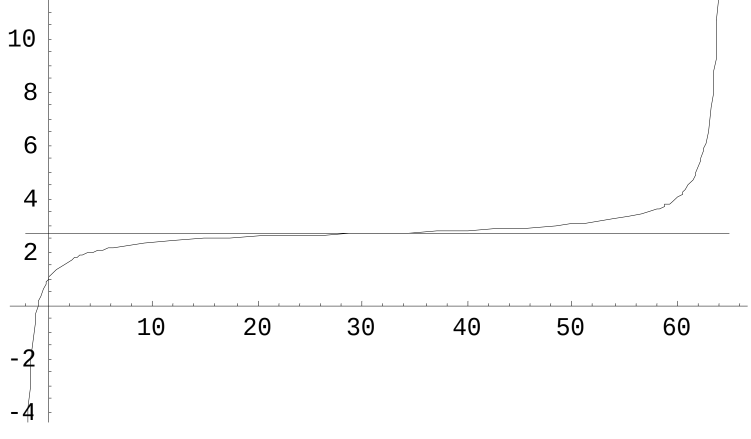
That's the main reason why I think we must accept that ${}^{\infty}1 = 1 \# \infty = 1$ (strictly) but, nevertheless, for $b \to 1$ from the right, there is also an infinite value (+∞), as limit. So, in the $1 < b < \sqrt[e]{e}$ range, we must indeed have two real infinite towers, which means two asymptotes in the (hypothetically smooth) ${}^{x}b$ plot.

With these facts in mind and with some basic assumptions similar to those adopted by Andrew, concerning the "critical interval" (at the level zero of his proposed method), it is possible to immediately build some simulations of the $y = {}^{x}b$ hypothetical functions, for various values of base *b*. Here are some examples.

For instance, in the range $b > \sqrt[e]{e} = 1.444...$ and, particularly, for $b = e = 2.718281828459045..$, a C^0 plot can be drawn of $y = {}^{x}e$, by supposing linear the critical path in $-1 < x \le 0$ (Andrew succeeded in obtaining a smooth C^{∞} solution. I am still looking with admiration at that. Bravo, Andrew!). Super-exponentially divergent for *x* → ∞, vertical asymptote for *x* = - 2.

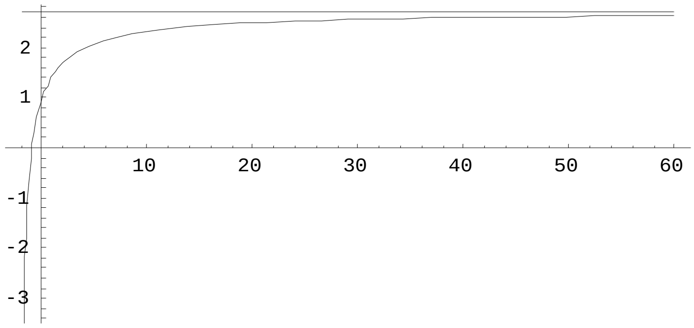


Again for $b > \sqrt[e]{e} = 1.444...$, but with values of base *b* near to $\sqrt[e]{e}$, ie.g.. for $b = 1.447$ and by assuming a linear behaviour (the critical interval) near *x* = 30, we get:



Superexponential at *x* > 60; "plateau" *y* = *e* around *x* = 30; vertical asymptote for *x* = 2..

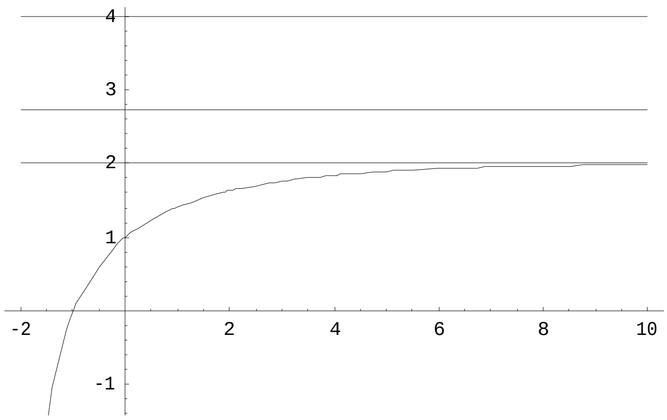
For exactly $b = \sqrt[e]{e}$, we can get a graphical simulation of high precision, by taking as critical interval a linear approximation of $y = {}^{x}b$ for high values of *x*. We get:



Horizontal asymptote for $x \to +\infty$ (at $y = e$); vertical asymptote ($-\infty$) for $b = -2$.

