ON ANALYTIC ITERATION AND ITERATED EXPONENTIALS

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ABSTRACT. Analytic iteration concerns $f^n(x)$ where n is a real or complex variable. We overview the theory of analytic iteration, and investigate $\exp_b^n(x) = b^{b^{\cdot^{\cdot^{b^x}}}}$ known as iterated exponentials. We present new power series and compare them to known power series. Some power series are shown to be related to the Lambert W function which has physical applications. In closing, we present a multiple-sum series expansion of $a_1^{a_2^{\cdot^{\cdot^{a_n^x}}}}$.

This research was performed at Montgomery College between 2006 and 2008. All of the theorems presented here are original work, except where noted. All of the following work was done purely for independent study, and not for credit (either past or future).

My interest in iteration started in 1998. A sequence I have since learned is called the hyper-operation sequence $\{x+y, xy, x^y, \cdots\}$ dominates the curriculum of elementary school through high school. When the next step of iterated exponentials (which seemed obvious to me) was forgotten in favor of calculus, I continued to research hyper-operations.

Around 2004, I found a method [53] for extending iterated exponentials to real heights (also called *hyper-exponents* [25]) which amounts to finding a power series of $\exp_b^n(1)$ about $n = 0$. This discovery led to more research to determine if it was well known. After reading many articles on the topic, I came across the work of Peter Walker [66], who had developed the same method for finding this power series, but for the special case $b = \boldsymbol{e}$. Learning this, I decided not to publish, as it was not new. Since then, I have found other ideas and formulae that might be new, some of which are in this article.

1. Analytic iteration

As a function of one variable, f can be iterated if it is a map from a set to itself, and in this case we write[1] $f^n(x)$ to refer to the n-fold application of f. As a function of two variables, $f^n(x)$ may be extended from integer n to real or complex n. If an extension[2] exists, then it makes sense to ask the questions: Is the extension unique? Where is it differentiable? What is the behavior of $f^n(x)$ as $n \to \infty$, $n = \frac{1}{2}$, or $n = \boldsymbol{i}$? The first ($n \to \infty$) is part of the study of asymptotic behaviour [67], the second ($n = \frac{1}{2}$) corresponds to the functional equation $f(x) = g(g(x))$ [42], and the third ($n = \boldsymbol{i}$) is part of the study of analytic iteration [41]. Analytic iteration theory is the subject of most of this paper.

One example of a problem faced in analytic continuation is the gamma function $\Gamma(x)$, which is a problem from functional equations (not technically iteration). The problem is that there are an infinite number of functions that satisfy $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(1) = 1$, but there is only one such function that satisfies $\frac{d^2}{dx^2} \ln \Gamma(x) \geq 0$ for all $z > 0$ [67].

Even differentiability is not strong enough for uniqueness of extensions to functional equations, since in many cases, any periodic function ϕ could be used to modify f in such a way that the functional equation satisfied by f is also satisfied by another function involving f and ϕ. This presents an issue when solving problems with analytic iteration, because at some level, the answer could be anything one wants. On the other hand, iteration is a much stronger framework than simple interpolation.

To illustrate how analytic iteration is different, there are an infinite number of functions that interpolate the orbit $\{x, f(x), f(f(x)), \cdots\}$, but as we shall see, there are methods (known as *regular* iteration and *natural* iteration) that produce a unique function that interpolates the same orbit. Although the results of these methods are well-determined, the uniqueness conditions associated with these results are not well-known.

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[1] To distinguish between the two, we use $f^n(x)$ for iteration, and $f(x)^n$ for powers.

[2] We use the term *extension* instead of *analytic continuation* because the tools used in analytic continuation are not used here, and some of the extensions discussed here are not analytic.

1.1. **Carleman matrices.** The *Carleman matrix transform* (*Bell matrix* for its transpose) is a fundamental tool in analytic iteration, first noticed by Koch [64], then mentioned by Bennett [9], and later developed by Erdös and Jabotinsky ([41], [20]). Bell and Carleman matrices were most recently investigated by Aldrovandi and Freitas ([2], [1]) who use the term *Bell matrix* named after E. T. Bell [8] and Kowalski, Gralewicz and Steeb ([44], [45]) who call its transpose a *Carleman matrix*, named after T. Carleman [12].

Bell and Carleman matrices obey similar composition formulae, obviously, since they are transpose to each other. A Bell matrix satisfies $\mathbf{B}[f \circ g] = \mathbf{B}[g]\mathbf{B}[f]$ and a Carleman matrix satisfies $\mathbf{M}[f \circ g] = \mathbf{M}[f]\mathbf{M}[g]$. Thus, Carleman matrices are functors[3] from a subcategory of **Set** to a subcategory of **Vec**, because they preserve the order of composition, and Bell matrices are *contravariant* functors, because they reverse the order of composition.

In the sections that follow, we often make reference to the power series of an arbitrary function f, and for this we use $f_k = f^{(k)}(0)/k!$ such that $f(x) = \sum_{k=0}^{\infty} f_k x^k$.

The Carleman matrix of f, denoted $\mathbf{M}[f]$, is a matrix whose rows are powers of $f(x)$, and whose columns are the coefficients of x^k in their power series. They are defined as

$$M_{jk}[f] = \frac{1}{k!}\left[\frac{\partial^k}{\partial x^k} f(x)^j\right]_{x=0} \tag{1}$$

so as to satisfy $f(x)^j = \sum_{k=0}^{\infty} M_{jk}[f]x^k$. This is the standard definition of Carleman matrices, found in [12], [44], [45], and Bell matrices, found in [1], [9], [2], [41], [20], [64].

There are two key observations which make these matrices useful. The first is that despite the fact that a Carleman matrix is an infinite matrix, it can be reasonably approximated by a finite $n \times n$ matrix, which will give an approximation to the order of $O(x^{n+1})$, because it forms a finite polynomial. The second is that these matrices convert composition into matrix multiplication, and thus convert iteration into matrix powers [1].

[3]Some definitions of the term *functor* require that they are also *covariant* [67].

To see this, let $\mathbf{V}(x) = (1, x, x^2, x^3, \cdots)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ f_0 & f_1 & f_2 & f_3 & \cdots \\ f_0^2 & 2f_0f_1 & f_1^2 + 2f_0f_2 & 2f_1f_2 + 2f_0f_3 & \cdots \\ f_0^3 & 3f_0^2f_1 & 3f_0(f_1^2 + f_0f_2) & f_1^3 + 6f_0f_1f_2 + 3f_0^2f_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ f(x) \\ f(x)^2 \\ f(x)^3 \\ \vdots \end{bmatrix} \tag{2}$$

so that $\mathbf{M}[f]\mathbf{V}(x) = \mathbf{V}(f(x))$. The next step is to show that $\mathbf{M}[f]^n = \mathbf{M}[f^n]$. If we start with the above, and substitute $f \to f^n$, then we obtain

$$\mathbf{M}[f^n]\mathbf{V}(x) = \mathbf{V}(f^n(x)) \tag{3}$$

If instead we were to substitute $x \to f^{n-1}(x)$, then we get

$$\mathbf{M}[f]\mathbf{V}(f^{n-1}(x)) = \mathbf{V}(f(f^{n-1}(x))) = \mathbf{V}(f^n(x)) \tag{4}$$

Using this recurrence equation, we can then continue substituting (4) until we are left with only $\mathbf{V}(x)$ on the *l.h.s.* and $\mathbf{V}(f^n(x))$ on the *r.h.s.*, which gives

$$\mathbf{M}[f]^n\mathbf{V}(x) = \mathbf{V}(f^n(x)) \tag{5}$$

and combining (5) with (3), then equating coefficients of x^k gives

$$\mathbf{M}[f]^n = \mathbf{M}[f^n] \tag{6}$$

1.2. **Regular iteration.** The term *regular* was once synonymous with *analytic* [67], so the term *regular iteration* used to mean analytic iteration, however, that has changed. Both Ritt [52] and Szekeres [61] use the term *regular iteration* for a specific method of finding a bivariate function $f^n(x)$. This method requires that f is analytic in an open ball around a fixed point, and the final results differ depending on the nature of the fixed point.

