Arborescent Numbers: Higher Arithmetic Operations and Division Trees

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1 Motivation

The original question leading to this work was: Why does the sequence of operations "addition", "multiplication", "exponentiation" (on $\mathbb{R}$) not continue and how can it be made to be continuable? This question is not new, instead it is repeatedly requested (by pupils and lay mathematicians) in mathematical newsgroups, (essence: "What is π times the power of π?") and there are also publications (see [4], [5], [6]) dealing with this theme. The completion of the program "arborescent numbers" could possibly give a satisfying answer to such questions. Let us become familiar with the subject:

Usually the next higher or successor operation with a natural-numbered operand is defined by repetition of the operation itself, for example multiplication and exponentiation are defined by

$$n \cdot x := \underbrace{x + \cdots + x}_{n \times x},$$
$$x^n := \underbrace{x \cdot \cdots \cdot x}_{n \times x}.$$

For an arbitrary binary operation $*$ one would hence define the successor operation $*'$ as

$$n *' x = \underbrace{x * \cdots * x}_{n \times x}.$$

Then $n \cdot x = n +' x$ and $x^n = n \cdot' x = n +'' x$. For addition and multiplication this natural-numbered repetition makes sense because the operations are associative and so bracketing does not matter. For non-associative operations one naturally would use binary trees, i.e. a structure that reflects the bracketing. For example

$$\begin{smallmatrix} \diagup & \diagdown \\ \diagup & \diagdown \end{smallmatrix} *' x = (1, ((1, 1), 1)) *' x := x * ((x * x) * x).$$

Let $\mathbb{B}$ be the set of binary trees in the above sense, i.e. the structure recursively built by taking ordered pairs, starting with 1. Then formally one would define $*' \colon \mathbb{B} \times X \to X$ for $* \colon X \times X \to X$ recursively by

$$\begin{aligned} 1 *' x &:= x, \\ (a_L, a_R) *' x &:= (a_L *' x) * (a_R *' x). \end{aligned}$$

So we can define an unlimited number of successor operations for each operation that is already defined on $X = \mathbb{B}$. We define, starting with the native pair-operation on the binary trees, the operations $\ltimes_n$ on $\mathbb{B}$ inductively by $a \ltimes_1 b := (a, b)$ and $a \ltimes_{n+1} b := a \ltimes'_n b$. We detect that only for the indices 1 (addition), 2 (multiplication) and 3 (exponentiation) can the operation $\ltimes_n$ be homomorphically defined on $\mathbb{N}$ (see proposition 12 and proposition 22). It may be mentioned that the idea of binary tree arithmetic (though with a different addition than under consideration here) was already intriguing to Loday (see [22]) who used it to construct the free dendriform algebra. The manifest idea, to define higher operations on binary trees, was explicated by Blondel[1] [6].

[1] As appeared little before the editorial deadline.

The other way to work around the obstacle of bracketing a repeated nonassociative operation is to use fixed bracketing. So the hyperpower (also called tetration) uses right-first bracketing:

$$
\begin{aligned}
{}^{1}x &:= x \\
{}^{n+1}x &:= x^{{}^{n}x}.
\end{aligned}
$$

There is much interest in hyperpowers, they are for example addressed in [2], [23] and [19]. The left-first bracketing for powers $n{*}x := ((x^x)\cdots)^x$ is not very interesting because it is equal to $x^{x^{n-1}}$. The original Ackermann's function (as defined in [1]) — though not used for investigation of higher operations — repeats a left-first bracketing for arbitrarily high operations, in the following way. Let the successor operation be defined by

$$
n *' x = \underbrace{((\cdots(x * x) * \cdots) * x)}_{n \times x}
$$

and the higher operations by

$$
\begin{aligned}
n *_0 x &:= n + x, \\
n *_{m+1} x &:= n *'_m x
\end{aligned}
$$

then the original Ackermann's function is mainly defined by $A(x, m, n) := m *_n x$. It is easily seen, that $m *_1 x = m \cdot x$, $m *_2 x = x^m$ and $m *_3 x = {}^{m}x$. (Because in $m *_2 x = x^m$ the operand order is swapped, left-bracketing of $*_2$ corresponds to right-bracketing of powers.) So we have definitions of arbitrarily high operations from the Ackermann's function. But the main hassle is that there is no unique or preferable way to (continuously) extend this definition to fractional and then real numbers, as it is possible with multiplication and exponentiation. This is reflected by some concurrent definitions of continuous hyperpowers scattered across the Internet (see [14], [13] and [25]). The question whether to extend the original Ackermann's function to a continuous one (presumably in "the" natural way) was also raised by Wolfram in [33].

Lets have a look why there is "the" unique extension of exponentiation and multiplication to fractional exponents and multipliers, respectively. Consider $n * x$ being nx on $X = \mathbb{R}$ or being x^n on $X = \mathbb{R}_+$ in

Proposition 1. *Let $*\colon \mathbb{N} \times X \to X$ be an operation which satisfies the multiplicative translation equation*

$$
\begin{aligned}
1 * x &= x \\
m * (n * x) &= (mn) * x,
\end{aligned}
$$

and that $f_n(x) := n * x$ is bijective for each $n \in \mathbb{N}$. Then there exists a unique extension of $*$ to $\circledast\colon \mathbb{Q}_+ \times X \to X$ which satisfies that translation equation.

The reader may prove that the following definition of extension is well-defined, valid and unique. For $p, q \in \mathbb{N}$, $x \in X$ define

$$
\frac{p}{q} \circledast x := (f_p \circ {f_q}^{-1})(x).
$$

Because this translation equation is satisfied for multiplication (i.e. $(ab)x = a(bx)$) and for exponentiation (i.e. $x^{ab} = (x^b)^a$) the respective extensions are unique (on $X = \mathbb{R}$ and $X = \mathbb{R}_+$ respectively).

Interestingly Frappier investigated in [12] an exponential operation $\boxdot\colon \mathbb{N} \times \mathbb{C} \to \mathbb{C}$ using another bracketing system rather than hyperpowers, namely

$$0 \boxdot z = z,$$
$$(n+1) \boxdot z = (n \boxdot z)^{n \boxdot z}.$$

It satisfies a similar equation (apart from the problem of uniqueness of exponentiation in the complex number plane here), the *additive* translation equation:

$$0 * x = x,$$
$$m * (n * x) = (m + n) * x.$$

It provides for unique extension to $\mathbb{Z}$ (under corresponding conditions of proposition 1). He additionally extends the operation to the rational, real and complex numbers, a problem that is well-known under the name fractional/real/complex iteration of functions (to see the equivalence let $f(x) = 1 * x$ then $n * x$ for $n \in \mathbb{N}$ is the n-th iterate of f usually written as $f^n(x)$, in Frappier's case for example $f(z) = z^z$ and $n \boxdot z = f^n(z)$). However, the additive translation equation often has the problem of non-unique extensions too (as $\boxdot$ has), see [31]. On the other hand see [21] and [28] for uniqueness conditions. For further references on this interesting topic see [18], [29] and [30] to selectively mention only a few contributions.

The lack of the multiplicative and even the additive translation equation for hyperpowers may be the reason that there is no unique (or natural) extension for them. This is different with higher operations on binary trees, where the multiplicative translation equation

$$(a \ltimes_2 b) \ltimes_n x = a \ltimes_n (b \ltimes_n x)$$

is satisfied for each $n \geq 2$ (see proposition 9). This kernel plants the motivation for an $\mathbb{N} \to \mathbb{Q}_+ \to \mathbb{R}_+$ similar construction beginning with the binary trees, whereto all higher operations can be uniquely extended. A first step for this aim is pointed out here, this step corresponds to the extension of $\mathbb{N}$ to $\mathbb{Q}_+$. We introduce the division structures *division binary trees* and *fractional trees* which are the corresponding extensions of the binary and the left-commutative binary trees. The main result presented here is the resolution of the word decision problem in the division binary trees and the fractional trees (the latter was the hard part, taking two years to solve).

The left-commutative binary trees are the binary trees under the equation $(a, (b, c)) = (b, (a, c))$. We will call them short *lcb-trees* and the pairing of two binary trees (a, b) as their addition. The concept of (labelled) left-commutative binary trees with this addition, was by the way already used in [10] to construct the free Novikov-Algebra. It seems that left/right-commutativity up to now was considered only for semigroups and the multiplication in an algebra or ring. Here the left-commutative addition plays a key role as the base operation (comparable to the *addition* of a ring) in the investigation of higher operations, because it is compatible with all higher operations (in the sense of proposition 17). Further the set of lcb-trees equipped with $\ltimes_1$ and $\ltimes_2$ is isomorphic to

the set of functions $f\colon \mathbb{R}_{>1} \to \mathbb{R}_{>1}$ generated from id by the process of raising a function to the power of another function equipped with the swapped power $\mathtt{y}^{\mathtt{x}}$ and function concatenation (see proposition 35). They allow easy inversion of multiplication by taking inverse functions, leading to another representation of the fractional trees.

To finish the motivation let us summarise the overall intentions in the following diagram. In the top row the free algebraic structures are listed which become specialised/factored by the equations listed in the left column.

$$
\begin{array}{ccccccc}
 & \text{1-magma} & \xrightarrow[\text{embed}]{\text{division}} & \text{coppice} & \xrightarrow[\text{embed}]{\text{convergence}} & \begin{array}{c}\text{complete}\\ \text{coppice}\end{array} \\
\text{no } \oplus\text{-equations} & \mathbb{B} & \longrightarrow & \mathbb{B}^\circ & \longrightarrow & \overline{\mathbb{B}^\circ}(?) \\
\downarrow \text{ factor} & \downarrow & & \downarrow & & \downarrow \\
a \oplus (b \oplus c) = b \oplus (a \oplus c) & \mathbb{P} & \longrightarrow & \mathbb{F} & \longrightarrow & \overline{\mathbb{F}}(?) \\
\downarrow \text{ factor} & \downarrow & & \downarrow & & \downarrow \\
(a \oplus b) \oplus c = a \oplus (b \oplus c) & \mathbb{N} & \longrightarrow & \mathbb{Q}_+ & \longrightarrow & \mathbb{R}_+
\end{array}
$$

The term "magma" was established by Bourbaki in [7] and denotes simply a set with an operation $\oplus$ on it. "1-magma" simply adds the constant 1 to the magma. *Coppice* is an algebraic structure introduced in this work to reflect invertible multiplication with 1-magmas/trees. $\mathbb{B}$ are the binary trees (same as the free 1-magma), $\mathbb{P}$ are the left-commutative trees, $\mathbb{F}$ stands for fractional trees. $\mathbb{B}^\circ$ is called the division binary trees. The addition $\oplus$ on $\mathbb{B}$ is the first operation $\ltimes_1$ in our sequence of arbitrarily high operations.

The constructions $\mathbb{N} \to \mathbb{Q}_+ \to \mathbb{R}_+$ are already well known. The overall program "arborescent numbers" aims to similarly repeat these constructions beginning with (specific) binary trees instead of the natural numbers. The trees mainly considered are $\mathbb{P}$ and $\mathbb{B}$. The program is completed if the higher operations ($\ltimes_m$, $m \geq 2$) defined at $\mathbb{P}$ (and $\mathbb{B}$) are extended to (possibly a subset of) $\overline{\mathbb{F}}$ (and $\overline{\mathbb{B}^\circ}$) so that they are continuous (and hence $x \mapsto a \ltimes_i x$ bijective for all a) and satisfying the defining equations on the extended set. For a more precise description see definition 45 and the following conjecture 21.

At last and at least the fractional trees can be seen in the company of quaternions, octonions and Conway numbers in that they are beautiful number systems, having some exotic flavour though.

1.1 Chapter Overview

Chapter 2 "Conventions and Preliminaries" clarifies the general notation used, introduces the multiset notation which is frequently used later and reviews the concept of a free algebraic structure which is needed in the later constructions.

In chapter 3 "Tree Arithmetics and Higher Operations", the binary trees $\mathbb{B}$ are properly introduced and the higher operations defined on them. A property of the higher operations is shown that naturally leads to the definition of the left-commutative binary trees (short lcb-trees) $\mathbb{P}$ on which all higher operations $\ltimes_i$ are still definable (contrary to the natural numbers which are the associative binary trees). A prime factorisation theorem for the binary and lcb-trees enables us to show that all higher operations are injective as functions in the right operation argument.

Chapter 4 "Power-Iterated Functions" shows that the power-iterated functions (with the swapped power as addition and function composition as multiplication) are isomorphic

to the lcb-trees. These power-iterated functions can be easily extended with inverses, i.e. embedding the multiplicative/compositional semigroup structure into a group structure by adding inverses.

This motivates the definition of a precoppice and a coppice and the question of embedding a precoppice into a coppice in chapter 5 "Coppices". The natural numbers/the lcb-trees/the binary trees are the free/free left-commutative/free associative precoppices and each is embeddable into the free/free left-commutative/free associative coppice which are the division binary trees/fractional trees/fractional numbers respectively. In each of these free coppices equality can be decided, the most difficult part however is to show it for the fractional trees in chapter 5.3.

In order to further follow the program "arborescent numbers", chapter 6 investigates order and topology as it is assumed to be necessary for the arborescent equivalent of the extension $\mathbb{Q}_+ \to \mathbb{R}_+$. Though this chapter has a noticeable lack of striking, constructive results, a negative result for the tree structures is already given, i.e. the tree structures can not be linearly ordered right-compatibly with all higher operations. However they can be non-trivially partially ordered compatibly with all higher operations.

Still on the hunt for orders and topology, in chapter 7 "Power-Inverse-Iterated Functions" we have a look at the coppice of the power-inverse-iterated functions regarding some conjectures. In particular the question is open as to whether the power-inverse-iterated functions are isomorphic to the fractional trees.

In chapter 8 "Prospects" attempts to render more precisely which tasks should be solved in the future, by introducing the concept of a higher operations coppice.

2 Conventions and Preliminaries

2.1 Conventions

Let us adopt the following most basic determinations. Further explanations can be found in the text after these points. You should also become acquainted with the non-standard indexing notation in 2.2.

- "free", without specification of a generating set, means "free by the empty generating set" (which is useful of course if there are constants in the signature). In category theory this property is also called "initial".
- A *factor* of an algebraic structure $\mathbf{A}$ is the image of an homomorphism from $\mathbf{A}$, or equivalently the quotient of $\mathbf{A}$ by some congruence relation.
- Expressions like $\mathtt{x} \oplus \mathtt{y}$, $\mathtt{xy}$, $\mathtt{y}^{\mathtt{x}}$, $\mathtt{x}^{\sim}$, ... denote the *whole* operation, and are short for $(x, y) \mapsto x \oplus y$, $(x, y) \mapsto xy$, $(x, y) \mapsto y^x$, $x \mapsto x^{\sim}$ respectively.
- f^n means multiplicative power as opposed to $f^{\circ n}$ which means the n-times iteration of f (for a function $f\colon X \to X$). However f^{-1} means the inverse function as opposed to $\frac{1}{f}$.
- "order" always means "partial order", as opposed to "linear order".
- $\mathbb{N}$, $\mathbb{Q}_+$ and $\mathbb{R}_+$ do *not* contain 0.

- r.h. means induction/recursion hypothesis
- $A \setminus^\circ B$ means $A \setminus B$ together with $B \subseteq A$.

Instead of writing only an operation symbol $+$, or $(x, y) \mapsto x + y$ we will write $\mathtt{x} + \mathtt{y}$, which denotes the whole operation and not a certain value. This is particularly useful when writing the invisible operation symbol $(x, y) \mapsto xy$ as $\mathtt{xy}$ or writing the inversion symbol $x \mapsto x^\sim$ as $\mathtt{x}^\sim$ for example in an algebraic structure $(X, 1, \mathtt{x} + \mathtt{y}, \mathtt{xy}, \mathtt{x}^\sim)$ we can see directly the arity and usage of the symbols (up to trinary operations with $\mathtt{x}, \mathtt{y}, \mathtt{z}$). It is also quite intuitive if we do not mistake it as a certain value.

The term *binary tree* is always used in short for *full ordered binary tree* which is also referred to as *planar binary rooted trees*. (*full binary* means that each node has exactly zero or two children; *ordered* that the order of the children matters).

$\mathbb{Q}_+$ denotes the fractional/rational numbers $x > 0$ and $\mathbb{R}_+$ denotes the real numbers $x > 0$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Application of the function f to x is mainly written as usual $f(x)$ but for homomorphism φ to avoid parentheses we will also write x^φ. For two functions $f, g \colon X \to X$ and an arbitrary operation $\mathtt{x} * \mathtt{y}$ on X let $(f * g)(x) := f(x) * g(x)$. $\mathrm{id} \colon X \to X$ is the identity function on X, defined by $\mathrm{id}(x) = x$ for all $x \in X$.