As we have seen, in the range $1 < b < \sqrt[e]{e}$, the $y = {}^{x}b$ plots should show two horizontal asymptotes. For example, for $b = \sqrt{2}$, they should exactly be $y = 2$ and $y = 4$. As a matter of fact, we have (3):

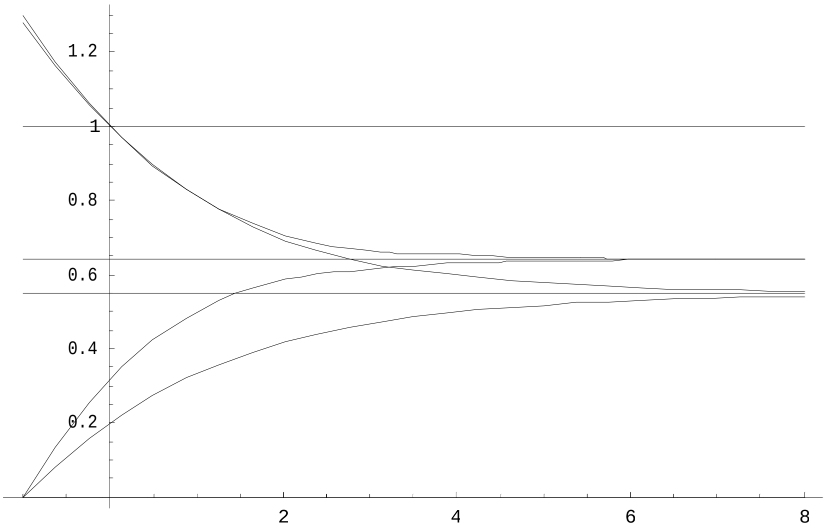
(8) $${}^{\infty}\left(\sqrt{2}\right) = \mathrm{plog}\left(-\ln\sqrt{2}\right)\Big/\left(-\ln\sqrt{2}\right) = \{2, 4\}$$



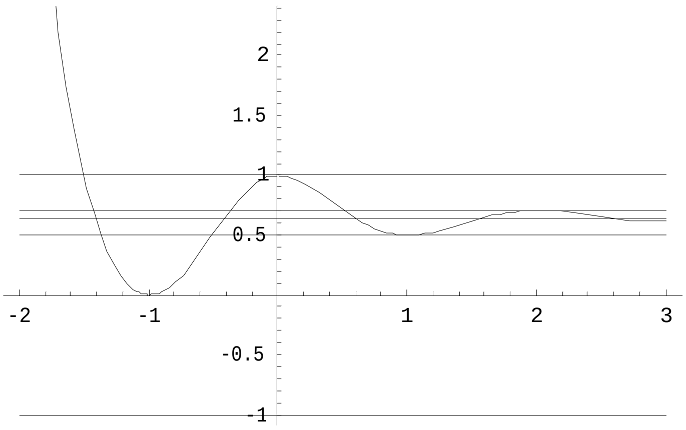
The plot quality is bad, because I didn't choose a critical interval sufficiently far in the direction $b \to +\infty$. However, the plot precision can be more and more accurate, with a more correct choice.
A vertical asymptote for $x = -2$ and two horizontal asymptotes for $x \to +\infty$, the lower one ($y = 2$) justifying the values obtained for integer *x* and the upper one ($y = 4$) still remaining to be explained. I agree with Enryk that the upper asymptote is not reachable from point (0, 1). My question is: can the upper asymptote be involved by a plot "coming down" from (0, $+\infty$)? In fact, we have seen that (7), i.e. $\lim_{b \to 1+} {}^{\infty}b = \{1, +\infty\}$. Why the value for integer x are not appearing? Two solutions? One real and one imaginary? Too strange ..!

In the range $0 < b < 1$, I have the impression (this is not mathematics, it is artistic imagination ..) that the $y = {}^{x}b$ is oscillating, up and down, around the curve of the infinite towers, like an old good sinus wave, slightly corrected in order to respect the constraints (see jaydfox). In fact, by using Mathematica, I tried to figure the possible interpolation of the $y = {}^{n}b$ solutions, for *n* odd or even. I tried it for two bases, $b = \frac{1}{2}$ and $b = \frac{1}{3}$, obtaining the following interpolations (the infinite towers calculated by (3)):