There are two main classes of fixed points used here.

(1) $|f_1| \neq 1$ and $f_1 \neq 0$. Bennett's Type I [9], or what Geisler calls *hyperbolic* [30].

(2) $f_1 = 1$. Bennett's Type II [9], or what Geisler calls a *parabolic* fixed point [30].

These two types of fixed points are important because the coefficients of x^k in the power series of $f^t(x)$ are naturally found by examining sums of terms of the form f_1^k. From elementary calculus [67], we know that the following holds for geometric series.

$$\sum_{k=0}^{t-1} f_1^k = \begin{cases} t & \text{if } f_1 = 1 \\ \frac{1-f_1^t}{1-f_1} & \text{otherwise} \end{cases} \tag{7}$$

Therefore, the solutions we find will appear different depending on what kind of fixed point we use. For a parabolic fixed point 0, we find that $f^t(x) = \sum_{k=1}^{\infty} g_k(t)x^k$ where

$$g_1(t) = 1 \tag{8}$$

$$g_2(t) = f_2 t \tag{9}$$

$$g_3(t) = f_3 t + f_2^2 t(t-1) \tag{10}$$

$$g_4(t) = f_4 t + \frac{5}{2} f_2 f_3 t(t-1) + \frac{1}{2} f_2^3 t(t-1)(2t-3) \tag{11}$$

and so on. The Lagrange interpolation of the orbit $\{x, f(x), f^2(x), f^3(x), \cdots\}$ will also give these results, because they are always finite polynomials in t. Furthermore, the coefficients of x^k are polynomials in t of degree $(k-1)$, so you only need at most k points.

For a hyperbolic fixed point 0, we find that $f^t(x) = \sum_{k=1}^{\infty} h_k(t)x^k$ where

$$h_1(t) = f_1^t \tag{12}$$

$$h_2(t) = f_2 f_1^{t-1} \frac{1-f_1^t}{1-f_1} \tag{13}$$

$$h_3(t) = f_3 f_1^{t-1} \frac{1-f_1^{2t}}{1-f_1^2} + 2f_2^2 f_1^{t-2} \frac{(1-f_1^t)(1-f_1^t)}{(1-f_1)(1-f_1^2)} \tag{14}$$

and so on. These cannot be obtained through Lagrange interpolation, and thus require more advanced methods. Since Carleman matrices transform iteration into matrix powers, one method to find the analytic iterate $f^t(x)$ is to find a noninteger matrix power of $\mathbf{M}[f]$.

It is well-known that noninteger matrix powers can be evaluated if a diagonalization of $\mathbf{M}[f]$ exists [67]. To diagonalize a matrix, one needs to solve one of the matrix equations $\mathbf{M}[f]\mathbf{P} = \mathbf{PD}$ or $\mathbf{SM}[f] = \mathbf{DS}$ where $\mathbf{D}$ is a diagonal matrix. Matrix diagonalization is equivalent to finding eigenvalues and eigenvectors associated with a matrix [67]. Also, given an eigenvalue, finding the associated eigenvector is relatively simple. So if we can find eigenvalues, then we can find eigenvectors, which means we can diagonalize, which means we can evaluate $\mathbf{M}[f]^t$ for noninteger t, which means we can find $f^t(x)$ for noninteger t.

According to Aldrovandi *et.al.* [2] and Kowalski *et.al.* [44], the eigenvalues of $\mathbf{M}[f]$ are always powers of f_1, giving the sequence $\{1, f_1, f_1^2, f_1^3, \cdots\}$, which would appear on the diagonal of $\mathbf{D}$. The same authors both relate this to a well-known functional equation.

Theorem 15 (Aldrovandi, Kowalski)**.** *The diagonalization of* $\mathbf{M}[f]$ *is equivalent to finding a Schröder function of* f*, a function* $S(x)$ *that satisfies* $S(f(x)) = f_1 S(x)$*.*

Proof. Taking the Carleman matrix of both sides gives $\mathbf{M}[S]\mathbf{M}[f] = \mathbf{M}_x[f_1 x]\mathbf{M}[S]$. □

Although eigenvalue decomposition is not the only way of evaluating matrix powers, it is the most well-known method. Theorem (15) shows that matrix powers and regular iteration coincide when the Carleman matrix is expanded about a fixed point, but otherwise one may try other methods, like using $\mathbf{A}^n = \exp(n \log \mathbf{A})$ which may provide an answer.

One method that should always be used first is pattern recognition, such as the case when $f(x) = x + c$ (which has no fixed points), then clearly $f^n(x) = x + cn$. However, when pattern recognition fails, a more general method such as regular iteration can provide a way to obtain an analytic continuation of $f^n(x)$ to real or complex number n. Having this analytic continuation allows us to use all of tools of analysis on both parameters of $f^n(x)$, perhaps suggesting solutions to unsolved problems.

1.3. **Natural iteration.** The logarithm of a Schröder function is called an Abel function, which satisfies the functional equation $A(f(x)) = A(x) + 1$. If we know the Schröder function $S(x)$, then $A(x) = \log_a(S(x))$, but if we do not, then we can proceed as follows.

First, define the following $n \times (n+1)$ and $(n+1) \times n$ matrices

$$\mathbf{J} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \tag{16}$$

Second, differentiate an Abel function, and let $x = 0$ and $\mathbf{F}[f] = (f_0, f_1, f_2, f_3, \cdots)$

$$\alpha(f(x)) - \alpha(x) = 1 \tag{17}$$

$$\frac{\partial^k}{\partial x^k} \sum_{j=0}^{\infty} \alpha_j f(x)^j - \frac{\partial^k}{\partial x^k} \sum_{j=0}^{\infty} \alpha_j x^j = \delta_{0k} \tag{18}$$

$$\sum_{j=0}^{\infty} \alpha_j \left[\frac{\partial^k}{\partial x^k} f(x)^j \right]_{x=0} - \sum_{j=0}^{\infty} \alpha_j \left[\frac{\partial^k}{\partial x^k} x^j \right]_{x=0} = \delta_{0k} \tag{19}$$

$$\sum_{j=0}^{\infty} \alpha_j \frac{1}{k!} \left[\frac{\partial^k}{\partial x^k} f(x)^j \right]_{x=0} - \sum_{j=0}^{\infty} \alpha_j \frac{1}{k!} \left[\frac{\partial^k}{\partial x^k} x^j \right]_{x=0} = \frac{\delta_{0k}}{k!} \tag{20}$$

$$\sum_{j=0}^{\infty} \alpha_j M_{jk}[f] - \sum_{j=0}^{\infty} \alpha_j M_{jk}[\mathrm{id}] = \frac{\delta_{0k}}{k!} \tag{21}$$

$$\sum_{j=0}^{\infty} \alpha_j (M_{jk}[f] - \delta_{jk}) = \frac{\delta_{0k}}{k!} \tag{22}$$

$$\mathbf{F}[\alpha]^{\mathrm{T}} (\mathbf{M}[f] - \mathbf{I}) = \mathbf{F}[1]^{\mathrm{T}} \tag{23}$$

$$(\mathbf{B}[f] - \mathbf{I}) \mathbf{F}[\alpha] = \mathbf{F}[1] \tag{24}$$

where $\mathbf{B}[f] = \mathbf{M}[f]^{\mathrm{T}}$ is a Bell matrix of f.