2.2 Multisets

We can regard a multiset M (with elements of X) as its characteristic function, mapping from the set of allowed elements X to their count/multiplicity in M. Let $M^\chi \colon X \to \mathbb{N}_0$ denote that characteristic function of the multiset M. We are only concerned with finite multisets here. Multisets are written using brackets instead of the curly braces used for sets, for example $[1, 2, 2] = [2, 1, 2] \neq [1, 2]$. We can convert multisets to sets by pairing each n-time occurring element with the indices $1 \ldots n$. For example $\sigma([a, b, b]) = \{(a, 1), (b, 1), (b, 2)\}$, generally $\sigma(M) := \bigcup_{x \in M} \{x\} \times \{1, \ldots, M^\chi(x)\}$. And we can convert back each set $P \subseteq S \times \mathbb{N}$ to the multiset $\mu(P)$ by $\mu(P)^\chi(x) := |\{i : (x, i) \in P\}|$. When using a set operation $\mathtt{x} * \mathtt{y}$ (particularly $\mathtt{x} \cup \mathtt{y}$, $\mathtt{x} \cap \mathtt{y}$, $\mathtt{x} \setminus \mathtt{y}$) on multisets L and M it means by default $L * M := \mu(\sigma(L) * \sigma(M))$ if applicable. And if using a set relation $R(\mathtt{x}, \mathtt{y})$ (particularly $\mathtt{x} \subset \mathtt{y}$, $\mathtt{x} \subseteq \mathtt{y}$) on multisets L, M it means $R(L, M) \iff R(\sigma(L), \sigma(M))$. Beside this we use the operation $\mathtt{x} \uplus \mathtt{y}$ on multisets defined by $(L \uplus M)^\chi := L^\chi + M^\chi$.

For finite sequences, multisets and sets we often have to deal with indexing. For convenience we use the *index-context* notation: For each indexable element a, mainly $a = \langle a_1, \ldots, a_k \rangle$ or $a = [a_1, \ldots, a_k]$ or $a = \{a_1, \ldots, a_k\}$, when writing a_* then let expand the index until the next indexable context. If we want to expand to a certain context, we name the index, for example with i, and mark the context with the same variable i. The following examples shall convey that notation:

$$
\begin{aligned}
\langle a_* \rangle &\quad \text{means} \quad \langle a_1, \ldots, a_k \rangle \\
\langle a_* b \rangle &\quad \text{means} \quad \langle a_1 b, \ldots, a_k b \rangle \\
\langle a_*, b_* \rangle &\quad \text{means} \quad \langle a_1, \ldots, a_k, b_1, \ldots, b_l \rangle \\
\Pi^*(a_* \oplus c) \text{ and } \Pi_i^*(a_i \oplus c) &\quad \text{means} \quad (a_1 \oplus c) * (a_2 \oplus c) * \cdots * (a_k \oplus c) \\
\Pi_i(a_i \oplus a_i) * b &\quad \text{means} \quad ((a_1 \oplus a_1)(a_2 \oplus a_2) \cdots (a_k \oplus a_k)) * b \\
\Pi_i^{\uplus}\{a_i \oplus b_*\} &\quad \text{means} \quad \{a_1 \oplus b_1, \ldots, a_1 \oplus b_l\} \uplus \cdots \uplus \{a_k \oplus b_1, \ldots, a_k \oplus b_l\}
\end{aligned}
$$

Of course it makes no sense to expand a set to a sequence (i.e. to write $\langle a_* \rangle$ for $a = \{a_1, \ldots, a_k\}$), because the index order is lost. The opposite expansion on the other hand is useful.

2.3 Free Algebras

From universal algebra we only need the following adapted content for the later considerations. We stick to the presentation in [8]. Particularly constants are considered to be first class (0-ary) operations, and are not separately treated. Outside this chapter we will use the term "algebraic structure" instead of "algebra" here (which is here meant in the sense of universal algebra, but which is ambiguous outside of this context).

In [8] the free algebra (over X) is defined for a class of algebras K, as being the term-algebra $\mathbf{T}$ (over X) factored by the intersection of all congruence relations θ on $\mathbf{T}$ that "map" $\mathbf{T}$ into some algebra of K. For the case of K being the class of all algebras satisfying a certain set of equations Σ and the set of generating variables being empty, we gather:

Definition 1 (free algebra). The *free algebra* $\mathbf{F}_{\mathfrak{F},\Sigma}$ of type $\mathfrak{F}$ with equations Σ is the quotient of the term algebra $\mathbf{T}_{\mathfrak{F}}$ of type $\mathfrak{F}$ (in no variables) by the congruence that is the intersection of all the congruences θ for which $\mathbf{T}_{\mathfrak{F}}/\theta$ satisfies Σ.

The previously mentioned congruence relation between the terms of the term algebra is the same as the following: Two terms s and t are congruent iff there is a sequence of terms $s = s_1 = \ldots = s_n = t$ such that s_{n+1} is made from s_n by substituting a subterm of s_n according to an equation of Σ.

For each algebra $\mathbf{A}$ of type $\mathfrak{F}$ the evaluation of a term by the operations in $\mathbf{A}$ yields a homomorphism $\mathbf{T}_{\mathfrak{F}} \to \mathbf{A}$. In the same way into each algebra $\mathbf{A}$ of type $\mathfrak{F}$ satisfying Σ exists an unique so called *universal homomorphism* φ from the corresponding free algebra. This is also known as the universal mapping property of a free algebra (here for the case of the empty generator set).

Definition 2 (generated). An algebra $\mathbf{A}$ of type $\mathfrak{F}$ is called *to be G-generated*, $G \subseteq A$, if every element of A is an evaluation of a term of type $\mathfrak{F}$ in the variables $\{x_g \colon g \in G\}$ by substituting the variable x_g by g. This is equivalent to $\mathbf{A}$ being a factor of the term algebra in the variables $\{x_g \colon g \in G\}$.

Proposition 2. *Let $\mathbf{A}$ be an algebra of type $\mathfrak{F}$ that satisfies Σ. If $\mathbf{A}$ is $\emptyset$-generated (i.e. a factor of $\mathbf{T}_{\mathfrak{F}}$) and there exists a homomorphism $\psi \colon \mathbf{A} \to \mathbf{F}_{\mathfrak{F},\Sigma}$ then $\mathbf{A}$ is isomorphic to $\mathbf{F}_{\mathfrak{F},\Sigma}$.*

Proof. From an $\emptyset$-generated algebra there is at most one homomorphism to another algebra because the homomorphism is already determined by the generation. Let $\varphi \colon \mathbf{F}_{\mathfrak{F},\Sigma} \to \mathbf{A}$ be the universal homomorphism. Then $\psi \circ \varphi \colon \mathbf{F}_{\mathfrak{F},\Sigma} \to \mathbf{F}_{\mathfrak{F},\Sigma}$ and $\varphi \circ \psi \colon \mathbf{A} \to \mathbf{A}$. By uniqueness $\psi \circ \varphi = \mathrm{id}$ and $\varphi \circ \psi = \mathrm{id}$. So φ and ψ are isomorphisms. $\square$

By the way it makes a difference whether to speak of a free algebra generated by $\{c\}$ or a free algebra with constant c generated by $\emptyset$. By the universal mapping property every map of the generating set of a free algebra satisfying Σ to an other algebra satisfying Σ

can be extended to a homomorphism between them. So if the generating set is empty generally not every map of the constant can be extended but only the map to the constant in the other algebra.

An *embedding* is not merely an injective map but an injective homomorphism.

3 Tree Arithmetic and Higher Operations

In this chapter we will construct the free, the free left-commutative and the free associative 1-magma. We define the higher operations on the binary trees (alias the free 1-magma) and notice that they are definable on the free left-commutative 1-magma (which is equivalent to the rooted trees of graph theory) but not on the natural numbers (alias the associative free 1-magma). We show that the higher operations are all injective in the right variable.

We start with some easy propositions to become familiar with the universal algebra preliminaries.

Definition 3 (1-magma). A set M equipped with a constant $1 \in M$ and a binary operation $\mathtt{x} \oplus \mathtt{y} \colon M \times M \to M$ — referred to as addition — is called an *1-magma*.

Proposition 3. $\mathbb{N}$ *(with* 1 *and* $\mathtt{x} + \mathtt{y}$*) is (isomorphic to) the free associative 1-magma.*

Proof. Let $(\mathcal{N}, 1, \mathtt{x} \oplus \mathtt{y})$ be the free associative 1-magma. We first show that $1 \oplus x = x \oplus 1$ in $\mathcal{N}$ by recursion over the terms of $\mathcal{N}$. For $x = 1$ it holds trivially. Let it be shown already for $x = a$ and $x = b$ and show it for $x = a \oplus b$ by

$$1 \oplus (a \oplus b) = (1 \oplus a) \oplus b = (a \oplus 1) \oplus b = a \oplus (1 \oplus b) = a \oplus (b \oplus 1) = (a \oplus b) \oplus 1.$$

By proposition 2 it suffices to present a homomorphism $\psi \colon \mathbb{N} \to \mathcal{N}$. Define ψ inductively by $\psi(1) := 1$ and $\psi(n+1) := \psi(n) \oplus 1$. Then we show $\psi(m+n) = \psi(m) \oplus \psi(n)$ for all $m \in \mathbb{N}$ by induction over n.

$$\begin{aligned}
n \mapsto 1 \qquad & \psi(m+1) = m^\psi \oplus 1 \\
n \mapsto n+1 \quad & \psi(m + (n+1)) = \psi((m+1)+n) = \psi(m+1) \oplus n^\psi = (m^\psi \oplus 1) \oplus n^\psi \\
& = m^\psi \oplus (1 \oplus n^\psi) = m^\psi \oplus (n^\psi \oplus 1) = m^\psi \oplus \psi(n+1)
\end{aligned}$$

□

Definition 4 ($\mathbb{B}$). Let $\mathbb{B}$ be the free 1-magma.

It has the usual representation as binary trees:

Definition 5 (binary trees $\mathbb{B}_B$,$\mathtt{x}_L$,$\mathtt{x}_R$). Let the set of *binary trees* $\mathbb{B}_B$ be the smallest set containing the element 1 and for each $a, b \in \mathbb{B}_B$ also containing the ordered pair (a, b). Equip $\mathbb{B}_B$ as 1-magma with the constant 1 and the addition $(\mathtt{x}, \mathtt{y})$. For each $a \in \mathbb{B}_B \setminus \{1\}$ define the indices L and R by $(a_L, a_R) := a$.

Proposition 4. $\mathbb{B}_B$ *is isomorphic to* $\mathbb{B}$.

Truly this was a trivial representation, lets have a look at a more interesting one (which we will use later): the representation as ordered trees.

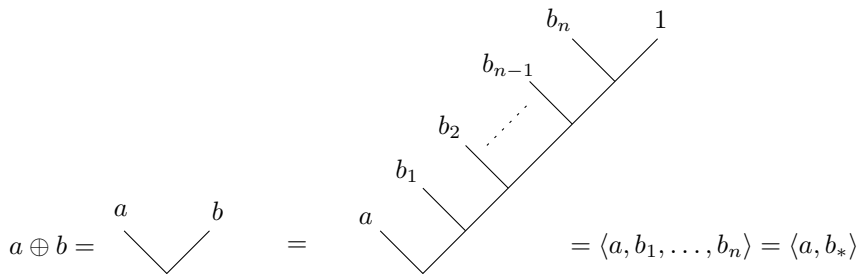
Definition 6 (ordered trees $\mathbb{B}_R$). Let the set of *ordered trees* $\mathbb{B}_R$ be the smallest set containing for each elements $a_1, \ldots, a_n \in \mathbb{B}_R$ ($n \in \mathbb{N}_0$) also the sequence $\langle a_1, \ldots, a_n \rangle$. Equip $\mathbb{B}_R$ as 1-magma with the constant $1 := \langle\rangle$ and the addition $a \oplus b := \langle a, b_* \rangle$.

Proposition 5. *$\mathbb{B}_R$ is isomorphic to $\mathbb{B}$ by the homomorphism $\varrho\colon \mathbb{B}_R \to \mathbb{B}$ defined recursively as $\varrho(\langle a_1, \ldots, a_n \rangle) = \varrho(a_1) \oplus (\cdots(\varrho(a_n) \oplus 1)\cdots)$.*

Proof. We prove the proposition by picture. Every binary tree b can be written in the form

$$b = (b_1, (b_2, (\ldots, (b_n, 1)\cdots)))$$

for some $n \geq 0$ which corresponds to $\langle b_1, \ldots, b_n \rangle$ in $\mathbb{B}_R$. The addition on $\mathbb{B}_R$ is depicted below. □



Definition 7 (successor operation $'$). For an operation $\mathtt{x} \circledast \mathtt{y}\colon X \times X \to X$ define the successor operation $\mathtt{x} \circledast' \mathtt{y}\colon \mathbb{B} \times X \to X$ by

$$\begin{aligned} 1 \circledast' x &:= x, \\ (a_L \oplus a_R) \circledast' x &:= (a_L \circledast' x) \circledast (a_R \circledast' x). \end{aligned}$$

Definition 8 ($\mathtt{x} \ltimes_n \mathtt{y}$, $\mathtt{xy}$, higher operation). For each $n \in \mathbb{N}$ we define the operation $\mathtt{x} \ltimes_n \mathtt{y}$ on $\mathbb{B}$ as the $(n-1)$-times application of $'$ to $\mathtt{x} \oplus \mathtt{y}$, or more formally by induction as

$$\begin{aligned} \mathtt{x} \ltimes_1 \mathtt{y} &:= \mathtt{x} \oplus \mathtt{y}, \\ \mathtt{x} \ltimes_{n+1} \mathtt{y} &:= \mathtt{x} \ltimes_n' \mathtt{y}. \end{aligned}$$

Define $\mathtt{xy} := \mathtt{x} \ltimes_2 \mathtt{y}$ and call it *multiplication*. By convention $\mathtt{xy}$ binds stronger than $\mathtt{x} \oplus \mathtt{y}$. All operations $\mathtt{x} \ltimes_k \mathtt{y}$ with $k \geq 3$ will be called *higher operations*.

Proposition 6. *Let $2 := 1 \oplus 1$ then $2 \ltimes_n 2 = 2 \oplus 2$ for each $n \in \mathbb{N}$.*

Proposition 7. *$1a = a1 = a$ for all $a \in \mathbb{B}$.*

Proposition 8. *$a \ltimes_n 1 = 1$ for all $n \geq 3$ and $a \in \mathbb{B}$.*

Proposition 9 (Translation Equation). *$(ab) \circledast' x = a \circledast' (b \circledast' x)$ for any operation $\mathtt{x} \circledast \mathtt{y}$ on X and all $a, b \in \mathbb{B}$ and $x \in X$.*

Proof. We prove by recursion over a. First let $a = 1$. Then

$$(1b) \circledast' x = b \circledast' x = 1 \circledast' (b \circledast' x).$$

Secondly let $a = a_L \oplus a_R$ and $(a_i b) \circledast' x = a_i \circledast' (b \circledast' x)$ for $i \in \{L, R\}$ already be shown. Then

$$\begin{aligned} & ((a_L \oplus a_R)b) \circledast' x \\ = \; & ((a_L b) \oplus (a_R b)) \circledast' x \\ = \; & ((a_L b) \circledast' x) \circledast ((a_R b) \circledast' x) \\ = \; & (a_L \circledast' (b \circledast' x)) \circledast (a_R \circledast' (b \circledast' x)) \\ = \; & (a_L \oplus a_R) \circledast' (b \circledast' x). \end{aligned}$$

□

In the special case $\mathtt{x} \circledast \mathtt{y} := \mathtt{x} \oplus \mathtt{y}$ we obtain

Proposition 10 (Associativity). *$(ab)c = a(bc)$ for all $a, b, c \in \mathbb{B}$.*

So we also can define a^n for $n \in \mathbb{N}$ in the usual way.

Definition 9 (numeric value $|\mathtt{x}|$). The numeric value of a binary tree $|\mathtt{x}| : \mathbb{B} \to \mathbb{N}$ is recursively defined by $|1| := 1$ and $|a \oplus b| := |a| + |b|$.

It is clear that $|a|$ counts the number of leaves of the binary tree a and that $|\mathtt{x}|$ is the universal epimorphism $\mathbb{B} \to \mathbb{N}$.