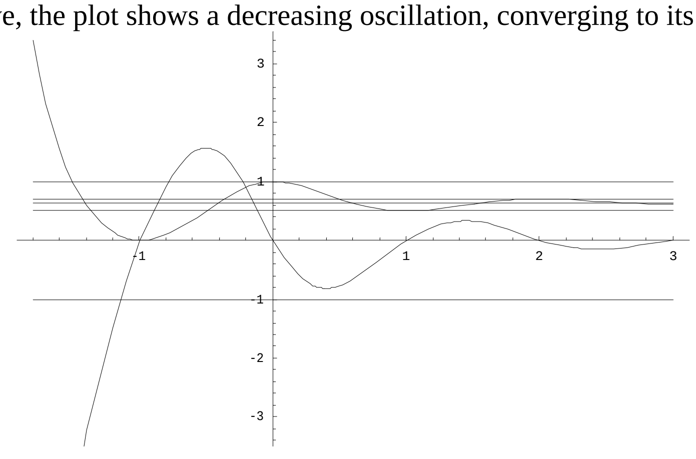
$${}^{\infty}\left(\frac{1}{3}\right) = 0.5478086216540974 \qquad \text{and} \qquad {}^{\infty}\left(\frac{1}{2}\right) = .0.641185744504986$$



The two Mathematica interpolations show a rather nice symmetrical display of the odd/even integer solutions, which corroborates my previous guessing. In the case of $b = 0.5$, firstly, I tried to assume a perfect sinusoidal approach for *x* ranges >> 4, so obtaining an about sine wave at $-1 < x < 0$. Then, I assumed an exact cosine curve in the $-1 < b < 0$ "critical interval" and obtained the following plot:

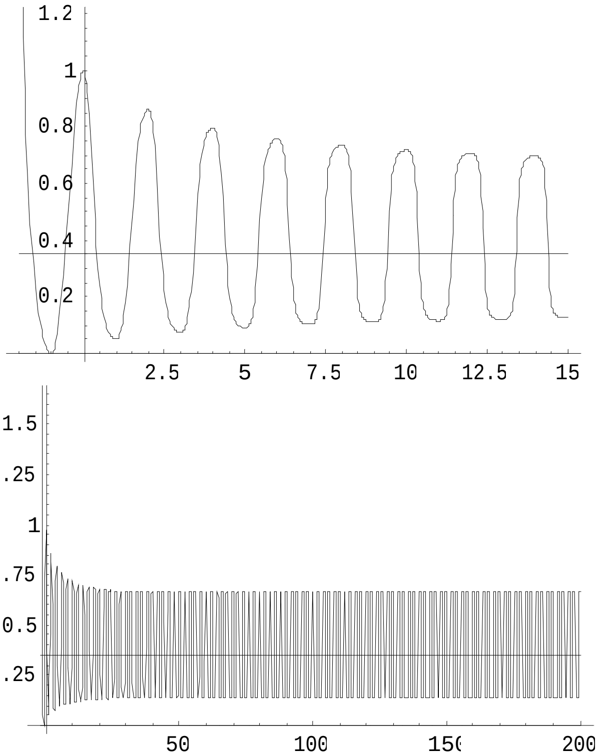


Its characteristics are: a vertical asymptote for *x* = -2 (*y* = + ∞), convergence at $y = {}^{\infty}\left(\frac{1}{2}\right)$, for $x \to +\infty$, and a justification of all the values obtained by $y = {}^{n}\left(\frac{1}{2}\right)$, for any integer *n*. Together with its first derivative, the plot shows a decreasing oscillation, converging to its infinite tower value:



and something like ... a class C^1 continuous function.

Finally, we saw that plots the family of the $y = {}^n b$ functions, for n integer > 0, try to systematically avoid what we called "the yellow zone" appearing in the range $0 < b < e^{-e}$. Practically, the perimeter of this area operates as an attractor for all the curves representing the tetrations (towers) with integer superexponents. My guessing is that, within this range of b, the oscillatory behaviour of the hypothetical $y = {}^x b$ continues, without any damping and, for $x \to \infty$, and shows permanent, quasi-sinusoidal waves, for $x \to \infty$. A rough simulation can be obtained by choosing as critical interval a square cosine curve in $-1 < x < 0$ and, then, by applying the ${}^{x+1}b = b \wedge \left({}^x b\right)$ formula. The following figures show the plots for the ranges $-2 < x < 15$ and $-2 < x < 200$



The hypothetical function $y = {}^x(0.05)$ could then have an oscillatory behaviour and, for $x \to +\infty$, and could show permanent oscillations, with upper and lower values described by the perimeter of the yellow zone.

Can we imagine that all this could be "explained" by the properties of one unique "function"? Perhaps. If this will not be possible, we have to find why. The consequence is the decision of accepting tetration as an "elementary hyperoperation". At the first sight, its right-inverse operation, the superlogarithm (slog) should very probably be limited to bases $b > \sqrt[e]{e}$, for avoiding b ranges where slog would have multiple values (with a ... variable, also infinite, multiplicity).

Thank you for your kind attention.

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