Notice that α_0 plays no part in these equations, or in other words, $(\mathbf{B}[f] - \mathbf{I})$ is not invertible. The essential theorem of natural iteration is that we can remove α_0 from these equations, at which point there is a chance for a unique solution. The matrix

$$\boxed{\mathbf{A}[f] = \mathbf{J}(\mathbf{B}[f] - \mathbf{I})\mathbf{K}} \tag{25}$$

was first described by Peter Walker [66] and then rediscovered by myself 40 years later. It is referred to as an **Abel matrix** by J. D. Fox [23] since it encodes the functional equation $A(f(x)) = A(x)+1$ in matrix form. To recover an Abel function from this matrix equation, we start with a simplification of (24) with α_0 removed and let $\alpha(x) = \alpha_0 + x\beta(x)$ so we can remove α_0 from the matrix equation by truncating the first column of $(\mathbf{B}[f] - \mathbf{I})$ to obtain the matrix equation $\mathbf{J}(\mathbf{B}[f] - \mathbf{I})\mathbf{K}\mathbf{F}[\beta] = \mathbf{F}[1]$. This is the same set of equations, only because $\mathbf{J}$ and $\mathbf{K}$ have truncated the first column and last row of the matrix, so we no longer have any equations in α_0. Using the definition of an Abel matrix given above, this is the same as $\mathbf{A}[f]\mathbf{F}[\beta]. = \mathbf{F}[1]$. Note also that $\alpha_k = \beta_{k-1}$, thus if the Abel matrix is invertible, then $\mathbf{F}[\beta] = \mathbf{A}[f]^{-1}\mathbf{F}[1]$ so the coefficients of $\beta(x)$ are just the entries in the first column of the inverse of $\mathbf{A}[f]$. We can now choose α_0 to be any constant (because it is not involved in the equations), so $\alpha(x) = \alpha_0 + x\beta(x)$ is an Abel function of f.

1.4. **Julia functions.** Where Abel functions are unique upto an addative constant, Julia functions are just unique, because they can be found using the derivative of an Abel function (which discards the constant). A *Julia function* $J(x)$ of f is a function that satisfies $J(f(x)) = f'(x)J(x)$. Most authors, such as Geisler [30] and Szekeres [61] define Julia functions by their functional equation, but Jabotinsky defines them by

$$L(x) = \left[\frac{\partial}{\partial t} f^t(x)\right]_{t=0} \tag{26}$$

then proceeds to show that $L(f(x)) = f'(x)L(x)$, which implies that $L(x) = J(x)$.

Let us take a step back and consider what this means.

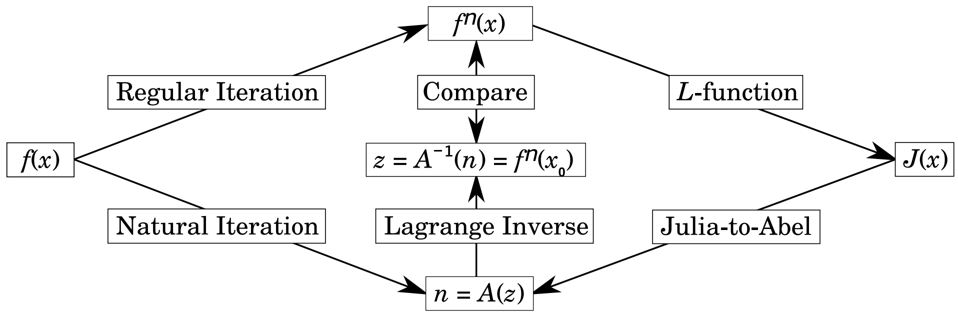

FIGURE 1. Overview of analytic iteration.

The accuracy of regular iteration and natural iteration are easy to find if $f^n(x)$ is known, in which case neither method provide any new information. If $f^n(x)$ is unknown, but approximations can be found by regular iteration and natural iteration, then it is important to know the accuracy of the corresponding power series. The two series $f^n(x) = \sum_{k=0}^{\infty} b_k(n)x^k$ and $f^n(x_0) = \sum_{k=0}^{\infty} c_k n^k$ are difficult to compare, since we cannot equate coefficients of x.

Knowing $L(x) = J(x)$ allows us to do this, by starting with regular iteration, taking the derivative of $f^t(x)$ with respect to t, evaluating at $t = 0$ (which gives us $J(x)$, then integrating the reciprocal from $x = x_0$ to $x = z$ gives an Abel function $A(z)$, which can be compared *directly* with the result of natural iteration.

The first step of this is to find a power series for a Julia function of f in terms of f_k, the Taylor coefficients of f. Using the power series of $f^n(x)$ we found in (8), differentiating with respect to t and evaluating it at $t = 0$ gives the expansion

$$J(x) = f_2 x^2 + (f_3 - f_2^2)x^3 + \left(\frac{3}{2}f_2^3 - \frac{5}{2}f_2 f_3 + f_4\right)x^4 + \cdots \tag{27}$$

for a parabolic fixed point 0, and

$$J(x) = \ln(f_1)\left(x + \frac{f_2}{f_1(f_1 - 1)}x^2 + \frac{2(f_1 f_3 - f_2^2)}{f_1^2(f_1^2 - 1)}x^3 + \cdots\right) \tag{28}$$

for a hyperbolic fixed point 0.

1.5. **Abel functions.** Not surprisingly, Abel functions have singularities at fixed points. Suppose $x_0 = f(x_0)$ is a fixed point. It follows that $A(f(x_0)) = A(x_0) = A(x_0) + 1$ or that $0 = 1$ which is a contradiction. However, since we have a power series expansion of $J(x)$ about $x = 0$, which we assumed was a fixed point of f, we can use this to find an *asymptotic* power series of $A(x)$ about $x = 0$. We find that $\frac{1}{J(x)}$ has a pole at $x = 0$ of residue 2, which means its asymptotic expansion about a parabolic fixed point 0 is

$$\frac{1}{J(x)} = \frac{1}{f_2 x^2}\left(1 + \left(f_2 - \frac{f_3}{f_2}\right)x + \left(\frac{f_3 - f_2^2}{2} + \frac{f_3^2}{f_2^2} - \frac{f_4}{f_2}\right)x^2 + \cdots\right) \tag{29}$$

$$= \frac{1}{f_2 x^2}\sum_{k=0}^{\infty}(k-1)A_k x^k = \frac{1}{f_2}\left(\frac{1}{x^2} + \frac{A_1}{x} + \sum_{k=1}^{\infty} k A_{k+1} x^{k-1}\right) \tag{30}$$

which means the power series of an Abel function about a parabolic fixed point 0 is

$$A(z) = \int_{z_0}^{z}\frac{dx}{J(x)} = \frac{1}{f_2}\left(-\frac{1}{z} + A_1 \ln(z) + \sum_{k=1}^{\infty} A_{k+1} z^k\right) + C \tag{31}$$

which Szekeres gives in [61], in which he also gives the formula

$$A(z) = \frac{1}{\ln(f_1)}\left(\ln(z) + \sum_{k=1}^{\infty} B_k z^k\right) + C \tag{32}$$

for a hyperbolic fixed point 0, and defines the *principal Abel function* to be where $C = 0$.

From these expansions, we obtain the sequence

$$A_0 = 1 \tag{33}$$

$$A_1 = f_2 - \frac{f_3}{f_2} \tag{34}$$

$$A_2 = -\frac{f_2^2}{2} + \frac{f_3}{2} + \frac{f_3^2}{f_2^2} - \frac{f_4}{f_2} \tag{35}$$

$$A_3 = \frac{f_2^3}{3} - \frac{7 f_2 f_3}{12} + \frac{f_4}{2} - \frac{f_3^2}{4 f_2} + \frac{f_3 f_4}{f_2^2} - \frac{f_3^3}{2 f_2^3} - \frac{f_5}{2 f_2} \tag{36}$$

which will be used in the next section.