Proposition 11. *$a \ltimes_3 x = x^{|a|}$ for all $a, x \in \mathbb{B}$.*

Proof. We prove the proposition by recursion over a.

$$\begin{aligned} 1 \ltimes_3 x &= x = x^1 \\ (a_L \oplus a_R) \ltimes_3 x &= (a_L \ltimes_3 x)(a_R \ltimes_3 x) = x^{|a_L|} x^{|a_R|} \\ &= x^{|a_L| + |a_R|} = x^{|a_L \oplus a_R|} \end{aligned}$$

□

Proposition 12. *$|\mathtt{x}|$ maps the first 3 operations on $\mathbb{B}$ to addition, multiplication and exponentiation on $\mathbb{N}$: $|a \oplus b| = |a| + |b|$, $|ab| = |a|\,|b|$ and $|a \ltimes_3 b| = |b|^{|a|}$.*

Proof. Addition by definition. Multiplication by the recursion

$$\begin{aligned} |1x| &= 1\,|x|\,, \\ |(a_L \oplus a_R)x| &= |a_L x| + |a_R x| \\ &= |a_L|\,|x| + |a_R|\,|x| = |a_L \oplus a_R|\,|x|\,. \end{aligned}$$

And exponentiation by use of proposition 11 together with the multiplication compatibility. □

The following two propositions show how multiplication and all higher operations can be directly defined on the right oriented representation $\mathbb{B}_R$ (see proposition 6). We avoid using back and forth isomorphisms by identifying $\mathbb{B}_R = \mathbb{B}_B$.

Proposition 13. *$ax = \langle a_* x, x_* \rangle$ for all $a, x \in \mathbb{B}_R$.*

Proof. Let $a = \langle a_1, \ldots, a_k \rangle$ and $x = \langle x_1, \ldots, x_m \rangle$ then

$$\begin{aligned}(a_1 \oplus (\cdots \oplus (a_k \oplus 1) \cdots))x &= a_1 x \oplus (\cdots \oplus (a_k x \oplus 1x) \cdots) \\ &= a_1 x \oplus (\cdots \oplus (a_k x \oplus (x_1 \oplus (\cdots \oplus (x_m \oplus 1) \cdots) \cdots)) \cdots) \\ &= \langle a_* x, x_* \rangle .\end{aligned}$$

□

Proposition 14. *$a *'' x = \Pi(a_* *'' x) *' x$ for any operation $\mathtt{x} * \mathtt{y}$ on $\mathbb{B}$ and $a, x \in \mathbb{B}_R$.*

Proof.

$$\begin{aligned} a *'' x &= a_1 \oplus (\cdots \oplus (a_k \oplus 1) \cdots) *'' x \\ &= (a_1 *'' x) *' (\cdots *' (a_k *'' x))) *' x \\ \text{by proposition 9:} \qquad &= \Pi(a_* *'' x) *' x \end{aligned}$$

□

Proposition 15 (Corollary). *$a \ltimes_n x = \Pi(a_* \ltimes_n x) \ltimes_{n-1} x$ for $n \geq 3$ and $a, x \in \mathbb{B}_R$.*

This gives rise to the following proposition, which may be easily proved by the reader.

Proposition 16. *Each higher operation $\mathtt{x} \ltimes_n \mathtt{y}$ can be written as*

$$a \ltimes_n x = x^{\kappa_n(a,x)}$$

where $\kappa_n \colon \mathbb{B} \times \mathbb{B} \to \mathbb{N}$ for $n \geq 3$ is recursively defined by

$$\begin{aligned} &\kappa_3(a, x) := |a| , \\ &\kappa_n(1, x) := 1, \\ &\kappa_n(a, x) := \kappa_{n-1} \left(x^{\sum \kappa_n(a_*, x)}, x \right) . \end{aligned}$$

Proposition 17 (Theorem). *$(a \oplus (b \oplus c)) \ltimes_n x = (b \oplus (a \oplus c)) \ltimes_n x$ for $n \in \mathbb{N}_{\geq 3}$ and for $a, b, c, x \in \mathbb{B}$.*

Proof.

$$\begin{aligned} \text{by proposition 9} \qquad (a \oplus (b \oplus c)) \ltimes_n x &= ((a \ltimes_n x)(b \ltimes_n x)) \ltimes_{n-1} (c \ltimes_n x) \\ \text{by proposition 16} \qquad &= x^{\kappa_n(a,x) + \kappa_n(b,x)} \ltimes_{n-1} (c \ltimes_n x) \\ &= ((b \ltimes_n x)(a \ltimes_n x)) \ltimes_{n-1} (c \ltimes_n x) \\ \text{by proposition 9} \qquad &= (b \oplus (a \oplus c)) \ltimes_n x \end{aligned}$$

□

Proposition 18 (Corollary). *For $n \geq 2$ and for all $a, b, c, x \in \mathbb{B}$*

$$(a \ltimes_{n+1} x) \ltimes_n ((b \ltimes_{n+1} x) \ltimes_n (c \ltimes_{n+1} x)) = (b \ltimes_{n+1} x) \ltimes_n ((a \ltimes_{n+1} x) \ltimes_n (c \ltimes_{n+1} x)).$$

3.1 Left-commutative binary trees

Because we are particularly interested in investigation of higher operations, it seems natural to regard the 1-magma $\mathbb{B}$ under the equation $a \oplus (b \oplus c) = b \oplus (a \oplus c)$ that was detected for higher operations in proposition 17.

Definition 10 (left-commutative, lcb-trees $\mathbb{P}$). An operation $\mathtt{x} * \mathtt{y}$ on X is called *left-commutative* if $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$. Define the *lcb-trees* $\mathbb{P}$ to be the free left-commutative 1-magma.

By repeated left-commutation we get

Proposition 19. *$a_1 \oplus (\cdots \oplus (a_n \oplus 1) \cdots) = a_{\alpha(1)} \oplus (\cdots \oplus (a_{\alpha(n)} \oplus 1) \cdots)$ for each permutation $\alpha\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ and $a_1, \ldots, a_n \in \mathbb{P}$.*

So we can use multisets to construct $\mathbb{P}$.

Definition 11 (rooted trees $\mathbb{P}_R$). Let the set of rooted (also called unordered) trees $\mathbb{P}_R$ be the smallest set that for each elements $a_1, \ldots, a_n \in \mathbb{P}_R$ ($n \in \mathbb{N}_0$) also contains the multiset $[a_1, \ldots, a_n]$ as an element. Equip $\mathbb{P}_R$ as 1-magma with the constant $1 := []$ and the addition $a \oplus b := [a, b_*]$.

Proposition 20. *$\mathbb{P}_R$ is isomorphic to $\mathbb{P}$.*

Proof. Define $\psi\colon \mathbb{P}_R \to \mathbb{P}$ (recursively) as $[a_1, \ldots, a_n]^\psi := a_1^\psi \oplus (\cdots \oplus (a_n^\psi \oplus 1) \cdots)$. By proposition 19 $\psi\colon \mathbb{P}_R \to \mathbb{P}$ is well-defined. It is easily seen too, that ψ is a homomorphism and $\mathtt{x} \oplus \mathtt{y}$ on $\mathbb{P}_R$ is left-commutative so the assertion follows from proposition 2. $\square$

We will see now that the multiplication and the higher operations on $\mathbb{B}$ can be compatibly defined on $\mathbb{P}_R$.

Definition 12 (π). Let π be the universal homomorphism from $\mathbb{B}_R$ to $\mathbb{P}_R$. It is recursively defined by $\pi(\langle a_* \rangle) := [\pi(a_*)]$.

Definition 13 (homomorphically definable). Let $\varphi\colon \mathbf{A} \to \mathbf{B}$ be an epimorphism from the algebraic structure $\mathbf{A}$ onto the algebraic structure $\mathbf{B}$. We say an operation $\mathtt{x} * \mathtt{y}$ on the set A is *homomorphically definable* on B if there exists an operation $\mathtt{x} \circledast \mathtt{y}$ on $\mathbf{B}$ such that $\varphi(x * y) = \varphi(x) \circledast \varphi(y)$ for each $x, y \in A$.

If for an operation $\mathtt{x}*\mathtt{y}$ on $\mathbb{B}$ there is an operation $\mathtt{x}\circledast\mathtt{y}$ on $\mathbb{P}$ with $\pi(a*b) = \pi(a)\circledast\pi(a)$, then the operation $\mathtt{x} \circledast \mathtt{y}$ is already uniquely determined. We have to show that there actually exists such an operation $\mathtt{x} \circledast \mathtt{y}$.

Proposition 21. *Addition, multiplication and all higher operations on $\mathbb{B}$ can be homomorphically defined on $\mathbb{P}$, in the following way.*

$$
\begin{aligned}
a \ltimes_1 x &= a \oplus x = [a, x_*] \\
a \ltimes_2 x &= ax = [a_* x, x_*] \\
a \ltimes_n x &= \Pi(a_* \ltimes_n x) \ltimes_{n-1} x \quad \textit{for} \quad n \geq 3
\end{aligned}
$$

Proof. The compatibility of the addition follows directly by application of π to the $\mathbb{B}_R$ addition (definition 6), that of the multiplication by proposition 13 and that of the higher operations by proposition 14. The higher operations on $\mathbb{P}$ are well-defined because of proposition 16. $\square$

Proposition 22. *The operation* $\mathtt{x} \ltimes_4 \mathtt{y}$ *(on* $\mathbb{B}$ *or* $\mathbb{P}$*) can not be homomorphically defined on* $\mathbb{N}$*.*

Proof. Regard the two trees $a = (1,(1,1))$ and $b = ((1,1),1)$ with $|a| = |b| = 3$ and suppose we could define $\mathtt{x} \ltimes_4 \mathtt{y}$ homomorphically on $\mathbb{N}$. Then $|a \ltimes_4 x| = |a| * |x| = |b| * |x| = |b \ltimes_4 x|$ (for $\mathtt{x} * \mathtt{y}$ being the homomorphic image of $\mathtt{x} \ltimes_4 \mathtt{y}$) and further

$$\begin{aligned} |a \ltimes_4 x| &= |x \ltimes_3 (x \ltimes_3 x)| = (|x|^{|x|})^{|x|} = |x|^{|x|^2}, \\ |b \ltimes_4 x| &= |(x \ltimes_3 x) \ltimes_3 x| = |x|^{|x|^{|x|}}, \\ |x|^{|x|^2} &= |x|^{|x|^{|x|}}, \end{aligned}$$

which is not true for example for $x = a$ or $x = b$. $\square$

We saw that the multiplication plays a unique role because its associativity and translation equation.

Proposition 23 (Right Cancellation for $\mathbb{B}$). $ac = bc \Longrightarrow a = b$ *for each* $a, b, c \in \mathbb{B}$*.*

Proof. If $a = 1$ and $b \neq 1$ then $|bc| > |c| = |ac|$ hence $b = 1$; vice versa for $b = 1$. So let $a = (a_L, a_R)$ and $b = (b_L, b_R)$ then $a_L c = b_L c$ and $a_R c = b_R c$ by distribution. By r.h. $a_L = b_L$ and $a_R = b_R$, yields $a = b$. $\square$

The following unique prime factorisation propositions are not only interesting in themselves, but provide us a tool for proving the injectivity (in the right variable) of all operations $\mathtt{x} \ltimes_n \mathtt{y}$ (proposition 30).

Definition 14 (prime). An (lc)binary tree $a \neq 1$ is called *prime* if there are no (lc)binary trees $b, c \neq 1$ such that $a = bc$.

Proposition 24 (Unique Prime Factorisation for $\mathbb{B}$). *For each binary tree* a *there is a sequence of prime binary trees* $p_1, \ldots, p_k$ *(*$k \geq 0$*) such that* $a = p_1 \cdots p_k$*, where the empty product is as usual defined as* 1*.* $k = m$ *and* $(p_1, \ldots, p_k) = (q_1, \ldots, q_m)$ *already for each other sequence of prime binary trees* $q_1, \ldots, q_m$ *(*$m \geq 0$*) with* $a = q_1 \cdots q_m$*.*

Proof. Existence: If $a = 1$ then we choose the empty sequence, if a is prime we choose the one-element sequence (a) as prime factorisation. Otherwise $a = a_1 a_2$ with $|a_1|, |a_2| < |a|$. By r.h. we have a prime factorisation for a_1 and for a_2 and we choose the concatenation of them as prime factorisation for a.

Uniqueness: If $a = 1$ then each prime factorisation $p_1 \cdots p_k$ with $k > 0$ would have a numeric value > 1 in contradiction to $|a| = 1$. For $a \neq 1$ let each of p, q be the last prime of a factorisation of a, say $bp = a = cq$. If $b = 1$ then $c = 1$ because p is prime and hence $p = q$, similar for $c = 1$. Otherwise let $b = (b_L, b_R)$ and $c = (c_L, c_R)$ then $b_L p = a_L = c_L q$. By r.h. a_L has a unique prime factorisation so $p = q$. $b = c$ by proposition 23 and they must have the same prime factorisation by r.h. (knowing that $|b|, |c| < |a|$). So all prime factorisations of a are equal. $\square$

Proposition 25 (Right Cancel in $\mathbb{P}$). *$ac = bc \Longrightarrow a = b$ for all $a, b \in \mathbb{P}$.*

Proof. If $[a_*c, c_*] = [b_*b, c_*]$ then $[a_*c] = [b_*c]$ so we have a bijection $i \mapsto j$ with $a_ic = b_jc$, then $a_i = b_j$ by r.h. and so $a = b$. □

Proposition 26 (Unique Prime Factorisation for $\mathbb{P}$). *For each lcb-tree a there is a sequence $p_1, \ldots, p_k$ ($k \geq 0$) of prime lcb-trees such that $a = p_1 \cdots p_k$. For each other sequence of prime lcb-trees with $q_1 \cdots q_m = a$ already $k = m$ and $(p_1, \ldots, p_k) = (q_1, \ldots, q_m)$.*

Proof. To show the existence is the same as in proposition 24. Uniqueness: Let again each of p and q be the last prime in a prime factorisation of a, say $bp = a = cq$, i.e. $[b_*p, p_*] = [c_*q, q_*]$. There are the cases $b_*p = c_*q$, $b_*p = q_*$, $p_* = c_*q$ and $p_* = q_*$ for contemplation. Assume now that there are i, j with $b_ip = c_jq$ then we already know by r.h. that $p = q$. If there are no such i, j but there are i, j with $p_i = c_jq$ then it were $|b_*p| > |p_i| = |c_jq| > |q_*|$ which indicates that $b_*p \neq q_*$ and forces $[b_*p] = \emptyset$ and so $b = 1$. Because p is prime then also $c = 1$ and further $p = q$. The same reasoning applies to the case $b_ip = q_j$ for some i, j. When we exclude all previous cases then already $[p_*] = [q_*]$ which is $p = q$ too. So we can always conclude that $p = q$. By proposition 25 we have $b = c$ and know again that b and c must have the same prime factorisation by r.h. (because $|b|, |c| < |a|$). □

Proposition 27 (Corollary). *In $\mathbb{B}$ and $\mathbb{P}$ we can cancel left elements, i.e. $ca = cb \Longrightarrow a = b$.*

Proposition 28. *If $a^m = b^n$ for some $a, b \in \mathbb{B}$ (or $a, b \in \mathbb{P}$), $n, m \in \mathbb{N}$, then there exists $c \in \mathbb{B}$ (or $c \in \mathbb{P}$) such that $a = c^{n/\gcd(m,n)}$ and $b = c^{m/\gcd(m,n)}$.*

Proof. If $1 < \gcd(m, n) =: d$ then we can consider $(a^d)^{m/d} = (b^d)^{n/d}$ with $\gcd(m/d, n/d) = 1$. So let $\gcd(m, n) = 1$. If $m = n$ then $m = n = 1$ and we finish by setting $c = a = b$. Otherwise let $m < n$ without restriction.