1.6. **Generating functions.** Now that we have all the background needed, we can present the major theorem of this paper. It pertains to the coefficients of $f^n(x)$ given in (8-11) but instead of considering coefficients of $n^j x^k$, we will be considering coefficients of $n^j x^{j+k}$, which are the "diagonals" of the matrix of coefficients of $n^j x^k$, so one way to find these power series is to make the substitution $n \to \frac{y}{x}$ in $f^n(x)$.

Theorem 37. *Given a function $f(x)$ with parabolic fixed point 0*

$$\boxed{\begin{aligned} f^n(x) &= \tfrac{x}{1-nxf_2} \\ &+ x^2 \tfrac{A_1 \ln(1-nxf_2)}{(1-nxf_2)^2} \\ &+ x^3 \tfrac{A_1^2 \ln(1-nxf_2)(\ln(1-nxf_2)-1)-nxf_2A_2}{(1-nxf_2)^3} + \cdots \end{aligned}} \tag{38}$$

Proof. We prove by expanding the power series of each term. The first diagonal is

$$\frac{x}{1-nxf_2} = \sum_{j=0}^{\infty} f_2^j n^j x^{j+1} = x + f_2 n x^2 + f_2^2 n^2 x^3 + f_2^3 n^3 x^4 + \cdots \tag{39}$$

which comprise all the coefficients of $n^j x^{j+1}$ in (8-11). The second diagonal is

$$x^2 \frac{A_1 \ln(1-nxf_2)}{(1-nxf_2)^2} = -\sum_{j=0}^{\infty} A_1 H_j^{(2)} f_2^j n^j x^{j+2} \tag{40}$$

$$= -(f_2^2 - f_3)nx^3 - \frac{5}{2}(f_2^3 - f_2 f_3)n^2 x^4 + \cdots \tag{41}$$

where the second order harmonic numbers [67] are defined by

$$H_k^{(2)} = \sum_{j=1}^{k} H_j = \sum_{j=1}^{k} \sum_{i=1}^{j} \frac{1}{i} \tag{42}$$

which are seen to be the coefficients of $n^j x^{j+2}$ in (8-11). The third diagonal can be written as the sum of two functions $x^3(A_1^2 G_1(nxf_2) + G_2(nxf_2)A_2)$ where

$$G_2(y) = -\frac{y}{(1-y)^3} = -\sum_{k=0}^{\infty} \frac{k}{2}(k+1)y^k = -y - 3y^2 - 6y^3 - 10y^4 + \cdots \tag{43}$$

and

$$G_1(y) = \frac{\ln(1-y)^2 - \ln(1-y)}{(1-y)^3} \tag{44}$$

$$= \sum_{k=0}^{\infty} \frac{y^k}{k!} \sum_{j=0}^{k} (-1)^{j+k} 3^j (j+1) \left((j+2) \left[{k \atop j+2} \right] - \left[{k \atop j+1} \right] \right) \tag{45}$$

$$= y + \frac{9}{2}y^2 + \frac{71}{6}y^3 + \frac{145}{6}y^4 + \frac{638}{15}y^5 + \frac{2443}{36}y^6 + \frac{14139}{140}y^7 + \cdots \tag{46}$$

Note that (45) can be found by subtracting A001711(n) from A001712($n-1$) [58].

With these expansions we can start comparing to (8-11). Note that the coefficient of f_4 will be the same as the coefficient of A_2 in the generating function, since A_2 is the only place where f_4 appears. We find from (8-11) that the terms involving f_4 are

$$-f_2 f_4 n x^4 - 3 f_2^2 f_4 n^2 x^5 - 6 f_2^3 f_4 n^3 x^6 - 10 f_2^4 f_4 n^4 x^7 + \cdots \tag{47}$$

which is the power series for the coefficients of f_4 in the expansion of $G_2(nxf_2)A_2$.

The other function (G_1) is not as easy to show, because $A_1^2 = \frac{f_3^2}{f_2^2} + \cdots$ and $A_2 = \frac{f_3^2}{f_2^2} + \cdots$, so the coefficients of f_3^2 in (8-11) will be described by the sum of $G_1 + G_2$. We find by continuing the coefficients in (8-11), those terms involving f_3^2 are

$$\frac{3}{2} f_2^2 n^2 x^5 + \frac{35}{6} f_2 f_3^2 n^3 x^6 + \frac{85}{6} f_2^2 f_3^2 n^4 x^7 + \cdots \tag{48}$$

$$= \left(\frac{9}{2} - 3 \right) f_2^2 n^2 x^5 + \left(\frac{71}{6} - 6 \right) f_2 f_3^2 n^3 x^6 + \cdots \tag{49}$$

which is exactly the expansion of $x^3(G_1(nxf_2) + G_2(nxf_2))\frac{f_3^2}{f_2^2}$. Using a similar argument for f_3 and f_2^k, the combined expansion of $x^3(A_1^2 G_1(nxf_2) + G_2(nxf_2)A_2)$ holds. □

Three immediate consequences of (38) are that $f^0(x) = x$, because all but the first term vanish, that $f^{1/xf_2}(x)$ is undefined because the denominators vanish, and that

$$f^{1/x}(x) = \frac{x}{1-f_2} + x^2 \frac{A_1 \ln(1-f_2)}{(1-f_2)^2} + x^3 \frac{A_1^2 \ln(1-f_2)(\ln(1-f_2) - 1) - f_2 A_2}{(1-f_2)^3} + \cdots \tag{50}$$

is a power series in x whose coefficients do not depend on x.

Corollary 51 (parabolic, $A_1 = 1$, $A_k = 0$ for $k > 1$)**.** *If*

$$f(x) = \frac{1}{W\left(\frac{1}{x}\exp\left(\frac{1}{x} - f_2\right)\right)} \quad \textit{then} \tag{52}$$

$$f^n(x) = \frac{1}{W\left(\frac{1}{x}\exp\left(\frac{1}{x} - f_2 n\right)\right)} \tag{53}$$

Proof. Assume $A_1 = 1$, $A_k = 0$ for $k > 1$. The Abel function of f would then be:

$$A(x) = \frac{x\ln(x) - 1}{f_2 x} \tag{54}$$

which means its inverse would be

$$A^{-1}(x) = \frac{1}{W(\exp(-f_2 x))} \tag{55}$$

which means the analytic iterate $f^n(x)$ is

$$f^n(x) = A^{-1}(A(x) + n) \tag{56}$$

$$= \frac{1}{W(\exp(\frac{1}{x} - \ln(x) - f_2 n))} \tag{57}$$

$$= \frac{1}{W(\frac{1}{x}\exp(\frac{1}{x} - f_2 n))} \tag{58}$$

and substituting $n = 1$ completes the proof. □

Corollary 59 (hyperbolic, $A_1 = 1$, $A_k = 0$ for $k > 1$)**.** *If*

$$f(x) = W(x\exp(x - f_2)) \quad \textit{then} \tag{60}$$

$$f^n(x) = W(x\exp(x - f_2 n)) \tag{61}$$

Proof. We know that f is topologically conjugate to the f in (51), so by corollary (51)

$$\frac{1}{f^n(\frac{1}{x})} = \frac{1}{W(\frac{1}{x}\exp(\frac{1}{x} - f_2 n))} \tag{62}$$

and substituting $x \to \frac{1}{x}$ completes the proof. □

2. Iterated Exponentials

An *iterated exponential* is a function of the form $\exp_b^n(x) = b^{b^{\cdot^{\cdot^{b^x}}}}$ where b occurs n times. The special case where $x = 1$ is written ${}^n b = \exp_b^n(1)$ [48] and is called *tetration* [33]. Iterated exponentials have been studied since the time of Euler [21] who proved that

$$ {}^\infty b = \lim_{n\to\infty} {}^n b \tag{63} $$

converges for all

$$ \boldsymbol{e}^{-\boldsymbol{e}} \le b \le \boldsymbol{e}^{1/\boldsymbol{e}} \tag{64} $$

Contemporaries of Euler, such as Eisenstein [19] and Lambert [46] also worked on the same inequality, and it has since been shown many times ([5], [3], [11], [14], [17], [32], [43]).

For complex number b, Thron estimated the region of convergence in [62], which was later determined exactly by Shell in [56] and [57]. Shell proved that the limit (63) exists where $b = h^{1/h}$ if and only if $|\ln h| \le 1$, shown below.

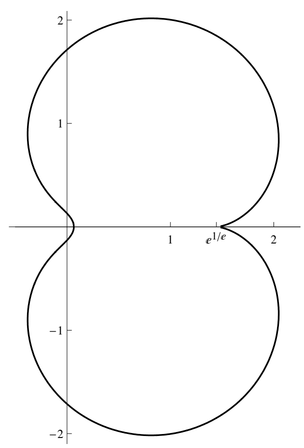

Figure 2. Convergence region of ${}^\infty b$.

This region, called the *Shell-Thron region* [29], is the most prominent feature of the *tetration fractal* [29], which debuted in [59], then later in [51] and was analysed in great detail with several different zoom levels in [30].

Two modern treatises on the topic include Knoebel [43] who coined the term *infinitely iterated exponential* (for ${}^{\infty}b$), and Galidakis [27] who was the first to describe an extension to real n where ${}^{n}b = n+1$ for $-1 \le n \le 0$, called the *linear extension* of tetration [63]. The linear extension has also been described by Hooshmand [39] using different notations[4].

There are three parameters of iterated exponentials $\exp_b^n(x)$: x, n and b. Power series about x are obtained through *regular iteration*, power series about n are obtained through *natural iteration*, and power series about b do not fit into any theory of analytic iteration, so these power series are tetration-specific, thus they are the topic of the next section.

If $x = 1$, then tetration can viewed as a binary operation, and just how exponentiation has terms for univariate functions, similar terms exist for each function of tetration. For constant n a function $(b \mapsto {}^{n}b)$ is called a *tetrate function* [49] (also *hyperpower* [47]), and for constant b a function $(n \mapsto {}^{n}b)$ is called a *tetrational function* [38].

2.1. **Tetration power series.** There are two common classes of power series of tetration about b, Taylor and Puiseux, both of which we will be using quite often in the next few sections. A Puiseux series is normally any power series involving logarithms [67], but here we will be using it in a more specific sense as follows.

Definition 65. A **Puiseux series** expansion of $f(x)$ is a Taylor series expansion of $f(\boldsymbol{e}^x)$.

These power series have great importance in combinatorics, where they are generating functions for Bell numbers,

Lemma 66. *The exponential coefficients of a Puiseux series expansion about* 0 *are the Stirling transform of the exponential coefficients of a Taylor series expansion about* 1.

Proof. Assume we have a Taylor series $f(x) = \sum_{k=0}^{\infty} t_k (x-1)^k$. It is well known that the generating functions for Stirling numbers of the second kind ($\left\{ {n \atop k} \right\}$) are:

$$\sum_{n=0}^{\infty} \left\{ {n \atop k} \right\} \frac{x^n}{n!} = \frac{(\boldsymbol{e}^x - 1)^k}{k!} \tag{67}$$

[4]Hooshmand uses $b^{\underline{n}}$ for tetration and $\mathrm{uxp}_b(n)$ for its linear extension.

so if we start with an expansion of $f(\boldsymbol{e}^x)$ as follows:

$$f(\boldsymbol{e}^x) = \sum_{k=0}^{\infty} t_k(\boldsymbol{e}^x - 1)^k \tag{68}$$

$$= \sum_{k=0}^{\infty} t_k k! \sum_{n=0}^{\infty} \left\{ {n \atop k} \right\} \frac{x^n}{n!} \tag{69}$$

$$= \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \frac{k!}{n!} t_k \tag{70}$$

so the Puiseux series expansion of $f(x)$ is

$$f(x) = \sum_{n=0}^{\infty} p_n \ln(x)^n = \sum_{n=0}^{\infty} \ln(x)^n \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \frac{k!}{n!} t_k \tag{71}$$

or in other words

$$\boxed{n! p_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k! t_k} \tag{72}$$

which completes the proof. □

Corollary 73. *Given* $f(x) = \sum_{k=0}^{\infty} p_k \ln(x)^k = \sum_{k=0}^{\infty} t_k (x-1)^k$

$$\boxed{n! t_n = \sum_{k=0}^{n} \left[{n \atop k} \right] k! p_k} \tag{74}$$

Proof. Trivial, Stirling numbers of the of the first kind ($\left[{n \atop k} \right]$) are inverses of Stirling numbers of the second kind. Since the inverse was proven in (72), so is this. □

These give us great freedom to transform power series without changing the function at all. Some power series transforms (such as differentiation) modify the function, but these do not. To illustrate how these two formulas work, we will experiment with a few simple examples of iterated exponentials, and investigate their properties on $f(x)$, $f^{-1}(x)$, $\ln(f(x))$, and $\ln(f^{-1}(x))$. The simplest of these is the identity function, $f(x) = x$.

$$x = 1 + (x - 1) \tag{75}$$

$$= \sum_{k=0}^{\infty} \frac{\ln(x)^k}{k!} \tag{76}$$

where (75) is a Taylor series and (76) is a Puiseux series. And $\ln f$ is

$$\ln(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(x-1)^k}{k} \tag{77}$$

$$= 0 + \ln(x) \tag{78}$$

where (77) is a Taylor series and (78) is a Puiseux series.

We can continue this on a nontrivial example of $f(x) = x^x = {}^2x$ which gives

$$x^x = 1 + (x-1) + (x-1)^2 + \frac{1}{2}(x-1)^3 + \cdots \tag{79}$$

$$= \sum_{k=0}^{\infty} \frac{(x-1)^k}{k!} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \sum_{i=0}^{j} \binom{j}{i} i^{j-i} \tag{80}$$

$$= 1 + \ln(x) + \frac{3}{2}\ln(x)^2 + \frac{5}{3}\ln(x)^3 + \cdots \tag{81}$$

$$= \sum_{k=0}^{\infty} \frac{\ln(x)^k}{k!} \sum_{j=1}^{k} \binom{k}{j} j^{k-j} \tag{82}$$