Let a be in (unique) prime factorisation of length k and b in prime factorisation of length l. Because $km = ln$ and relative primeness of m and n, n divides k and m divides l. So we can group each k/n prime elements of a into a_i, such that $a = a_0 \cdots a_{n-1}$, and do it for $b = b_0 \cdots b_{m-1}$ similarly. The a_i and b_i have the same length $k/n = l/m$. Now let us compare a^m with b^n. The element a_0 occurs at the places 0, n, $2n$, $\ldots$, $(m-1)n$ in a^m. For illustration let us look at an example $m = 2$, $n = 3$.

$$(a_0a_1a_2)(a_0a_1a_2) = (b_0b_1)(b_0b_1)(b_0b_1)$$

At the place i in b^n we find the element $b_{i \bmod m}$ (where $i \bmod m$ computes the remainder in $\{0, \ldots, m-1\}$ of i divided by m as usual). So particularly $a_0 = b_{in \bmod m}$ for $i = 0, \ldots, m-1$. Because m and n are relatively prime $f(i) := in \bmod m$ is a bijection on $\{0, \ldots, m-1\}$. And hence $a_0 = b_0 = \ldots = b_{m-1} =: c$ and further $c = a_0 = \ldots = a_{n-1}$, $a = c^n$ and $b = c^m$. □

Proposition 29 (κ_n Exponent Monotony). *For trees $a, x \in \mathbb{B}$ (or $a, x \in \mathbb{P}$) and $p, q, k, l \in \mathbb{N}$ and for all $n \geq 3$*

$$p \leq q, k \leq l \Longrightarrow \kappa_n(a^p, x^k) \leq \kappa_n(a^q, x^l).$$

Proof. We prove the proposition by the induction of the definition (proposition 16) of κ. The induction base is

$$\kappa_3(a^p, x^k) = |a^p| = |a|^p \leq |a|^q = \kappa_3(a^q, x^l),$$
$$\kappa_n(1, x^k) = 1 = \kappa_3(1, x^l).$$

First notice that $a^p = \langle a_* a^{p-1}, \ldots, a_* a^1, a_* \rangle$, where we can use the r.h. on the elements because $|a_* a^{p-i}| < |a^p|$ for each a_* and each $i \in \{1, \ldots, p\}$.

$$\begin{aligned}
\kappa_n(a^p, x^k) &= \kappa_{n-1}\left(x^{k\left(\sum \kappa_n(a_* a^{p-1}, x^k) + \cdots + \sum \kappa_n(a_*, x^k)\right)}, x^k\right) \\
&\leq \kappa_{n-1}\left(x^{l\left(\sum \kappa_n(a_* a^{p-1}, x^l) + \cdots + \sum \kappa_n(a_*, x^l)\right)}, x^l\right) \\
&\leq \kappa_{n-1}\left(x^{l\left(\sum \kappa_n(a_* a^{q-1}, x^l) + \cdots + \sum \kappa_n(a_* a^{p-1}, x^l) + \cdots + \sum \kappa_n(a_*, x^l)\right)}, x^l\right) \\
&= \kappa_n(a^q, x^l)
\end{aligned}$$

□

Proposition 30 (Operations Injectivity[2]). *All maps $x \mapsto a \ltimes_n x$ are injective (on $\mathbb{B}$ and $\mathbb{P}$, $n \geq 1$).*

Proof. For $n = 1$ it is clear by definition, for $n = 2$ we have proposition 27. Now take a look on the higher operations. By proposition 16 we have to show: if $x_1^{\kappa_n(a, x_1)} = x_2^{\kappa_n(a, x_2)}$ then $x_1 = x_2$. By our previous proposition 28 we know $x_1 = y^k$, $x_2 = y^l$. If $x_1 \neq x_2$ assume without restriction $k < l$. To satisfy the precondition must $k\kappa_n(a, y^k) = l\kappa_n(a, y^l)$. But by the exponent monotony of κ_n (proposition 29) we have $\kappa_n(a, y^k) \leq \kappa_n(a, y^l)$ and so $k\kappa_n(a, y^k) < l\kappa_n(a, y^l)$. □

Of course the other-sided statement that $x \mapsto x \ltimes_n a$ is injective is not true simply by taking $a = 1$. $(1, (1, 1)) \ltimes_3 a = a^3 = ((1, 1), 1) \ltimes_3 a$ is true even independent of a. So it seems interesting to regard the equivalence relations $\mathtt{x} \simeq_n \mathtt{y}$ on $\mathbb{B}$ defined by: $a \simeq_n b$ iff $a \ltimes_n x = b \ltimes_n x$ for all $x \in \mathbb{B}$.

It is already clear that

$$\mathbb{B}/_{\simeq_1} = \mathbb{B}, \qquad \mathbb{B}/_{\simeq_2} = \mathbb{B}, \qquad \mathbb{B}/_{\simeq_3} = \mathbb{N}$$

and that $\mathbb{B}/_{\simeq_n}$ for $n \geq 4$ is a factor of $\mathbb{P}$ (by proposition 17). The next chapter shows that for each $a, b \in \mathbb{P}$ if $a \ltimes_4 x = b \ltimes_4 x$ for all $x \in \mathbb{N}$, then $a = b$. So this is particularly valid when taking elements $x \in \mathbb{B}$. So we can already provide $\mathbb{B}/_{\simeq_4} = \mathbb{P}$.

Conjecture 1 (medium[3]). $\mathbb{B}/_{\simeq_i} = \mathbb{P}$ *for all* $i \geq 3$.

[2] As turned out little before the editorial deadline, a similar result (though merely for the binary trees) was independently found by Duchon [9].

[3] The difficulties of some conjectures (as they appear to the author) are estimated by the keywords "straight" (sophisticated exercise), "medium" (exercise with some more effort) or "difficult" (needs some good investigation).

4 Power-Iterated Functions

Definition 15 (power-iterated functions $\mathbb{P}_I$, swapped power $\curlywedge$). Let the set of power-iterated functions $\mathbb{P}_I$ be the smallest set of functions $f\colon \mathbb{R}_{>1} \to \mathbb{R}_{>1}$ that contains $\mathrm{id}\colon \mathbb{R}_{>1} \to \mathbb{R}_{>1}$ and for each $f, g \in \mathbb{P}_I$ contains g^f. Equip $\mathbb{P}_I$ as 1-magma with the constant $\mathrm{id} \in \mathbb{P}_I$ and the swapped power $\mathtt{y}^{\mathtt{x}}$ (as operation on functions) as addition. To also have an operation symbol we introduce the swapped power symbol $\mathtt{x} \curlywedge \mathtt{y} := \mathtt{y}^{\mathtt{x}}$.

In this chapter we consider the relation between the power-iterated functions and the lcb-trees. It is clear that there is an epimorphism $\overline{\mathtt{x}}$ from (the free 1-magma) $\mathbb{P}_R$ to $\mathbb{P}_I$ because $\mathbb{P}_I$ is an $\emptyset$-generated left-commutative 1-magma: $f \curlywedge (g \curlywedge h) = g \curlywedge (f \curlywedge h)$.

Definition 16 ($\overline{\mathtt{x}}$). Let $\overline{\mathtt{x}}\colon \mathbb{P}_R \to \mathbb{P}_I$ be the universal epimorphism.

Proposition 31. *This epimorphism $\overline{\mathtt{x}}\colon \mathbb{P}_R \to \mathbb{P}_I$ is explicitly given by*

$$\overline{a} := (\Pi\overline{a_*}) \curlywedge \mathrm{id} = \mathrm{id}^{\Pi\overline{a_*}}.$$

Proof. We simply translate 1 and $\mathtt{x} \oplus \mathtt{y}$ on $\mathbb{P}_R$ to id and $\mathtt{x} \curlywedge \mathtt{y}$ on $\mathbb{P}_I$ in

$$\begin{aligned}\overline{[a_1, \dots, a_n]} &= \overline{(a_1 \oplus (\cdots \oplus (a_n \oplus 1) \cdots)} \\ &= (\cdots (\mathrm{id}^{\overline{a_n}}) \cdots)^{\overline{a_1}} \\ &= \mathrm{id}^{\overline{a_n} \cdots \overline{a_1}}.\end{aligned}$$

□

As an example consider $\overline{[1, 1, [1]]} = (\mathrm{id}^2(\mathrm{id} \curlywedge \mathrm{id})) \curlywedge \mathrm{id} = \mathrm{id}^{\mathrm{id}^2 \mathrm{id}^{\mathrm{id}}}$.

Note however that we equivalently could have defined $\overline{\mathtt{x}}\colon \mathbb{B} \to \mathbb{P}_I$ by $\overline{a}(x) = a \curlywedge' x$ (see definition 7, for example $(1, (1, 1)) \curlywedge' x = x \curlywedge (x \curlywedge x)$), and then recognised that $a \curlywedge' x = x^{\Pi(a_* \curlywedge' x)}$. This is independent of the order of the a_* and so conferred to $\mathbb{P}$. By defining this so, it is immediately seen that $\overline{1} = \mathrm{id}$ and $\overline{a \oplus b} = \overline{a} \curlywedge \overline{b}$ on $\mathbb{P}$ because it is the definition in $\mathbb{B}$.

Now we want to prove that the epimorphism is injective too (yielding an isomorphism). To distinguish the elements we recursively equip $\mathbb{P}_R$ with the reverse lexicographic order. This means when we sort the elements of a and b with the greatest first (therefrom the naming "reverse"), we compare lexicographically by cutting off the first respectively equal elements and then respectively looking at the first remaining element if any. Where there remains no more element or where there is the smaller element, this is the smaller lcb-tree, for example $[[1, 1], 1] <_M [[1, 1], 1, 1]$ and $[[1, 1], 1] <_M [[1, 1], [1]]$. Putting this into notation:

Definition 17 ($<_M$). For $a, b \in \mathbb{P}_R$ define recursively $a <_M b$ as: one of the following cases occurs.

1. $a \subset b$.
2. There exist $a_0 \in a \setminus b$ and $b_0 \in b \setminus a$ such that $x \leq_M a_0 <_M b_0$ for each $x \in a \setminus b$.

Proposition 32. *The relation $<_M$ on $\mathbb{P}_R$ is a linear order. Particularly either $a <_M b$, $b <_M a$ or $a = b$.*

Proof. First notice that the second case in the definition is equivalent to $\max(a \setminus b) <_M \max(b \setminus a)$ using the linear order from the r.h. Now let us show the two conditions of a linear order. For trichotomy assume that $a \neq b$ and show either $a <_M b$ or $b <_M a$. If $b \not\subset a$ and $a \not\subset b$ (in each case already $a <_M b$ or $b <_M a$) then $(a \setminus b) \neq \emptyset \neq (b \setminus a)$. We can choose $a_0 := \max(a \setminus b)$ and $b_0 := \max(b \setminus a)$ while the maximum exist by r.h. Further by r.h. we know that either $a_0 <_M b_0$, $b_0 <_M a_0$ or $a_0 = b_0$. The last case cannot occur because $(a \setminus b) \cap (b \setminus a) = \emptyset$ and the first both cases imply $a <_M b$ or $b <_M a$.

To show transitivity, we regard the four cases:

1. $a \subset b$ and $b \subset c$:
   Then $a \subset c$.

2. $a \subset b$ and $\max(b \setminus c) <_M \max(c \setminus b)$:
   Then $\max(a \setminus c) \leq_M \max(b \setminus c) <_M \max(c \setminus b) \leq_M \max(c \setminus a)$.

3. $\max(a \setminus b) <_M \max(b \setminus a)$ and $b \subset c$:
   Then $\max(a \setminus c) \leq_M \max(a \setminus b) <_M \max(b \setminus a) \leq_M \max(c \setminus a)$.

4. $\max(a \setminus b) <_M \max(b \setminus a)$ and $\max(b \setminus c) <_M \max(c \setminus b)$:
   Let $d = a \cap b \cap c$ and $(x, y) \in \{(a, b), (b, c)\}$. If $\max(x \setminus y) = \max(x \setminus d)$ then $\max((x \cap y) \setminus d) \leq_M \max(x \setminus y) <_M \max(y \setminus x) = \max(y \setminus d)$. If otherwise $\max(y \setminus x) <_M \max(y \setminus d)$ then $\max(x \setminus y) <_M \max(y \setminus x) <_M \max((x \cap y) \setminus d) = \max(y \setminus d) = \max(x \setminus d)$. We get that only either $\max(x \setminus d) <_M \max(y \setminus d)$ or $\max(x \setminus d) = \max(y \setminus d)$ can occur. The case $\max(a \setminus d) = \max(b \setminus d) = \max(c \setminus d) =: m$ is not possible because then would $m \in d$. The remaining 3 cases all yield $\max(a \setminus d) <_M \max(c \setminus d)$. We see that $\max((a \cap c) \setminus d) \leq_M \max(c \setminus a)$ because otherwise would $\max(c \setminus d) = \max((a \cap c) \setminus d) \leq_M \max(a \setminus d)$. Hence $\max(c \setminus d) = \max(c \setminus a)$ and finally $\max(a \setminus c) \leq_M \max(a \setminus d) <_M \max(c \setminus d) = \max(c \setminus a)$.

□

We want to establish an equivalent order on $\mathbb{P}_I$. The idea is that in $f = \mathrm{id}^{f_1 \cdots f_k}$ the maximal f_i dominates the behaviour of f for large x. Be aware that mainly for the seamless use of the constant function 1, we define the order on $\mathbb{R}_+$ and not on $\mathbb{R}_{>1}$ which is the domain of the functions of $\mathbb{P}_I$.

Definition 18 ($<_\uparrow$). For $f, g \in \mathbb{R}_+ \to \mathbb{R}_+$ define $f <_\uparrow g$ as: there exists an $x_0 \in \mathbb{R}_+$ so that $f(x) < g(x)$ for each $x > x_0$. We also say then f is *ultimately smaller* than g. We define $f \leq_\uparrow g$ as $f <_\uparrow g$ or $f = g$.

Proposition 33. *$<_\uparrow$ is a strict partial order on $\{f \colon \mathbb{R}_+ \to \mathbb{R}_+\}$, which is compatible with the (function) multiplication, and which has the following properties for all $f, g \colon \mathbb{R}_{>1} \to \mathbb{R}_{>1}$.*

1. *$h^f <_\uparrow h^g$ is equivalent to $f <_\uparrow g$ for $1 <_\uparrow h$ and $f, g, h \colon \mathbb{R}_+ \to \mathbb{R}_+$.*

2. *$c <_\uparrow f$ for each constant function c and any $f \in \mathbb{P}_I$.*

Proof. Showing $<_\uparrow$ to be a strict partial order compatible with multiplication is left to the reader. For assertion 1 consider $r > 1$ and $s, t > 0$:

$$r^s < r^t \iff s\log(r) < t\log(r) \iff s < t.$$

Assertion 2 follows from strict increase of all the functions of $\mathbb{P}_I$. $\square$

Proposition 34 (Lemma). *For each $a, b \in \mathbb{P}$ if $a <_M b$ then $\overline{a}^p <_\uparrow \overline{b}$ for each $p \in \mathbb{R}_+$.*

Proof. We prove the proposition by recursion over $\mathbb{P}$. We show that it holds again for a and b assuming it is already pairwise proved for all $c_* \in a \uplus b =: c$.

In the case $a \subset b$ we show $\overline{a}^p <_\uparrow \overline{b}$ by the following equivalences.

$$\overline{a}^p <_\uparrow \overline{b} \iff \mathrm{id}^{p\Pi\overline{a_*}} <_\uparrow \mathrm{id}^{\Pi\overline{b_*}} \iff p \cdot \Pi\overline{a_*} <_\uparrow \Pi\overline{b_*} \iff p <_\uparrow \Pi\overline{(b \setminus^\circ a)_*}$$

The last statement is true because $b \setminus^\circ a \neq \emptyset$ and all $\overline{(b \setminus^\circ a)_*} \in \mathbb{P}_I$.

In the remaining case there are $a_0 \in a$ and $b_0 \in b$ with $a_0 <_M b_0$ and $(a \setminus b)_* \leq_M a_0$. Let $f = \overline{a_0}$ and $g = \overline{b_0}$, by recursion assumption we know then that $f^n <_\uparrow g$ for each $n \in \mathbb{N}$ and $\overline{(a \setminus b)_*} \leq_\uparrow f$. That in mind make the following conclusions for each $p \in \mathbb{R}_+$, where $n = \mathrm{card}(a \setminus b) + 1$.

$$\begin{aligned}
& & f^n &<_\uparrow g \\
\text{use} \quad & p <_\uparrow f : & p \cdot f^{n-1} &<_\uparrow g \\
& \overline{(a \setminus b)_*} \leq_\uparrow f : & p \cdot \Pi\overline{(a \setminus b)_*} &<_\uparrow g \\
\text{multiply} \quad & \Pi\overline{(a \cap b)_*} : & p \cdot \Pi\overline{a_*} &<_\uparrow g \cdot \Pi\overline{(a \cap b)_*} \\
\text{multiply} \quad & 1 \leq_\uparrow \overline{(b \setminus a)_*} : & p \cdot \Pi\overline{a_*} &<_\uparrow \Pi\overline{b_*} \\
& \curlywedge \mathrm{id} : & \overline{a}^p &<_\uparrow \overline{b}
\end{aligned}$$

$\square$

For $p = 1$ we gain:

Proposition 35 (Theorem). *For each $a, b \in \mathbb{P}$ if $a <_M b$ then $\overline{a} <_\uparrow \overline{b}$ (and vice versa). Hence $\mathbb{P}$ is isomorphic to $\mathbb{P}_I$.*

Proposition 36 (Corollary). *$<_\uparrow$ is a linear order on $\mathbb{P}_I$.*[4]

The interesting fact of representing $\mathbb{P}$ as (bijective) functions is that we can easily embed it into a structure with invertible concatenation/multiplication by adding inverses.