The inverse function of $y^y = x$ is $y = \mathrm{srt}_2(x)$ (*super-root*) so these are expansions of f^{-1}

$$\mathrm{srt}_2(x) = \frac{1}{{}^{\infty}(1/x)} = \frac{\ln(x)}{W(\ln(x))} = \exp(W(\ln(x))) \tag{83}$$

$$= 1 + (x-1) - (x-1)^2 + \frac{3}{2}(x-1)^3 + \cdots \tag{84}$$

$$= \sum_{k=0}^{\infty} \frac{(x-1)^k}{k!} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (1-j)^{j-1} \tag{85}$$

$$= 1 + \ln(x) - \frac{1}{2}\ln(x)^2 + \frac{2}{3}\ln(x)^3 + \cdots \tag{86}$$

$$= \sum_{k=0}^{\infty} \frac{\ln(x)^k}{k!} (1-k)^{k-1} \tag{87}$$

where $0^0 = 1$. The Lagrange inverse series of (87) then gives expansions of $\ln f$

$$\ln(x^x) = 0 + (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \cdots \tag{88}$$

$$= \sum_{k=0}^{\infty} \frac{(x-1)^k}{k!} \sum_{j=1}^{k} \begin{bmatrix} k \\ j \end{bmatrix} j = \sum_{k=2}^{\infty} \frac{(-1)^k (x-1)^k}{k(k-1)} \tag{89}$$

$$= 0 + \ln(x) + \ln(x)^2 + \frac{1}{2}\ln(x)^3 + \cdots \tag{90}$$

$$= \sum_{k=1}^{\infty} \frac{\ln(x)^k}{k!} k = \sum_{k=1}^{\infty} \frac{\ln(x)^k}{(k-1)!} \tag{91}$$

Going back to (82), its inverse series gives the expansions of $\ln f^{-1}$

$$\ln(\mathrm{srt}_2(x)) = W(\ln(x)) = 0 + (x-1) - \frac{3}{2}(x-1)^2 + \frac{17}{6}(x-1)^3 + \cdots \tag{92}$$

$$= \sum_{k=0}^{\infty} \frac{(x-1)^k}{k!} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (-j)^{j-1} \tag{93}$$

$$= 0 + \ln(x) - \ln(x)^2 + \frac{3}{2}\ln(x)^3 - \frac{8}{3}\ln(x)^4 + \cdots \tag{94}$$

$$= \sum_{k=0}^{\infty} \frac{\ln(x)^k}{k!} (-k)^{k-1} \tag{95}$$

Notice that we knew closed form expressions for each of the functions listed above, even the inverse functions. So each of these we could have obtained by differentiating these well-known functions. For x^{x^x} on the other hand, no such closed form is known for the inverse function, so letting $f(x) = x^{x^x} = {}^3x$ we find the expansions are:

$$x^{x^x} = 1 + (x-1) + \frac{1}{2}(x-1)^2 + \frac{3}{2}(x-1)^3 + \frac{4}{3}(x-1)^4 + \frac{3}{2}(x-1)^5 + \cdots \tag{96}$$

$$= 1 + \ln(x) + \frac{3}{2}\ln(x)^2 + \frac{8}{3}\ln(x)^3 + \frac{101}{24}\ln(x)^4 + \frac{63}{10}\ln(x)^5 + \cdots \tag{97}$$

Taking the inverse series of (96) gives expansions of f^{-1}

$$\mathrm{srt}_3(x) = 1 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{7}{6}(x-1)^4 - \frac{17}{4}(x-1)^5 + \cdots \tag{98}$$

$$= 1 + \ln(x) - \frac{1}{2}\ln(x)^2 - \frac{1}{3}\ln(x)^3 + \frac{11}{8}\ln(x)^4 - \frac{23}{15}\ln(x)^5 + \cdots \tag{99}$$

The inverse series of (99) then gives expansions of $\ln f$

$$\ln(x^{x^x}) = 0 + (x-1) + \frac{1}{2}(x-1)^2 + \frac{5}{6}(x-1)^3 + \frac{1}{12}(x-1)^4 + \frac{11}{30}(x-1)^5 + \cdots \tag{100}$$

$$= 0 + \ln(x) + \ln(x)^2 + \frac{3}{2}\ln(x)^3 + \frac{5}{3}\ln(x)^4 + \frac{41}{24}\ln(x)^5 + \cdots \tag{101}$$

Going back, the inverse series of (97) gives expansions of $\ln f^{-1}$

$$\ln(\mathrm{srt}_3(x)) = 0 + (x-1) - \frac{3}{2}(x-1)^2 + \frac{11}{6}(x-1)^3 - \frac{13}{12}(x-1)^4 + \cdots \tag{102}$$

$$= 0 + \ln(x) - \ln(x)^2 + \frac{1}{2}\ln(x)^3 + \frac{5}{6}\ln(x)^4 - \frac{59}{24}\ln(x)^5 + \cdots \tag{103}$$

These are all new power series that we could not have found by differentiating a closed form, since no closed form is known for the inverse function of $f(x) = x^{x^x}$. Thus equations (72) and (74) were the only way to find these new power series. Now that we have shown the usefulness of these equations, we can return to more well-known power series, those associated with the infinite tetrate (or the infinitely iterated exponential [43]) $f(x) = {}^\infty x$.

$$ {}^{\infty}x = \frac{W(-\ln(x))}{-\ln(x)} \tag{104} $$

$$ = 1 + (x-1) + (x-1)^2 + \frac{3}{2}(x-1)^3 + \cdots \tag{105} $$

$$ = \sum_{k=0}^{\infty} \frac{(x-1)^k}{k!} \sum_{j=1}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (j+1)^{j-1} \tag{106} $$

$$ = 1 + \ln(x) + \frac{3}{2}\ln(x)^2 + \frac{8}{3}\ln(x)^3 + \cdots \tag{107} $$

$$ = \sum_{k=1}^{\infty} \frac{\ln(x)^k}{k!}(k+1)^{k-1} \tag{108} $$

The inverse of (106) gives expansions of f^{-1}

$$ x^{1/x} = \mathrm{srt}_{\infty}(x) = 1 + (x-1) - (x-1)^2 + \frac{1}{2}(x-1)^3 + \cdots \tag{109} $$

$$ = \sum_{k=0}^{\infty} \frac{(x-1)^k}{k!} \sum_{j=1}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \sum_{i=0}^{j} \binom{j}{i} (-i)^{j-i} \tag{110} $$

$$ = 1 + \ln(x) - \frac{1}{2}\ln(x)^2 - \frac{1}{3}\ln(x)^3 + \cdots \tag{111} $$

$$ = \sum_{k=1}^{\infty} \frac{\ln(x)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} (-j)^{k-j} \tag{112} $$

The inverse of (112) gives expansions of $\ln f$

$$ {}^{\infty}x \ln(x) = 0 + (x-1) + \frac{1}{2}(x-1)^2 + \frac{5}{6}(x-1)^3 + \cdots \tag{113} $$

$$ = \sum_{k=0}^{\infty} \frac{(x-1)^k}{k!} \sum_{j=1}^{k} \begin{bmatrix} k \\ j \end{bmatrix} j^{j-1} \tag{114} $$

$$ = 0 + \ln(x) + \ln(x)^2 + \frac{3}{2}\ln(x)^3 + \frac{8}{3}\ln(x)^4 \cdots \tag{115} $$

$$ = \sum_{k=1}^{\infty} \frac{\ln(x)^k}{k!} k^{k-1} \tag{116} $$

And finally, the inverse of (108) gives expansions of $\ln f^{-1}$

$$\ln(x^{1/x}) = 0 + (x-1) - \frac{3}{2}(x-1)^2 + \frac{11}{6}(x-1)^3 + \cdots \tag{117}$$

$$= \sum_{k=0}^{\infty} \frac{(x-1)^k}{k!} \sum_{j=1}^{k} \begin{bmatrix} k \\ j \end{bmatrix} j(-1)^{j-1} \tag{118}$$

$$= 0 + \ln(x) - \ln(x)^2 + \frac{1}{2}\ln(x)^3 - \frac{1}{6}\ln(x)^4 \cdots \tag{119}$$

$$= \sum_{k=1}^{\infty} \frac{\ln(x)^k}{k!} k(-1)^{k-1} \tag{120}$$

Theorem 121. *The Taylor series of integer tetration* ($^n x = \exp_x^n(1), n \in \mathbb{Z}$) *is*

$$^n x = \sum_{k=0}^{\infty} t_{nk}(x-1)^k \quad \textit{where} \tag{122}$$

$$t_{nk} = \begin{cases} 1 & \textit{if } n \geq 0 \textit{ and } k = 0, \\ 0 & \textit{if } n = 0 \textit{ and } k > 0, \\ 1 & \textit{if } n = 1 \textit{ and } k = 1, \\ 0 & \textit{if } n = 1 \textit{ and } k > 1, \\ & \textit{otherwise:} \end{cases} \tag{123}$$

$$\boxed{t_{nk} = \frac{1}{k} \sum_{j=1}^{k} \frac{1}{j} \sum_{i=j}^{k} i(-1)^{j-1} t_{n(k-i)} t_{(n-1)(i-j)}} \tag{124}$$

Proof. According to Galidakis [27], the Puiseux series expansion of $^n x$ about $x = 0$ is

$$^n x = \sum_{k=0}^{\infty} p_{nk} \ln(x)^k \quad \text{where} \quad p_{nk} = \frac{1}{k} \sum_{j=1}^{k} j p_{n(k-j)} p_{(n-1)(j-1)} \tag{125}$$

with different initial conditions. This, together with corollary (74) allow us to transform the Puiseux coefficients into Taylor coefficients as follows.

$$p_{nk} = \frac{1}{k}\sum_{j=1}^{k} j p_{n(k-j)} p_{(n-1)(j-1)} \tag{126}$$

$$t_{nm} = \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix} \frac{k!}{m!} \frac{1}{k} \sum_{j=1}^{k} j p_{n(k-j)} p_{(n-1)(j-1)} \tag{127}$$

$$= \sum_{k=0}^{m}\sum_{j=1}^{k} \begin{bmatrix} m \\ k \end{bmatrix} \frac{k!}{m!} \frac{j}{k} \left(\sum_{i=0}^{\infty} \begin{Bmatrix} k-j \\ i \end{Bmatrix} \frac{i! t_{ni}}{(k-j)!} \right) \left(\sum_{i=0}^{\infty} \begin{Bmatrix} j-1 \\ i \end{Bmatrix} \frac{i! t_{(n-1)i}}{(j-1)!} \right) \tag{128}$$

which is recurrence equation in t_{nk} only. This proves that such a recurrence equation exists. It is left as an exercise to the reader to simplify (128) into (124). □

2.2. **Exponential commutator.** The equation $x^y = y^x$ was first studied by Hengel [65], then by [15], [16], [22], [24], [36], [40], and most recently investigated by Knoebel [43]. The interest seems to be in that multiplication is commutative, but exponentiation is not. Also, there seems to be interest in finding intervals of x and y over which $(x^y - y^x)$ has constant sign, for example $2^x - x^2 < 0$ for all $2 < x < 4$, and $4^x - x^4 > 0$ for all $2 < x < 4$.

The only integers that satisfy this equation are 2 and 4 ($2^4 = 4^2$), however, there are an infinite number of rational solutions, which were first studied by Euler [21], then Bernoulli and Goldbach [31], and reviewed by Knoebel [43].

Definition 129. Let $E(x)$ satisfy $E(\boldsymbol{e}) = \boldsymbol{e}$ and $E(x)^x = x^{E(x)}$ where $E(x) \neq x$. Let $H(x) = {}^{\infty}x$ and $H_k(x) = \frac{W_k(-\ln(x))}{-\ln(x)}$ where W is the Lambert W function [13]. $E(x)$ can also be defined as $E(x) = H_0(x^{1/x})$ for $x > \boldsymbol{e}$ and $E(x) = H_{-1}(x^{1/x})$ for $x < \boldsymbol{e}$.

Theorem 130. *A closed form of the derivative of $E(x)$ is*

$$E'(x) = \frac{E(x)^2(1 - \ln(x))}{x^2(1 - E(x)\ln(x^{1/x}))} \tag{131}$$

Proof. According to Corless *et.al.* [13], we know that $W'(x) = W(x)/x(1 + W(x))$, from which we can show that $H'(x) = H(x)^2/x(1 - H(x)\ln(x))$. Using the chain rule, the main branch (where $k = 0$), and $E(x) = H(x^{1/x})$ as above, it follows that

$$E'(x) = (H(x^{1/x}))' = H'(x^{1/x})(x^{1/x})' \tag{132}$$

$$= \frac{H(x^{1/x})^2}{x^{1/x}(1 - H(x^{1/x})\ln(x^{1/x}))} \frac{x^{1/x}(1 - \ln(x))}{x^2} \tag{133}$$

$$= \frac{E(x)^2(1 - \ln(x))}{x^2(1 - E(x)\ln(x^{1/x}))} \tag{134}$$

which completes the proof. □

The differential equation (131) and $E(4) = 2$ can be used to find a power series of $E(x)$:

$$E(x) = 2 + \frac{2\ln(2) - 1}{4(\ln(2) - 1)}(x - 4) + \frac{2\ln(2)^2 - 6\ln(2) + 3}{64(\ln(2) - 1)^3}(x - 4)^2 + \cdots \tag{135}$$

but only about $x \neq \boldsymbol{e}$, since $x = \boldsymbol{e}$ would make the denominator zero. For $x = \boldsymbol{e}$, multiply both sides of (131) by the denominator, and the method of unknown coefficients gives:

$$E(x) = \boldsymbol{e} - (x - \boldsymbol{e}) + \frac{5}{3\,\boldsymbol{e}}(x - \boldsymbol{e})^2 - \frac{25}{9\,\boldsymbol{e}^2}(x - \boldsymbol{e})^3 + \frac{1243}{270\,\boldsymbol{e}^3}(x - \boldsymbol{e})^4 + \cdots \tag{136}$$

and since the nearest singularity is $x = 1$, this would have radius of convergence $\boldsymbol{e} - 1$.

After some inspection, we can rearrange the series to have rational coefficients:

$$\boxed{\frac{1}{\boldsymbol{e}}E(\boldsymbol{e}(x + 1)) = 1 - x + \frac{5}{3}x^2 - \frac{25}{9}x^3 + \frac{1243}{270}x^4 - \frac{1229}{162}x^5 + \cdots} \tag{137}$$

which would have radius of convergence $1 - \frac{1}{\boldsymbol{e}}$. Neither this method nor these coefficients seem to be in the literature. Power series (136) also permits the approximation

$$E(x) = \frac{\boldsymbol{e}(2x + \boldsymbol{e})}{5x - 2\boldsymbol{e}} + O((x - \boldsymbol{e})^4) \tag{138}$$

which can be used as an approximation near $x = \boldsymbol{e}$, especially when evaluating $E(x)$ using $H(x^{1/x})$ since it has singularity at $x = \boldsymbol{e}$, which makes computation difficult.

2.3. **Bifurcation point of $^{\infty}x$ at $x = \boldsymbol{e}^{-\boldsymbol{e}}$.** Both Galidakis [26] and Knoebel [43] have studied 2-periodic points of exponentials. Also, according to Geisler [30], the largest region of bases with 2-periodic points are found approximately where $|x| < \boldsymbol{e}^{-\boldsymbol{e}}$.

Definition 139. Let $B(x)$ satisfy $B\left(\frac{1}{\boldsymbol{e}}\right) = \boldsymbol{e}^{-\boldsymbol{e}}$ and $x = B(x)^{B(x)^x}$ where $x \neq B(x)^x$, or in other words, x is a strictly 2-periodic point of the base-B exponential function.

Three points that are known to lie on this function are $B\left(\frac{1}{\boldsymbol{e}}\right) = \boldsymbol{e}^{-\boldsymbol{e}}$ (by definition), $B\left(\frac{1}{2}\right) = \frac{1}{16}$ and $B\left(\frac{1}{4}\right) = \frac{1}{16}$, because $(1/16)^{1/2} = 1/4$ and $(1/16)^{1/4} = 1/2$.