Definition 19 (power-inverse-iterated functions $\mathbb{P}_I^\circ$). Let the set of *power-inverse-iterated functions* $\mathbb{P}_I^\circ$ be the smallest set containing $\mathrm{id}\colon \mathbb{R}_{>1} \to \mathbb{R}_{>1}$ and for each $f, g \in \mathbb{P}_I^\circ$ also containing f^g, the function concatenation $f \circ g$ and the inverse function f^{-1}. Equip $\mathbb{P}_I^\circ$ with the constant id, the binary operations $\mathtt{y}^{\mathtt{x}}$, $\mathtt{x} \circ \mathtt{y}$ and the unary operation $\mathtt{x}^{-1}$.

[4]This proposition may also be concluded from paragraph III.2 in G. H. Hardy's Monograph [16].

If $f, g\colon \mathbb{R}_{>1} \to \mathbb{R}_{>1}$ are continuous, strictly increasing and (above and below) unbounded (for bounds in $\mathbb{R}_{>1}$), then f^g is too (strictly increasing because: $f(x_1)^{g(x_1)} < f(x_2)^{g(x_1)} < f(x_2)^{g(x_2)}$ for $x_1 < x_2$, previous inequality because $f(x_2) > 1$). A continuous, strictly increasing and unbounded function $f\colon \mathbb{R}_{>1} \to \mathbb{R}_{>1}$ is bijective. The inverse of a continuous and strictly increasing and unbounded function is again continuous, strictly increasing and unbound and so is the concatenation. So the above definition is valid.

Proposition 37 (Theorem). $\mathbb{P}$ *(even with the multiplication* $\mathtt{xy}$*) is embeddable into* $\mathbb{P}_I^\circ$ *via* $\overline{\mathtt{x}}$.

5 Coppices

In generalisation of the process of making the multiplication invertible we define:

Definition 20 (precoppice, coppice). $\mathbf{X} = (X, 1, \mathtt{x} \oplus \mathtt{y}, \mathtt{xy}, \mathtt{x}^\sim)$ is called a *coppice* iff $(X, 1, \mathtt{xy}, \mathtt{x}^\sim)$ is a group and the multiplication is right-distributive over the addition, i.e. $(x \oplus y)z = xz \oplus yz$ for all $x, y, z \in X$. If $\mathtt{x} \oplus \mathtt{y}$ has a property *abc* we call $\mathbf{X}$ an *abc* coppice. A *precoppice* is just a coppice without $\mathtt{x}^\sim$, i.e. $(X, 1, \mathtt{xy})$ is a monoid and multiplication is right-distributive over addition. The operation $\mathtt{x} \oplus \mathtt{y}$ is called the *addition*, $\mathtt{xy}$ is called the *multiplication* and $\mathtt{x}^\sim$ is called the *inversion* of the coppice.

The relation "precoppice to coppice" resembles the relations "commutative ring to field", "ring to skew field", "semiring to semifield", "near-ring to near-field", i.e. the associative multiplication becomes a group. However the pair precoppice/coppice is more general than the other pairs mentioned, in that there are no restrictions — not even associativity — on the addition. A near-ring/near-field is a precoppice/coppice where the addition forms a group (and the multiplicative inversion is defined only on the nonzero elements) a semiring/semifield is a precoppice/coppice where the addition is associative and the multiplication is additionally left-distributive over the addition.

For overview reasons here the defining set of equations of a coppice. A precoppice just omits (4).

$$(r \oplus s)t = rt \oplus st \tag{1}$$
$$(rs)t = r(st) \tag{2}$$
$$1r = r \tag{3}$$
$$r^\sim r = 1 \tag{4}$$

As it is already well-known for groups the following equations are consequences in a coppice.

$$r1 = r \tag{5}$$
$$rr^\sim = 1 \tag{6}$$
$$(rs)^\sim = s^\sim r^\sim \tag{7}$$

Our first example of a coppice is $\mathbb{P}_I^\circ$.

Proposition 38. $\mathbb{P}_I^\circ$ *is a left-commutative coppice.*

We embedded the precoppice $\mathbb{P}_I$ into $\mathbb{P}_I^\circ$. In this chapter we will also perform the embedding of the free precoppice into the free coppice for our 3 main types:

1. Associative: The natural numbers $\mathbb{N}$ into the fractional numbers $\mathbb{Q}_+$.
2. Left-commutative: The lcb-trees $\mathbb{P}$ into the fractional trees $\mathbb{F}$.
3. No equations: The binary trees $\mathbb{B}$ into the division binary trees $\mathbb{B}^\circ$.

But before doing that let us cast our eyes on some precoppice/coppice examples. Though verifying the embeddability of an associative system into a group can already be a difficult task (see [24]), there is a whole class of interesting precoppices where the embedding is possible and straight. Recall the construction of $\mathbb{P}_I$ and $\mathbb{P}_I^\circ$ (definition 19).

Definition 21 ($\mathfrak{U}(I)$, ciu-, C^A, $\mathfrak{U}(I,A)$, $\mathfrak{U}^\circ(I,A)$). Let $\mathfrak{U}(I)$ be the set of continuous, strictly increasing and above and below (in I) unbounded functions $: I \to I$, where I is one of $\mathbb{R}$, $\mathbb{R}_+$ or $\mathbb{R}_{>1}$. Call $\mathfrak{U}(I)$ the *ciu-functions of* I. Call a binary operation $\mathtt{x} \oplus \mathtt{y}$ on I a ciu-operation if $f \oplus g$ is a ciu-function for any two ciu-functions f, g. For a set A of ciu-operations on I and a subset F of $\mathfrak{U}(I)$ let $\mathrm{C}^A(F)$ be the set generated by application of the operations of A to members of F (the closure of F with respect to the operations of A). Let $\mathfrak{U}(I,A) := \mathrm{C}^A(\{\mathrm{id}\})$ and let $\mathfrak{U}^\circ(I,A)$ be the set generated by id, application of operations of A, application of the function concatenation and taking the inverse function.

The definition of $\mathfrak{U}^\circ(I,A)$ is valid, because the ciu-functions are closed under concatenation and inversion and A are ciu-operations. The constructions of $\mathbb{P}_I$ and $\mathbb{P}_I^\circ$ are just a special case, namely

$$\mathbb{P}_I = \mathfrak{U}(\mathbb{R}_{>1}, \{\mathtt{y}^{\mathtt{x}}\}) \qquad\qquad \mathbb{P}_I^\circ = \mathfrak{U}^\circ(\mathbb{R}_{>1}, \{\mathtt{y}^{\mathtt{x}}\}).$$

Of course $\mathfrak{U}(I)$ is itself a precoppice for every ciu-operation as addition on it ($\circ$ as multiplication). And $\mathfrak{U}^\circ(I,A)$ is a coppice for every operation of A as addition. Note that the function concatenation is right-distributive over every pointwise operation of functions $((f * g) \circ h)(x) = (f \circ h)(x) * (g \circ h)(x)$. Every operation $*\colon I \times I \to I$ is a ciu-operation if (but *not* only if) $x \mapsto x * y$ and $y \mapsto x * y$ are ciu-functions. Particularly the operations $\mathtt{x} + \mathtt{y}$ on $\mathbb{R}$, $\mathtt{xy}$ on $\mathbb{R}_+$ and $\mathtt{x}^{\mathtt{y}}$ on $\mathbb{R}_{>1}$ are ciu-operations.

Proposition 39. *The precoppice $\mathfrak{U}(I, \{\mathtt{x} \oplus \mathtt{y}\})$ (with the operations* id, $\mathtt{x} \oplus \mathtt{y}$, $\mathtt{x} \circ \mathtt{y}$*) is embeddable into the coppice $\mathfrak{U}^\circ(I, \{\mathtt{x} \oplus \mathtt{y}\})$ (with the operations* id, $\mathtt{x} \oplus \mathtt{y}$, $\mathtt{x} \circ \mathtt{y}$, $\mathtt{x}^{-1}$*) for each ciu-operation $\mathtt{x} \oplus \mathtt{y}$.*

We can construct the $\mathfrak{U}^\circ(I, \{\mathtt{x} \oplus \mathtt{y}\})$ in a more sophisticated manner (rather than simply throwing all operations $\mathrm{id}, \mathtt{x} \oplus \mathtt{y}, \mathtt{x} \circ \mathtt{y}, \mathtt{x}^{-1}$ over each other) by the following induction. The idea is to apply right-distributivity until no more is possible.

Proposition 40. $\mathfrak{U}^\circ(I,A) = \bigcup_{n=0}^\infty U_n$ *for the following by induction defined U_n, $n \in \mathbb{N}_0$.*

$$\begin{aligned} U_0 &:= \{\mathrm{id}\} \\ U_{n+1} &:= \{f^{-1} \circ g \colon f, g \in D^A(U_n)\} \end{aligned}$$

Where $D^A(X)$ is the set of elements that are built by terms that contain at least one operation and that contains id*, i.e. $D^A(X) = \mathrm{C}^A\{a(x,y) \colon a \in A, x, y \in X\} \cup \{\mathrm{id}\}$.*

Proof. Because every U_n only contains generated elements, we merely need to show the closure of $\bigcup_{n=0}^{\infty} U_n$. Each U_n is closed under inversion as $(f^{-1} \circ g)^{-1} = g^{-1} \circ f$. Next we assure ourselves that D^A is a monotone operator and show that $U_k \subseteq U_l$ whenever $k \leq l$ by induction on $U_n \subseteq U_{n+1}$. We get the induction base by $\mathrm{id} \in D^A(U_0)$:

$$U_0 = \{\mathrm{id}^{-1} \circ \mathrm{id}\} \subseteq \{f^{-1} \circ g \colon f, g \in D^A(U_0)\} = U_1.$$

The induction step $U_{n+1} \subseteq U_{n+2}$ or

$$\{f^{-1} \circ g \colon f, g \in D^A(U_n)\} \subseteq \{f^{-1} \circ g \colon f, g \in D^A(U_{n+1})\}$$

follows then by applying monotony of D^A on the induction assumption.

Next we show closure under addition. If there are some $h_1 \in U_k$ and $h_2 \in U_l$ then $h_1, h_2 \in U_m$ where m is the maximum of k and l. So $a(h_1, h_2) = \mathrm{id}^{-1} \circ a(h_1, h_2) \in U_{m+1}$ for any $a \in A$.

At last we show the closure under composition. We want to show that $h_1 \circ h_2 \in U_{k+l}$ by induction over k. The induction base at $k = 0$ is satisfied because $h_1 = \mathrm{id}$ and so $h_1 \circ h_2 = h_2 \in U_{0+l}$. Let now $k > 0$ and $h_1 = {f_1}^{-1} \circ g_1$ with $f_1, g_1 \in D^A(U_{k-1})$. Then $g_1 \circ h_2 \in D^A(U_{k-1+l})$ by r.h. and distributivity (for $g_1 \neq \mathrm{id}$) or monotony (for $g_1 = \mathrm{id}$). So ${f_1}^{-1} \circ (g_1 \circ h_2) \in U_{k+l}$ by $f_1 \in D^A(U_{k-1}) \subseteq D^A(U_{k-1+l})$. □

With this lens we can take a closer look at some examples for coppices by choosing appropriate additions A and an appropriate domain I for $\mathfrak{U}^\circ(I, A)$:

1. $I = \mathbb{R}$, $A = \{\mathtt{x} + \mathtt{y}\}$. The coppice is entirely comprised of the functions $\frac{m}{n}x$ $(m, n \in \mathbb{N})$ and so isomorphic to $\mathbb{Q}_+$.

2. $I = \mathbb{R}_+$, $A = \{\mathtt{xy}\}$. The restriction to $\mathbb{R}_+$ is necessary because multiplying negative or null values destroys strict increase. The coppice is entirely comprised of the functions $x^{\frac{m}{n}}$ $(m, n \in \mathbb{N})$ and so isomorphic to $\mathbb{Q}_+$ too.

3. $I = \mathbb{R}_+$, $A = \{\sqrt{\mathtt{x}^2 + \mathtt{y}^2}\}$. The restriction to $\mathbb{R}_+$ is again necessary. The coppice is entirely comprised of the functions $\sqrt{\frac{m}{n}}x$ $(m, n \in \mathbb{N})$ and so again isomorphic to $\mathbb{Q}_+$. As the reader might already have recognised, whenever the operation $\mathtt{x} \oplus \mathtt{y}$ is associative then $\mathfrak{U}^\circ(I, \{\mathtt{x} \oplus \mathtt{y}\})$ is isomorphic to $\mathbb{Q}_+$ (see proposition 48).

4. $I = \mathbb{R}$, $A = \{2\mathtt{x} + \mathtt{y}\}$. This is now a non-associative operation, but it is left-commutative.

   $$f \oplus (g \oplus h) = 2f + 2g + h = 2g + 2f + h = g \oplus (f \oplus h)$$

   It turns out that the functions of the coppice are the linear functions with an uneven fraction as slope

   $$\frac{2m+1}{2n+1}x \quad \text{for} \quad m, n \in \mathbb{N}_0.$$

   The proof is left to the reader. Indeed we show (proposition 56) that every element of a left-commutative coppice can take the form

   $$(r_1 \oplus \cdots \oplus r_k \oplus 1)^{\sim}(s_1 \oplus \cdots \oplus s_l \oplus 1).$$

   For a coppice not isomorphic to $\mathbb{Q}_+$ it is rare to have a commutative multiplication.

5. $I = \mathbb{R}_{>1}$, $A = \{\mathtt{y}^{\mathtt{x}}\}$. Our well known power-inverse-iterated functions which are left-commutative too.

6. $I = \mathbb{R}_{+}$, $A = \{\sqrt{\mathtt{x}} + \sqrt{\mathtt{y}}\}$. This time the operation is commutative but not associative. The 1-magma already is an intricate structure consisting of certain nested sums of roots. I would conjecture it is isomorphic to the commutative binary trees (i.e. the free commutative 1-magma.)

7. $I = \mathbb{R}_{>1}$, $A = \{\mathtt{x}^{\mathtt{y}}\mathtt{y}^{\mathtt{x}}\}$ yields a commutative coppice too. Though we will *not* regard non-associative commutative coppices here anymore. Each associative $\emptyset$-generated coppice is by the way commutative, because every associative $\emptyset$-generated coppice is either $\mathbb{Q}_{+}$ or the trivial coppice (see proposition 48 and proposition 50).

8. $I = \mathbb{R}_{+}$, $A = \{\mathtt{x} + \mathtt{y}, \mathtt{xy}\}$. In this case instead of generating only with one binary operation $\mathtt{x} \oplus \mathtt{y}$ we generate with $\mathtt{x} \oplus_1 \mathtt{y} := \mathtt{x} + \mathtt{y}$ and $\mathtt{x} \oplus_2 \mathtt{y} := \mathtt{xy}$. $\mathfrak{U}(I, A)$ consists of all polynomials with natural numbered coefficients and lacking constant. The coppices of the examples 1, 2 and 4 are all subsets of this multi-coppice (i.e. it is a coppice regarding each addition).

9. $I = \mathbb{R}_{>1}$, $A = \{\mathtt{xy}, \mathtt{y}^{\mathtt{x}}\}$. There is the bijection $f \mapsto \mathrm{id}^f$ between $\mathfrak{U}(\mathbb{R}_{>1}, \{\mathtt{xy}, \mathtt{y}^{\mathtt{x}}\})$ and $\mathbb{P}_I = \mathfrak{U}(\mathbb{R}_{>1}, \{\mathtt{y}^{\mathtt{x}}\})$ and the injection $f \mapsto \mathrm{id}^{\mathrm{id}^f}$ of $\mathfrak{U}(\mathbb{R}_{+}, \{\mathtt{x} + \mathtt{y}, \mathtt{xy}\})$ into $\mathbb{P}_I$.

Conjecture 2 (medium). *Each function $f \in \mathfrak{U}^{\circ}(\mathbb{R}_{+}, \{\mathtt{x} + \mathtt{y}, \mathtt{xy}\})$ can be assigned a polynomial $P_f(x, y)$ with integer coefficients such that f is the only function of $\mathfrak{U}^{\circ}(\mathbb{R}_{+}, \{\mathtt{x}+\mathtt{y}, \mathtt{xy}\})$ with $f \subseteq \{(x, y) \colon P_f(x, y) = 0\}$. Particularly $f \neq \mathrm{id}$ can have at most a finite number of fixed points.*

The above conjecture should be no big problem to prove with some background in algebraic geometry.

Conjecture 3 (difficult). *Every function of $\mathfrak{U}^{\circ}(\mathbb{R}_{>1}, \{\mathtt{xy}, \mathtt{x}^{\mathtt{y}}\}) \setminus \{\mathrm{id}\}$ has merely a finite number of fixed points.*

These fixed point propositions are connected with the linearity of the $<_{\uparrow}$ order (see section 7.1) which in turn points towards the topological completion of those coppices (see proposition 115).

Now we finish considering examples and address the aim of this chapter to computably construct the free associative coppice, the free coppice, and the free left-commutative coppice, respectively and to embed the corresponding free precoppices into them. But before doing that we have to show that $\mathbb{B}$, $\mathbb{P}$ and $\mathbb{N}$ are indeed the corresponding free precoppices.

Proposition 41. $\mathbb{B}$ *together with the multiplication* $\mathtt{xy}$ *is the free precoppice.*

Proof. Let us consider the term algebra T (in no variables) of the operations $\{1, \mathtt{x} \oplus \mathtt{y}, \mathtt{xy}\}$. We show recursively that every term ab of T is equal to a term of $\mathbb{B}$ (i.e. only consisting of operations $\{1, \mathtt{x} \oplus \mathtt{y}\}$) in the free precoppice: So let the subterms a and b already shown to be in $\mathbb{B}$, but (3) and (1) are a recursive definition for $\mathtt{xy}$ in $\mathbb{B}$, hence $ab \in \mathbb{B}$. On the other hand $\mathbb{B}$ is actually a precoppice ($\mathtt{xy}$ satisfies associativity). $\square$

Proposition 42 (Corollary). *Each $\emptyset$-generated precoppice is already generated by its addition (and 1). Or, in other words, each $\emptyset$-generated precoppice is isomorphic to a $\emptyset$-generated 1-magma.*

Proof. Let $\mathbf{P}$ be the precoppice and $\varphi\colon \mathbb{B} \to \mathbf{P}$ the universal precoppice epimorphism. Then for every element $y \in \mathbf{P}$ there is an x with $\varphi(x) = y$ and hence y is made up of additions of 1. □

Proposition 43 (Corollary). *The free 1-magmas $\mathbb{B}$, $\mathbb{P}$ and $\mathbb{N}$ are equal to their corresponding free precoppices.*

Proposition 44. *Let $\mathbf{P}$ be a $\emptyset$-generated precoppice and $\mathbf{C}$ be a (pre)coppice. Each homomorphism h from the 1-magma of $\mathbf{P}$ to $\mathbf{C}$ is a homomorphism from the precoppice $\mathbf{P}$ to $\mathbf{C}$.*

Proof. We prove $h(ab) = h(a)h(b)$, $a, b \in \mathbf{P}$ by induction over $\mathbb{B}$. Let $\varphi\colon \mathbb{B} \to \mathbf{P}$ the universal epimorphism, $\varphi(\mathbb{B}) = \mathbf{P}$.

$$
\begin{aligned}
& & h(1b) &= h(b) = 1h(b) \\
& & h((a_L \oplus a_R)^\varphi b^\varphi) &= h((a_L b)^\varphi \oplus (a_R b)^\varphi) \\
& & &= h((a_L b)^\varphi) \oplus h((a_R b)^\varphi) \\
& \text{by r.h.} & &= h(a_L^\varphi)h(b^\varphi) \oplus h(a_R^\varphi)h(b^\varphi) \\
& \text{by distributivity of } \mathbf{C} & &= (h(a_L^\varphi) \oplus h(a_R^\varphi))h(b^\varphi) \\
& & &= h((a_L \oplus a_R)^\varphi)h(b^\varphi)
\end{aligned}
$$

□

Proposition 45 (Corollary). *A precoppice is embeddable into a coppice if (and only if) its 1-magma is embeddable.*

5.1 Fractional Numbers

Definition 22 ($\mathcal{Q}$). Let $(\mathcal{Q}, 1, \mathtt{x} \oplus \mathtt{y}, \mathtt{xy}, \mathtt{x}^\sim)$ temporarily be the free associative coppice.

Because of associativity of $\mathtt{x} \oplus \mathtt{y}$ we have also left-distributivity:

Proposition 46 (Left-Distributivity). *$r(s \oplus t) = rs \oplus rt$ for all $r, s, t \in \mathcal{Q}$.*

Proof. Assume that it is already shown for $r = r_1, r_2$ and arbitrary $s, t \in \mathcal{Q}$. Show that

it is also valid for $r = 1, r_1 \oplus r_2, r_1 r_2, {r_1}^\sim$.

$$\begin{aligned}
1(s \oplus t) &= s \oplus t \\
&= 1s \oplus 1t \\
(r_1 \oplus r_2)(s \oplus t) &= r_1(s \oplus t) \oplus r_2(s \oplus t) \\
&= (r_1 s \oplus r_1 t) \oplus (r_2 s \oplus r_2 t) \\
&= (r_1 s \oplus r_2 s) \oplus (r_1 t \oplus r_2 t) \\
&= (r_1 \oplus r_2)s \oplus (r_1 \oplus r_2)t \\
(r_1 r_2)(s \oplus t) &= r_1 r_2 (s \oplus t) \\
&= r_1(r_2 s \oplus r_2 t) \\
&= r_1 r_2 s \oplus r_1 r_2 t \\
&= (r_1 r_2)s \oplus (r_1 r_2)t \\
{r_1}^\sim(s \oplus t) &= {r_1}^\sim(r_1 {r_1}^\sim s \oplus r_1 {r_1}^\sim t) \\
&= {r_1}^\sim r_1({r_1}^\sim s \oplus {r_1}^\sim t) \\
&= {r_1}^\sim s \oplus {r_1}^\sim t
\end{aligned}$$

□

Proposition 47 (Commutativity). *$rs = sr$ for all $r, s \in \mathcal{Q}$.*

Proof. Assume that it is already shown for $r = r_1, r_2$ and arbitrary $s \in \mathcal{Q}$. Show that it is valid for $r = 1, r_1 \oplus r_2, r_1 r_2, {r_1}^\sim$.

$$\begin{aligned}
&\text{by (3) and (5)} & 1s &= s = s1 \\
&\text{by (1), r.h. and proposition 46} & (r_1 \oplus r_2)s &= r_1 s \oplus r_2 s = s r_1 \oplus s r_2 = s(r_1 \oplus r_2) \\
&\text{by r.h. and associativity} & (r_1 r_2)s &= r_1(s r_2) = s(r_1 r_2) \\
&\text{by r.h. and (7)} & {r_1}^\sim s &= (s^\sim r_1)^\sim = (r_1 s^\sim)^\sim = s {r_1}^\sim
\end{aligned}$$

□

Proposition 48 (Theorem). *The coppice of (positive) fractional numbers $\mathbb{Q}_+$ is (isomorphic to) the free associative coppice.*

Proof. That $\mathbb{Q}_+$ is an associative coppice is already known. By proposition 2 it suffices to show that there exists a homomorphism $\psi \colon \mathbb{Q}_+ \to \mathcal{Q}$. We first define ψ on $\mathbb{N} \subset \mathbb{Q}_+$ inductively by

$$\begin{aligned}
1^\psi &:= 1, \\
(n+1)^\psi &:= n^\psi \oplus 1.
\end{aligned}$$

Then $(m+n)^\psi = n^\psi \oplus m^\psi$ (proof similar to proposition 3) and $(nm)^\psi = n^\psi m^\psi$ by

$$\begin{aligned}
((n+1)m)^\psi &= (nm + m)^\psi = (nm)^\psi \oplus m^\psi \\
&= n^\psi m^\psi \oplus m^\psi = (n^\psi + 1)m^\psi \\
&= (n+1)^\psi m^\psi.
\end{aligned}$$

Now define ψ on $\mathbb{Q}_+$ by $(\frac{m}{n})^\psi := m^\psi (n^\psi)^\sim$ (for $m, n \in \mathbb{N}$). ψ is well-defined because $(\frac{km}{kn})^\psi = (km)^\psi (kn)^{\psi^\sim} = m^\psi k^\psi (n^\psi k^\psi)^\sim = m^\psi k^\psi k^{\psi^\sim} n^{\psi^\sim} = m^\psi n^{\psi^\sim}$. We show that ψ is a homomorphism by the following calculations.

$$\left(\frac{1}{1}\right)^\psi = 11^\sim = 1$$

$$\begin{aligned}
\left(\frac{m_1}{n_1} + \frac{m_2}{n_2}\right)^\psi &= \left(\frac{m_1 n_2 + m_2 n_1}{n_1 n_2}\right)^\psi \\
&= (m_1 n_2 + m_2 n_1)^\psi (n_1 n_2)^{\psi^\sim} \\
&= \left((m_1 n_2)^\psi \oplus (m_2 n_1)^\psi\right) (n_1 n_2)^{\psi^\sim} \\
&= \left(m_1^\psi n_2^\psi \oplus m_2^\psi n_1^\psi\right) n_2^{\psi^\sim} n_1^{\psi^\sim} \\
&= m_1^\psi n_2^\psi n_2^{\psi^\sim} n_1^{\psi^\sim} \oplus m_2^\psi n_1^\psi n_2^{\psi^\sim} n_1^{\psi^\sim} \\
&= m_1^\psi n_1^{\psi^\sim} \oplus m_2^\psi n_2^{\psi^\sim} \\
&= \left(\frac{m_1}{n_1}\right)^\psi \oplus \left(\frac{m_2}{n_2}\right)^\psi
\end{aligned}$$

$$\begin{aligned}
\left(\frac{m_1}{n_1}\frac{m_2}{n_2}\right)^\psi &= \left(\frac{m_1 m_2}{n_1 n_2}\right)^\psi = (m_1 m_2)^\psi (n_1 n_2)^{\psi^\sim} \\
&= m_1^\psi m_2^\psi n_2^{\psi^\sim} n_1^{\psi^\sim} \\
&= m_1^\psi n_1^{\psi^\sim} m_2^\psi n_2^{\psi^\sim} \\
&= \left(\frac{m_1}{n_1}\right)^\psi \left(\frac{m_2}{n_2}\right)^\psi
\end{aligned}$$

$$\left(\frac{1}{\left(\frac{m_1}{n_1}\right)}\right)^\psi = \left(\frac{n_1}{m_1}\right)^\psi = n_1^\psi m_1^{\psi^\sim} = \left(m_1^\psi n_1^{\psi^\sim}\right)^\sim = \left(\left(\frac{m_1}{n_1}\right)^\psi\right)^\sim$$

□

Proposition 49 (Theorem). *The free associative precoppice $\mathbb{N}$ is embeddable into the free associative coppice $\mathbb{Q}_+$ which can be computably constructed as cancelled fractions.*

Proof. This is already well-known. □

Proposition 50. *The only proper factor coppice of $\mathbb{Q}_+$ is the trivial coppice.*

Proof. Assume we have a coppice $\mathbf{C}$ and an epimorphism $\varphi\colon \mathbb{Q}_+ \to \mathbf{C}$ such that there are different elements $p_0, q_0 \in \mathbb{Q}_+$ with $p := \varphi(p_0) = \varphi(q_0) =: q$. We can use all laws of

$\mathbb{Q}_+$ (as a coppice) in $\mathbf{C}$ for the following conclusions.

$$
\begin{aligned}
& & p &= q \\
&\text{choose } m_0, n_0 \in \mathbb{N} & 1 &= q/p =: \varphi(n_0/m_0) \\
& & \varphi(m_0) &= \varphi(n_0) \\
&\text{without restriction } m_0 < n_0 & m_0 + k_0 &= n_0 \\
&m := \varphi(m_0), k := \varphi(k_0) & m + k &= m \\
&\text{divide by } m & 1 + k/m &= 1 \\
&\text{repeated auto application} & 1 + ik/m &= 1 \quad \forall i \in \varphi(\mathbb{N}) \\
&\text{substitute } i \text{ by } mi & 1 + ik &= 1 \quad \forall i \in \varphi(\mathbb{N}) \\
&\text{keep for } k_0 = 1 \text{ otherwise add } \varphi(k_0 - 1) & k + ik &= k \quad \forall i \in \varphi(\mathbb{N}) \\
&k + ik = (i+1)k & ik &= k \quad \forall i \in \varphi(\mathbb{N}_{\geq 2}) \\
&\text{divide by } k & 1 &= i \quad \forall i \in \varphi(\mathbb{N}_{\geq 2}) \\
& & 1 &= r \quad \forall r \in \varphi(\mathbb{Q}_+) = \mathbf{C}
\end{aligned}
$$

□

5.2 Division Binary Trees

Definition 23 ($\mathbb{B}^\circ$). Let $\mathbb{B}^\circ$ be the free coppice. Call its elements the *division binary trees.*

In search for a normal form we may encounter the possibility to reduce every addition to multiplication and the (right) addition of 1.

$$v \oplus w = (vw^\sim \oplus 1)w$$

More generally we can even state

Proposition 51. *For each group $(X, 1, \mathtt{xy}, \mathtt{x}^\sim)$ there is a bijection between all operations $\mathtt{x} \oplus \mathtt{y}$ on X for which $(X, 1, \mathtt{x} \oplus \mathtt{y}, \mathtt{xy}, \mathtt{x}^\sim)$ is a coppice and all functions $[\mathtt{x}]$ on X.*

Proof. Let A be the set of all binary operations $a\colon X \times X \to X$ such that the structure $(X, 1, a(\mathtt{x}, \mathtt{y}), \mathtt{xy}, \mathtt{x}^\sim)$ is a coppice and let I be the set of all functions $i\colon X \to X$. We choose the following two mappings $\varphi\colon I \to A$ and $\psi\colon A \to I$ given by

$$
\begin{aligned}
i^\varphi(x, y) &:= i(xy^\sim)y, \\
a^\psi(x) &:= a(x, 1).
\end{aligned}
$$

First we prove that φ is well-defined, i.e. that distributivity is satisfied for the resulting operation i^φ.

$$
\begin{aligned}
i^\varphi(x, y)z &= (i(xy^\sim)y)z \\
&= i(xzz^\sim y^\sim)(yz) \\
&= i((xz)(yz)^\sim)(yz) \\
&= i^\varphi(xz, yz)
\end{aligned}
$$

To show that φ (and ψ) is a bijection it suffices to show that $\varphi \circ \psi = \mathrm{id}$ and $\psi \circ \varphi = \mathrm{id}$.

$$(a^\varphi)^\psi(x,y) = i^\psi(xy^\sim)y = a(xy^\sim,1)y = a(x,y)$$

$$(i^\psi)^\varphi(x) = a^\varphi(x,1) = a^\varphi(x1^\sim,1)1 = i(x)$$

□

There is also the other-sided bijection between A and I given by the φ' and ψ' below.

$$i^{\varphi'}(x,y) := i(yx^\sim)x$$
$$a^{\psi'}(x) := a(1,x)$$

We favour the original bijection because $a(x,a(y,1)) = a(y,a(x,1))$ for left-commutative coppices.

So we can present a coppice also by the signature $(X,1,[\mathtt{x}],\mathtt{xy},\mathtt{x}^\sim)$ only requiring that $(X,1,\mathtt{xy},\mathtt{x}^\sim)$ is a group. One says that both structures (as varieties) are *termwise definitionally equivalent*, or that they have the same *clone*. This implies for example that they have the same subalgebras, congruences and homomorphisms.

Definition 24 (additive/incremental coppice). To nevertheless distinguish between the two, we call the usual coppice with addition an *additive* coppice and a group equipped with the increment $(X,1,[\mathtt{x}],\mathtt{xy},\mathtt{x}^\sim)$ an *incremental* coppice.

Definition 25 (words, letters, $\mathrm{W}(X)$, $\mathrm{cW}(X)$). For a set X and an element $x \in X$ call the syntactic construct $[x]$ and $[x]^\sim$ a letter of X. For letters $l_1,\dots,l_n$ ($n \geq 0$) call the sequence $l_1 \cdots l_n$ a word of X. Let $\mathrm{W}(X)$ be the set of words of X. $\mathrm{W}(X)$ is equipped with the empty word 1 (which always belongs to $\mathrm{W}(X)$), the usual *concatenation* $\mathtt{x} \centerdot \mathtt{y}$, and the *inversion* $\mathtt{x}^\sim\colon \mathrm{W}(X) \to \mathrm{W}(X)$ which is defined by

$$([a])^\sim := [a]^\sim,$$
$$([a]^\sim)^\sim := [a],$$
$$(w_1 \cdots w_n)^\sim := {w_n}^\sim \cdots {w_1}^\sim.$$

We say a word $w_1 \cdots w_n$ is *cancelled* if there is no $i \in \{1,\dots,n-1\}$ with $w_i = {w_{i+1}}^\sim$, let $\mathrm{cW}(X)$ be the cancelled words of $\mathrm{W}(X)$.