Theorem 140. *A closed form of the derivative is*

$$B'(x) = \frac{B(x)(1 - x\ln(x)\ln(B(x)))}{x(B(x)^x + x\ln(x))} \tag{141}$$

Proof. Starting with the definition and differentiating, we find:

$$x = B^{B^x} \tag{142}$$

$$1 = \frac{(B^{B^x})B^x(B\ln(B)^2 + (1 + x\ln(B))B')}{B} \tag{143}$$

$$B = (B^{B^x})B^x(B\ln(B)^2 + (1 + x\ln(B))B') \tag{144}$$

$$= xB\ln(x)\ln(B) + x(B^x + x\ln(x))B' \tag{145}$$

and after solving for $B'(x)$

$$B' = \frac{B(1 - x\ln(x)\ln(B))}{x(B^x + x\ln(x))} \tag{146}$$

we obtain (141). □

Shortly after this derivation [54], Romerio discovered that $B(x) = \exp(W(x\ln(x))/x)$ [55] which was then verified to satisfy (141) and expand to the same power series

$$B(x) = \boldsymbol{e}^{-\boldsymbol{e}} + 2\,\boldsymbol{e}^{2-\boldsymbol{e}}\left(x - \frac{1}{\boldsymbol{e}}\right) + \frac{4\,\boldsymbol{e} - 5}{2}\,\boldsymbol{e}^{3-\boldsymbol{e}}\left(x - \frac{1}{\boldsymbol{e}}\right)^2 + \cdots \tag{147}$$

which has a radius of convergence of $\frac{1}{\boldsymbol{e}}$ because of the singularities at 0 and 1.

3. Nested Exponentials

A *nested exponential* is the solution E_1 of a recurrence equation of the form $E_k = a_k^{E_{k+1}}$ together with a given value of E_n. This encompasses a wide variety of functions and expressions related to tetration and iterated exponentials. We use the notation

$$\mathop{\mathrm{T}}_{k=1}^{n} a_k = \mathop{\mathrm{T}}_{k=1}^{n-1} (a_k; a_n) = a_1^{a_2^{\cdot^{\cdot^{a_{n-1}^{a_n}}}}} \tag{148}$$

s which is a synthesis of notations used by Barrow [7] and Shell ([56], [57]), who used the letter E (*epsilon*), with Brunson [10] and Thron [62] who used the letter T (*tau*).

One of the first occurances of nested exponentials appears to be in Hardy [35] (hence the term *Hardy space*) and they have since been mentioned in [4], [18], [34], [60] and numerous articles by Galidakis ([26], [28]) and Yukalov *et.al.* ([50], [68], [69], [70], [71]).

In [7], Barrow gives an expansion of *infinite exponentials* in a specific form

$$\mathop{\mathrm{T}}_{k=1}^{\infty} \boldsymbol{e}^{x b_k} = \sum_{n=0}^{\infty} x^n \sum_{\substack{k_j \geq 0 \\ 1 \leq j \leq n}} \prod_{i=1}^{n} \frac{(k_{i-1} b_i)^{k_i}}{(k_i)!} \tag{149}$$

where $\sum_{j=1}^{n} k_j = n$ and $k_0 = 1$. It can be generalized to nested exponentials as follows.

Theorem 150. *Nested exponentials have the multiple-sum expansion*

$$\boxed{\mathop{\mathrm{T}}_{k=1}^{n} a_k = \sum_{\substack{k_j \geq 0 \\ 1 \leq j \leq n}} \prod_{i=1}^{n} \frac{(k_{i-1} \ln(a_i))^{k_i}}{(k_i)!}} \tag{151}$$

where $k_0 = 1$.

Proof. We prove by induction. For clarity, we substitute $a_k = \boldsymbol{e}^{b_k}$ without loss of generality. As a basis (case 1) we know that:

$$\mathop{\mathrm{T}}_{k=1}^{1} (\boldsymbol{e}^{b_k}; z) = \boldsymbol{e}^{b_1 z} = \sum_{k_1=0}^{\infty} \frac{(b_1 z)^{k_1}}{(k_1)!} = \sum_{k_1=0}^{\infty} z^{k_1} \frac{(b_1)^{k_1}}{(k_1)!} \tag{152}$$

If we assume the following (case n)

$$\mathop{\mathrm{T}}_{k=1}^{n}(\boldsymbol{e}^{b_k}; z) = \sum_{\substack{k_j \ge 0 \\ 1 \le j \le n}} z^{k_n} \prod_{i=1}^{n} \frac{(k_{i-1}b_i)^{k_i}}{(k_i)!} \tag{153}$$

where $k_0 = 1$, then it follows that:

$$\begin{aligned} \mathop{\mathrm{T}}_{k=1}^{n+1}(\boldsymbol{e}^{b_k}; z) &= \mathop{\mathrm{T}}_{k=1}^{n}(\boldsymbol{e}^{b_k}; \boldsymbol{e}^{b_{(n+1)}z}) \\ &= \sum_{\substack{k_j \ge 0 \\ 1 \le j \le n}} \left(\boldsymbol{e}^{b_{n+1}z}\right)^{k_n} \prod_{i=1}^{n} \frac{(k_{i-1}b_i)^{k_i}}{(k_i)!} \\ &= \sum_{\substack{k_j \ge 0 \\ 1 \le j \le n}} \boldsymbol{e}^{k_n b_{n+1} z} \prod_{i=1}^{n} \frac{(k_{i-1}b_i)^{k_i}}{(k_i)!} \\ &= \sum_{\substack{k_j \ge 0 \\ 1 \le j \le n}} \sum_{k_{n+1} \ge 0} \frac{(k_n b_{n+1} z)^{k_{n+1}}}{(k_{n+1})!} \prod_{i=1}^{n} \frac{(k_{i-1}b_i)^{k_i}}{(k_i)!} \\ &= \sum_{\substack{k_j \ge 0 \\ 1 \le j \le n}} \sum_{k_{n+1} \ge 0} z^{k_{n+1}} \frac{(k_n b_{n+1})^{k_{n+1}}}{(k_{n+1})!} \prod_{i=1}^{n} \frac{(k_{i-1}b_i)^{k_i}}{(k_i)!} \end{aligned} \tag{154}$$

which simplifies to (153)

$$\mathop{\mathrm{T}}_{k=1}^{n+1}(\boldsymbol{e}^{b_k}; z) = \sum_{\substack{k_j \ge 0 \\ 1 \le j \le (n+1)}} z^{k_{n+1}} \prod_{i=1}^{n+1} \frac{(k_{i-1}b_i)^{k_i}}{(k_i)!} \tag{155}$$

By induction, substituting $z = 1$ and $b_k = \log(a_k)$ completes the proof. □

The primary difference between these two expansions is that (149) is well suited for infinite exponentials, whereas (151) is well suited for finite nested exponentials. For example, if one wanted to find an alternate expansion of $\cos(x)$ (as Barrow did in [7]), then one would use (149), but if one wanted expansions of ${}^{n}x$ for finite n, then one would use (151). Both of these power series are tools for expansions of nested exponentials.

Appendix A. Mathematica code

```
CarlemanMatrix[series:SeriesData[x_, x0_, _, _, n_, 1]] :=
    Table[D[Normal[series]^j, {x, k}]/k! /. {x -> x0}, {j, 0, n}, {k, 0, n}];

Iterate[series:SeriesData[x_, x0_, _, _, n_, 1], t_] :=
    SeriesData[x, x0, MatrixPower[CarlemanMatrix[series], t][[2]], 0, n, 1];
```

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