Definition 26 (cancelled multiplication $\mathtt{xy}$ on $\mathrm{cW}(X)$). We equip $\mathrm{cW}(X)$ with the multiplication $\mathtt{xy}$ by concatenation with cancelling: Let $v = v_1 \cdots v_m, w = w_1 \cdots w_n \in \mathrm{cW}(X)$, $m,n \in \mathbb{N}_0$, let $k \in \mathbb{N}_0$ be the maximal index $i \in \mathbb{N}_0$ where $v_{m-j+1} = {w_j}^\sim$ for all $j \leq i$. Then define

$$vw := v_1 \cdots v_{m-k} \centerdot w_{k+1} \cdots w_n.$$

Clearly this is a cancelled word and so again element of $\mathrm{cW}(X)$.

Proposition 52. $(\mathrm{cW}(X),1,\mathtt{xy},\mathtt{x}^\sim)$ *is a group.*

Proof. $\mathtt{x}^\sim$ of $\mathrm{W}(X)$ is restrictable to $\mathrm{cW}(X)$. It is already well known that $\mathrm{cW}(X)$ with the above operations is the free group generated by X. □

Definition 27 (cancelled recursive words $\mathbb{B}_w^\circ$). Let $W_0 := \emptyset$ and $W_{n+1} := \mathrm{cW}(W_n)$, it is clear by induction that $W_n \subseteq W_{n+1}$ (Imagine the W_k as onion with W_i, $i < k$ as subonions and W_1 as core. The application of cW shifts onion i, $i \leq k$, to onion $i+1$ and a new core appears) and each of the operations $\mathtt{x}^\sim$ and $\mathtt{xy}$ on W_n is the restriction of the operation on W_{n+1}. So let $\mathbb{B}_w^\circ = \bigcup_{i=1}^\infty W_n$ and equip it with the empty word 1, the increment $[\mathtt{x}]$, the multiplication $\mathtt{xy}$ and the inversion $\mathtt{x}^\sim$ inherited from the W_i.

As example we explicitly write down

$$W_1 = \{1\},\ W_2 = \{1\} \cup \{[1]^n \colon 1 \leq n\} \cup \{[1]^{-n} \colon 1 \leq n\}.$$

Proposition 53 (Theorem). $\mathbb{B}_w^\circ \equiv \mathbb{B}^\circ$.

Proof. We first see that $\mathbb{B}_w^\circ$ is actually a coppice, because the group properties are inherited from the W_i. Because $\mathbb{B}_w^\circ$ is $\emptyset$-generated, by proposition 2 we only need to show that there is a homomorphism $\psi \colon \mathbb{B}_w^\circ \to \mathbb{B}^\circ$. It is however evident how to define this homomorphism, namely by

$$\psi([a]) := [\psi(a)], \qquad \psi([a]^\sim) := [\psi(a)]^\sim, \qquad \psi(w) := \Pi_i^\psi(w_i).$$

The constant 1 and $[\mathtt{x}]$ are compatible by definition, $\mathtt{xy}$ is compatible by equation (4) and (6) (cancelling), and $\mathtt{x}^\sim$ by (7). □

Proposition 54 (Theorem). $\mathbb{B}$ *is embeddable into* $\mathbb{B}_w^\circ$.

Proof. We have to show that there is an injective homomorphism $\iota \colon \mathbb{B} \to \mathbb{B}_w^\circ$. We define it recursively by

$$\begin{aligned} \iota(1) &:= 1, \\ \iota(a_L \oplus a_R) &:= \iota(a_L) \oplus \iota(a_R) := [\iota(a_L)\iota(a_R)^\sim]\iota(a_R) \end{aligned}$$

compatible with $\mathtt{x} \oplus \mathtt{y}$. We have to show that ι is injective. To do this we verify that each $\iota(a)$ has only positive letters, and is empty iff $a = 1$. This is easily done by recursion, $\iota(1) = 1$ is trivially valid, in the other case

$$\begin{aligned} \iota(a_L \oplus a_R) &= \iota(a_L) \oplus \iota(a_R) \\ &= [\iota(a_L)\iota(a_R)^\sim]\iota(a_R) \\ &= [\iota(a_L)\iota(a_R)^\sim] \centerdot \iota(a_R) \\ &\neq 1. \end{aligned}$$

By r.h. the right multiplicand $\iota(a_R)$ consists only of positive letters, so this multiplication is a simple concatenation and $\iota(a)$ again consists of only positive letters. Let us now finish the proof by showing $a = b$ whenever $\iota(a) = \iota(b)$. We discern the following cases.

1. $a = 1$ and $b = 1$: Then the conclusion $a = b$ is already true.
2. $a = a_L \oplus a_R$ and $b = 1$: then $a \neq b$ and already $\iota(a) \neq 1 = \iota(b)$.
3. $a = 1$ and $b = b_L \oplus b_R$: as for the last case.

4. $a = a_L \oplus a_R$ and $b = b_L \oplus b_R$: Comparing

$$[\iota(a_L)\iota(a_R)^\sim] \centerdot \iota(a_R) = [\iota(b_L)\iota(b_R)^\sim] \centerdot \iota(b_R)$$

yields $\iota(a_R) = \iota(b_R)$ and $\iota(a_L)\iota(a_R)^\sim = \iota(b_L)\iota(b_R)^\sim$ and hence $\iota(a_L) = \iota(b_L)$. By r.h. $a_R = b_R$ and $a_L = b_L$ and hence $a = b$.

□

Proposition 55 (Power Injectivity). *If $r^n = s^n$ for some $r, s \in \mathbb{B}^\circ$ and $n \in \mathbb{N}$ then $r = s$.*

Proof. First let us have a look at cancellation when raising to powers. For example in r^2 let p be the elements, which are removed by cancellation, i.e. $r = r_1 \centerdot p$ and $r = p^\sim \centerdot r_2$ such that $r^2 = r_1 \centerdot r_2$. The two representations of r can be put into one: $r = p^\sim \centerdot v \centerdot p$ for some cancelled v. The same holds for r^n (with $n > 1$). So the problem is reduced to

$$p^\sim \centerdot v^{\centerdot n} \centerdot p = r^n = s^n = q^\sim \centerdot w^{\centerdot n} \centerdot q.$$

Without restriction q is longer than p and so $q = a \centerdot p$ for some (possibly empty) a.

$$\begin{aligned} p^\sim \centerdot v^{\centerdot n} \centerdot p &= p^\sim \centerdot a^\sim \centerdot w^{\centerdot n} \centerdot a \centerdot p \\ v^{\centerdot n} &= a^\sim \centerdot w^{\centerdot n} \centerdot a \end{aligned}$$

We show that a is empty (and hence $v = w$ and $r = s$). If v is longer than a we have $v = a^\sim \centerdot v_1$ and $v = v_2 \centerdot a$ for some words v_1 and v_2. But this means that $v^{\centerdot n}$ was not cancelled (and it must be cancelled as part of r^n) unless a was already empty. Otherwise $a = a_1 \centerdot v$ and $a^\sim = v \centerdot a_2{}^\sim$ for some a_1, a_2. But this would mean that $v = v^\sim$ and hence $v = 1$ and $a = 1$. □

5.3 Fractional Trees

Definition 28 (fractional trees $\mathbb{F}$,$|\mathtt{x}|$). Let $\mathbb{F}$ be the free left-commutative coppice. Let $|\mathtt{x}| : \mathbb{F} \to \mathbb{Q}_+$ be the universal epimorphism (because $\mathbb{Q}_+$ is also left-commutative).

Because of left-commutativity of $\mathbb{F}$ we can use a similar bracket notation as in $\mathbb{P}$, i.e. write

$$[a_1, \ldots, a_k] \quad \text{for} \quad a_1 \oplus (\cdots \oplus (a_k \oplus 1) \cdots).$$

As we know from proposition 53 each element of $\mathbb{F}$ is equal to a product of (possibly inverted) terms $r_i \oplus 1$, or written as $[r_i]$ and $[r_i]^\sim$ ($r_i \in \mathbb{F}$). While each element of $\mathbb{B}^\circ$ has a unique representation as such a reduced product, the elements of $\mathbb{F}$ have not. For example if we apply the left-commutativity

$$\begin{aligned} s \oplus (r \oplus 1) = [r, s] &= r \oplus (s \oplus 1) \\ s \oplus [r] &= r \oplus [s] \\ (s[r]^\sim \oplus 1)[r] &= (r[s]^\sim \oplus 1)[s] \end{aligned}$$

we get

$$[s[r]^{\sim}][r] = [r[s]^{\sim}][s] \tag{8}$$

If we further multiply with inverses we get (see also proposition 83)

$$[r][s]^{\sim} = [s[r]^{\sim}]^{\sim}[r[s]^{\sim}]. \tag{9}$$

Despite the inconspicuous appearance of this formula, it is the key for transforming expressions into a normal form in $\mathbb{F}$. Equation (9) gives us a way to swap inverses in a word to the left. In a word we can change each sequence of two letters of the form $[r][s]^{\sim}$ into a sequence of the form $[r_2]^{\sim}[s_2]$.

By repeated application of this change, we can transform each word into the form $[s_k]^{\sim} \cdots [s_1]^{\sim}[t_1] \cdots [t_l]$. By distributivity $[s_1] \cdots [s_k]$ can be written as $[u_1, \ldots, u_k]$, similarly for t_i. So we finally can write all elements in $\mathbb{F}$ as $[u_1, \ldots, u_k]^{\sim}[v_1, \ldots, v_l]$ where $u_i, v_j \in \mathbb{F}$. So we realise that $\mathbb{F}$ consists of (quasi) fractions! From this observation the name *fractional trees* is derived.

Proposition 56. *For each $r \in \mathbb{F}$ there are $k, l \geq 0$ and $a_1, \ldots, a_k, b_1, \ldots, b_l \in \mathbb{F}$ such that $r = [a_1, \ldots, a_k]^{\sim}[b_1, \ldots, b_l]$.*

On the other hand it is not always possible to write an $r \in \mathbb{F}$ as

$$r = [a_1, \ldots, a_k][b_1, \ldots, b_l]^{\sim}$$

($a_i, b_j \in \mathbb{F}$) or swap inverses to the right (see proposition 81).

The next step is to decide whether two fractions are equal. For fractional *numbers* one way to do this is by $a/b = c/d \iff ad = cb$. This doesn't work for fractional *trees* because they are not commutative, we perhaps would get $b^{\sim}a = d^{\sim}c \iff db^{\sim} = ca^{\sim}$ which leads to nowhere.

Another possibility is to extend the fractions to share a common denominator and then to compare the extended nominators. This is indeed also possible for fractional *trees* and in some way equivalent to the third possibility: to cancel each and then directly compare both nominators and denominators, respectively. Though we have to start moderately by defining the multiset fractions with appropriate operations and congruence on it.

Definition 29 (multiset fractions $\mathrm{F}(X)$, lower/upper elements, $[\mathtt{x}\rfloor$`$[\mathtt{x}\rceil$, φ, $\breve{\mathtt{x}}$, value). Let $(X, 1, [\mathtt{x}], \mathtt{xy}, \mathtt{x}^{\sim})$ be a left-commutative coppice. Define $\mathrm{F}(X)$ as the set of the (formal) multiset fractions

$$r := \frac{[y_*]}{[x_*]},$$

where $x_*, y_* \in X$. The brackets as usual denote multisets (i.e. sequences where order does not matter). We write $r\rfloor$ for x_* and $r\rceil$ for y_*. Call $r\rfloor$ the *lower* elements and $r\rceil$ the *upper* elements of the multiset fraction r. We have the *value* homomorphism $\varphi\colon \mathrm{F}(X) \to X$ defined by

$$\varphi(r) := [r\rfloor]^{\sim}[r\rceil]$$

for all $r \in \mathrm{F}(X)$. For more readability we write also $\breve{r}$ for $\varphi(r)$ and for better layout we also write the multiset fraction r as $[x_*]\,\grave{}\,[y_*]$ inside text.

For convenience we define all multiset operations and relations also for multiset fractions in the natural way $(a\grave{}b) \circledast (c\grave{}d) = (a \circledast c)\grave{}(b \circledast d)$ and $(a\grave{}b)R(c\grave{}d) \iff aRc \wedge bRd$.

To become familiar with the new notation let us start with verifying some simple computation laws.

Proposition 57. *For all $r, s \in \mathrm{F}(X)$ the following two rules apply.*

$$\breve{r}^{\sim} = [r\restriction]^{\sim}[r\downharpoonleft] \tag{10}$$
$$\breve{r}\breve{s} = [s\downharpoonleft\breve{r}^{\sim}, r\downharpoonleft]^{\sim}[r\restriction\breve{s}, s\restriction] \tag{11}$$

Proof. Equation (10) is trivial:

$$([r\downharpoonleft]^{\sim}[r\restriction])^{\sim} = [r\restriction]^{\sim}[r\downharpoonleft]^{\sim\sim} = [r\restriction]^{\sim}[r\downharpoonleft].$$

To show (11) we use an extended variant of equation (9), let $a = [a_*], b = [b_*], a_*, b_* \in \mathbb{F}$:

$$ab^{\sim} = [b_*a^{\sim}]^{\sim}[a_*, b_*]b^{\sim} = [b_*a^{\sim}]^{\sim}[a_*b^{\sim}]. \tag{12}$$

The proof is left as an exercise. Then we finish (11) with

$$\begin{aligned}
\breve{r}\breve{s} &= [r\downharpoonleft]^{\sim}[r\restriction][s\downharpoonleft]^{\sim}[s\restriction] \\
&= [r\downharpoonleft]^{\sim}[s\downharpoonleft[r\restriction]^{\sim}]^{\sim}[r\restriction[s\downharpoonleft]^{\sim}][s\restriction] \\
&= ([s\downharpoonleft[r\restriction]^{\sim}][r\downharpoonleft])^{\sim}[r\restriction[s\downharpoonleft]^{\sim}][s\restriction] \\
&= [s\downharpoonleft[r\restriction]^{\sim}[r\downharpoonleft], r\downharpoonleft]^{\sim}[r\restriction[s\downharpoonleft]^{\sim}[s\restriction], s\restriction] \\
&= [s\downharpoonleft\breve{r}^{\sim}, r\downharpoonleft]^{\sim}[r\restriction\breve{s}, s\restriction].
\end{aligned}$$

□

We model coppice operations on $\mathrm{F}(X)$ on the previous laws in

Definition 30. On $\mathrm{F}(X)$ define:

$$\begin{array}{llll}
1 \in \mathrm{F}(X) & [\mathtt{x}]\colon \mathrm{F}(X) \to \mathrm{F}(X) & \mathtt{x} * \mathtt{y}\colon \mathrm{F}(X) \times \mathrm{F}(X) \to \mathrm{F}(X) & \mathtt{x}^{\sim}\colon \mathrm{F}(X) \to \mathrm{F}(X) \\
1 := \dfrac{[]}{[]} & [r] := \dfrac{[\breve{r}]}{[]} & r * s := \dfrac{[r\restriction\breve{s}, s\restriction]}{[s\downharpoonleft\breve{r}^{\sim}, r\downharpoonleft]} & r^{\sim} := \dfrac{[r\downharpoonleft]}{[r\restriction]}
\end{array}$$

Proposition 58. $\varphi\colon (\mathrm{F}(X), 1, [\mathtt{x}], \mathtt{x} * \mathtt{y}, \mathtt{x}^{\sim}) \to (X, 1, [\mathtt{x}], \mathtt{xy}, \mathtt{x}^{\sim})$ *is a homomorphism.*

Proof. $\varphi(1) = 1$, $\varphi[r] = [\varphi(r)]$, $\varphi(r * s) = \varphi(r)\varphi(s)$ and $\varphi(r^{\sim}) = \varphi(r)^{\sim}$ is easily seen by the previously said. □

These operations on $\mathrm{F}(X)$ do not constitute a coppice. The only coppice condition that is *not* satisfied is $r^{\sim} * r = 1$ because we have no suitable equivalence defined. We will do that after verifying the other coppice conditions.

Proposition 59. $(r * s)^{\sim} = s^{\sim} * r^{\sim}$ *for all $s, r \in \mathrm{F}(X)$.*

Proof.

$$(r * s)^{\sim} = \left(\frac{[r\restriction \breve{s}, s\restriction]}{[s\lfloor \breve{r}^{\sim}, r\lfloor]} \right)^{\sim} = \frac{[s\lfloor \breve{r}^{\sim}, r\lfloor]}{[r\restriction \breve{s}, s\restriction]} = s^{\sim} * r^{\sim}$$

□

Proposition 60 (F(X) Associativity). $(r * s) * t = r * (s * t)$ *for all* $r, s, t \in \mathrm{F}(X)$.

Proof.

$$\begin{aligned}
(r * s) * t &= \frac{[r\restriction \breve{s}, s\restriction]}{[s\lfloor \breve{r}^{\sim}, r\lfloor]} * t \\
&= \frac{[(r\restriction \breve{s})\breve{t}, s\restriction \breve{t}, t\restriction]}{[t\lfloor (\breve{r}\breve{s})^{\sim}, s\lfloor \breve{r}^{\sim}, r\lfloor]} \\
&= \frac{[r\restriction (\breve{s}\breve{t}), s\restriction \breve{t}, t\restriction]}{[(t\lfloor \breve{s}^{\sim})\breve{r}^{\sim}, s\lfloor \breve{r}^{\sim}, r\lfloor]} \\
&= r * \frac{[s\restriction \breve{t}, t\restriction]}{[t\lfloor \breve{s}^{\sim}, s\lfloor]} = r * (s * t)
\end{aligned}$$

□

Proposition 61 (F(X) Left-Commutativity). $r \oplus (s \oplus t) = s \oplus (r \oplus t)$ *for all* $r, s, t \in \mathrm{F}(X)$, *where* $r \oplus s := [r * s^{\sim}] * s$.

Proof. For easier verification we first calculate the defining term of $\mathtt{x} \oplus \mathtt{y}$ as a multiset fraction.

$$r \oplus s = \frac{[\breve{r}\breve{s}^{\sim}]}{[]} * s = \frac{[\breve{r}, s\restriction]}{[s\lfloor [\breve{r}\breve{s}^{\sim}]^{\sim}]}$$

And now we apply it twice to the ternary term.

$$\begin{aligned}
r \oplus (s \oplus t) &= r \oplus \frac{[\breve{s}, t\restriction]}{\left[t\lfloor \left[\breve{s}\breve{t}^{\sim}\right]^{\sim}\right]} \\
&= \frac{[\breve{r}, \breve{s}, t\restriction]}{\left[t\lfloor \left[\breve{s}\breve{t}^{\sim}\right]^{\sim} \left[\breve{r}\left(\left[\breve{s}\breve{t}^{\sim}\right] \breve{t}\right)^{\sim}\right]^{\sim}\right]} \\
&= \frac{[\breve{r}, \breve{s}, t\restriction]}{\left[t\lfloor \left[\breve{s}\breve{t}^{\sim}\right]^{\sim} \left[\breve{r}\breve{t}^{\sim} \left[\breve{s}\breve{t}^{\sim}\right]^{\sim}\right]^{\sim}\right]} \\
&= \frac{[\breve{r}, \breve{s}, t\restriction]}{\left[t\lfloor \left(\left[\breve{r}\breve{t}^{\sim} \left[\breve{s}\breve{t}^{\sim}\right]^{\sim}\right] \left[\breve{s}\breve{t}^{\sim}\right]\right)^{\sim}\right]} \\
&= \frac{[\breve{r}, \breve{s}, t\restriction]}{\left[t\lfloor \left[\breve{r}\breve{t}^{\sim}, \breve{s}\breve{t}^{\sim}\right]^{\sim}\right]}
\end{aligned}$$

The last term is symmetric in r, s. □

Let us now have a look at the equality of fractions with respect to their value. As we already saw, there are fractions with equal value, that are not equal as multiset fractions (for example $1 \neq r^{\sim} * r$ but $1 = \breve{r}^{\sim}\breve{r} = \varphi(r^{\sim} * r)$). Let us have a look at this issue. Let $\breve{r} = \breve{s}$ $(r, s \in \mathrm{F}(X))$, then

$$1 = \breve{r}^{\sim}\breve{s} = \varphi\left(r^{\sim} * s\right) = \varphi\frac{[r\lfloor\breve{s}, s\lceil]}{[s\lfloor\breve{r}, r\lceil]} = [s\lfloor\breve{r}, r\lceil]^{\sim}[r\lfloor\breve{s}, s\lceil]$$
$$\iff [s\lfloor\breve{r}, r\lceil] = [r\lfloor\breve{s}, s\lceil].$$

Before continuing let us mention that this result can also be achieved by extending r and s to get a common denominator.

$$\begin{aligned}
[r\lfloor]^{\sim}[r\lceil] &= [r\lfloor]^{\sim}[s\lfloor[r\lfloor]^{\sim}]^{\sim}[s\lfloor[r\lfloor]^{\sim}][r\lceil] \\
&= ([s\lfloor[r\lfloor]^{\sim}][r\lfloor])^{\sim}[s\lfloor[r\lfloor]^{\sim}[r\lceil], r\lceil] \\
\breve{r} &= [s\lfloor, r\lfloor]^{\sim}[s\lfloor\breve{r}, r\lceil] \\
\breve{s} &= [r\lfloor, s\lfloor]^{\sim}[r\lfloor\breve{s}, s\lceil]
\end{aligned}$$

If now $\breve{r} = \breve{s}$ then the nominators must be equal:

$$[s\lfloor\breve{r}, r\lceil] = [r\lfloor\breve{s}, s\lceil].$$

Under the slight assumption (though we can not show it until proposition 67) that both sides must then be equal as multisets, there are 4 cases of element equality:

1. $r\lceil = s\lceil$.
2. $r\lceil = r\lfloor\breve{s}$. It is equivalent to $r\lfloor^{\sim}r\lceil = \breve{s}$.
3. $s\lfloor\breve{r} = s\lceil$. It is equivalent to $s\lfloor^{\sim}s\lceil = \breve{r}$.
4. $s\lfloor\breve{r} = r\lfloor\breve{s}$. It is equivalent to $s\lfloor = r\lfloor$.

If we remove the equal pairs of case 4 and case 1, there must be a multiset bijection between the remaining $r\lfloor$ and $r\lceil$ satisfying case 2, and a multiset bijection between the remaining $s\lfloor$ and $s\lceil$ satisfying case 3. In other words: between the lower and upper elements of $r \setminus s$ there must be a multiset bijection satisfying case 2, and with $s \setminus r$ correspondingly. This motivates the next definition.

Definition 31 (x-distant, $\rhd$, $\simeq$). Call $r \in \mathrm{F}(X)$ u-distant $(u \in X)$ — symbolic $r \rhd u$ — iff $[r\lfloor u] = [r\lceil]$. (By definition the empty multiset fraction 1 is u-distant for every $u \in X$). For $r, s \in \mathrm{F}(X)$ define $r \simeq s$ as $\breve{r} = \breve{s} =: u$ and $s \setminus r$ and $r \setminus s$ are u-distant.

$r \rhd u$ induces a multiset bijection $\delta\colon [r\lfloor] \to [r\lceil]$ with $x^{\sim}y = u$ for all $(x, y) \in \delta$, we say $r \rhd u$ via δ. For convenience we define mixed set operations between multiset fractions and multiset bijections by first converting each multiset bijection $\delta\colon [a_*] \to [b_*]$ to the multiset fraction $[a_*] \setminus [b_*]$.

Note, that from the conventions chapter we will regard a multiset bijection δ between multisets a and b as a bijection between $\sigma(a)$ and $\sigma(b)$. Which in turn is equivalent to regarding δ as a multiset of pairs (x, y) such that $a = [\pi_1(\delta_*)]$ and $b = [\pi_2(\delta_*)]$. (π_1 and π_2 are the projections defined by $\pi_1((x, y)) = x$ and $\pi_2((x, y)) = y$.)

Proposition 62. *If r and s are u-distant then $r \cap s$, $r \cup s$ and $r \setminus s$ are also u-distant.*

Proof. Because $f(x) := xu$ is a bijection on X the multiplicities in a multiset are conserved under applying f to the elements, i.e. for $a_* \in X$:

$$[a_* u]^\chi(x) = [a_*]^\chi(xu^\sim) \quad \text{for all } x \in X.$$

Further we derive

$$[a_* u] \cap [b_* u] = [(a \cap b)_* u], \qquad [a_* u] \cup [b_* u] = [(a \cup b)_* u], \qquad [a_* u] \setminus [b_* u] = [(a \setminus b)_* u]$$

by

$$\begin{aligned}
([a_* u] \cap [b_* u])^\chi(x) &= \min\{[a_* u]^\chi(x), [b_* u]^\chi(x)\} \\
&= \min\{[a_*]^\chi(xu^\sim), [b_*]^\chi(xu^\sim)\} \\
&= (a \cap b)^\chi(xu^\sim) \\
&= [(a \cap b)_* u]^\chi(x),
\end{aligned}$$

$$\begin{aligned}
([a_* u] \cup [b_* u])^\chi(x) &= \max\{[a_* u]^\chi(x), [b_* u]^\chi(x)\} \\
&= \max\{[a_*]^\chi(xu^\sim), [b_*]^\chi(xu^\sim)\} \\
&= (a \cup b)^\chi(xu^\sim) \\
&= [(a \cup b)_* u]^\chi(x),
\end{aligned}$$

and

$$(a \setminus b)^\chi(x) = (a \setminus^\circ (a \cap b))^\chi(x) = a^\chi(x) - (a \cap b)^\chi(x).$$

Hence for $\bigcirc$ being any of the 3 operations we get

$$[(r \bigcirc s)\lfloor u] = [r\lfloor u] \bigcirc [s\lfloor u] = [r\lceil] \bigcirc [s\lceil] = [(r \bigcirc s)\lceil].$$

□

Proposition 63. $\mathtt{x} \simeq \mathtt{y}$ *is an equivalence relation on* $\mathrm{F}(X)$.

Proof. That $r \simeq r$ and $(r \simeq s \iff s \simeq r)$ are trivial. We show transitivity ($r \simeq s \wedge s \simeq t \implies r \simeq t$ for all $r, s, t \in \mathrm{F}(X)$). From $r \simeq s$ and $s \simeq t$ we conclude $\breve{r} = \breve{s} = \breve{t} =: u$. Then $r \setminus s$, $s \setminus r$, $t \setminus s$ and $s \setminus t$ are each u-distant and so are both the following terms.

$$\begin{aligned}
r \setminus t &= ((r \setminus s) \setminus (t \setminus s)) \cup ((s \setminus t) \setminus (s \setminus r)) \rhd u \\
t \setminus r &= ((t \setminus s) \setminus (r \setminus s)) \cup ((s \setminus r) \setminus (s \setminus t)) \rhd u
\end{aligned}$$

□

Proposition 64. $\mathtt{x} \simeq \mathtt{y}$ *is a congruence relation on* $\mathrm{F}(X)$.

Proof. We have to show the compatibility with the 4 operations 1, $[\mathtt{x}]$, $\mathtt{x}^\sim$ and $\mathtt{x} * \mathtt{y}$. For the constant 1 is nothing to show, $[r'] = [r]$ holds trivially for $r' \simeq r$. Let now $r \simeq r'$ by $\breve{r} = \breve{r'} =: u$, $r \setminus r' \rhd u$, $r' \setminus r \rhd u$. From $x \rhd u \iff x^\sim \rhd u^\sim$ for arbitrary x follows $r^\sim \simeq r'^\sim$ by substituting x with $r \setminus r'$ and $r' \setminus r$.

At last we show the compatibility of $\mathtt{x} * \mathtt{y}$. It suffices to show $r * s \simeq r' * s$ for arbitrary s, because the right side compatibility follows then from $s * r = (r^\sim * s^\sim)^\sim \simeq (r'^\sim * s^\sim)^\sim = s * r'$ (by proposition 59).

$$r * s \setminus r' * s = \frac{[r\restriction\breve{s}, s\restriction]}{[s\lfloor u^\sim, r\lfloor]} \setminus \frac{[r'\restriction\breve{s}, s\restriction]}{[s\lfloor u^\sim, r'\lfloor]} = \frac{[(r \setminus r')\restriction\breve{s}]}{[(r \setminus r')\lfloor]} \rhd u\breve{s}$$

Similarly $r' * s \setminus r * s \rhd u\breve{s}$ yields $r * s \simeq r' * s$. □

Proposition 65. $\mathrm{F}(X)/_\simeq$ *is an lc-coppice.*

Proof. Associativity and left-commutativity are already shown (in proposition 60 and proposition 61) even with $\mathtt{x} = \mathtt{y}$ instead of $\mathtt{x} \simeq \mathtt{y}$; $1 * r = r$ is trivial. Mainly we have to show that $r^\sim * r \simeq 1$ (this was the equation for which we defined $\mathtt{x} \simeq \mathtt{y}$):

$$1 \simeq \frac{[r\lfloor\breve{r}, r\restriction]}{[r\lfloor\breve{r}, r\restriction]}.$$

The bijection between the lower and upper elements is obvious and it is always guaranteed that $r\restriction_i{}^\sim r\restriction_i = 1$ and $(r\lfloor_j\breve{r})^\sim r\lfloor_j\breve{r} = 1$. □

Proposition 66 (Theorem). $\mathbb{F} \equiv \mathrm{F}(\mathbb{F})/_\simeq$.

Proof. By proposition 2 we only need to show that there is a homomorphism $\mathrm{F}(\mathbb{F})/_\simeq \to \mathbb{F}$. But $\mathtt{x} \simeq \mathtt{y}$ has additional requirements to $\varphi(\mathtt{x}) = \varphi(\mathtt{y})$ (i.e. possibly more pairs are unequal). So $\varphi/_\simeq$ is the searched for homomorphism. □

Proposition 67 (Corollary). *If* $[a_*]$ *and* $[b_*]$ *(*$a_*, b_* \in \mathbb{F}$*) are equal as elements of* $\mathbb{F}$ *then they are equal as multisets. Particularly if* $[v] = [w]$ *then* $v = w$ *for all* $v, w \in \mathbb{F}$.

Proof. Let $r := [] \setminus [a_*]$ and $s := [] \setminus [b_*]$ then equality as elements of $\mathbb{F}$ means $\breve{r} = \breve{s} =: u$. By proposition 66 we know that then $r \setminus s, s \setminus r \rhd u$. Unfortunately neither $r \setminus s$ nor $s \setminus r$ contains any pairs, so they must be empty. Hence $r = s$. □

We are on the way to a cancel concept for $\mathbb{F}$, i.e. is there a unique smallest subfraction $s \subseteq t \in \mathrm{F}(\mathbb{F})$ with $s \simeq t$? Let us first have a look at the opposite direction, how we can expand a fraction (without changing its value).

Proposition 68 (Fraction Expansion).

$$v^\sim w = [a_*]^\sim [b_*] \Longrightarrow v^\sim w = [v, a_*]^\sim [w, b_*]$$

for elements v, w, a_*, b_* *of an lc-coppice.*

Proof. Consider $[u, c_*]$ for some lc-coppice elements u, c_*. By (8) we can "pull out" u to the left:

$$[u, c_*] = [uc^\sim]c$$

and apply this to $[v, a_*]$ and $[w, b_*]$:

$$\begin{aligned}[v, a_*]^\sim[w, b_*] &= ([va^\sim]a)^\sim([wb^\sim]b) \\ &= a^\sim[va^\sim]^\sim[wb^\sim]b.\end{aligned}$$

The middle two elements vanish because from the precondition we conclude:

$$\begin{aligned}v^\sim w &= a^\sim b \\ w &= va^\sim b \\ wb^\sim &= va^\sim.\end{aligned}$$

So finally

$$[v, a_*]^\sim[w, b_*] = a^\sim b = v^\sim w.$$

□

Definition 32 (removable, remove, added, reduced, $\mathrm{cF}(X)$). We call $(v, w) \in X \times X$ a *removable pair* in $t \in \mathrm{F}(X)$, or say that (v, w) can be *removed* from t, iff removing the pair keeps the value of the fraction, i.e. iff $\varphi(t) = \varphi(t \setminus^\circ [v] \setminus [w])$. We call $(v, w) \in X \times X$ an *added pair* in $t \in \mathrm{F}(X)$, iff it was added by fraction expansion, i.e. iff $v^\sim w = \varphi(t \setminus^\circ [v] \setminus [w])$. We call $t \in \mathrm{F}(X)$ *reduced* iff there are no added pairs in t. Let $\mathrm{cF}(X)$ be the reduced elements of $\mathrm{F}(X)$.

First we see directly from proposition 68 that every added pair is also a removable pair (whether every removable pair is also an added pair will be answered for $X = \mathbb{F}$ in proposition 69.), i.e. if we have any multiset fraction r we can remove any added pair $(r\lfloor_i, r\lceil_j)$. So we can always remove added pairs until the fraction is reduced. The problem we will solve next is whether the result reduced fraction depends on which added pairs and in which order we remove, and whether we can remove several added pairs at once. For example assume we have 2 added pairs in $[a_*] \setminus [b_*]$:

$$\begin{aligned}a_1{}^\sim b_1 &= (a \setminus^\circ [a_1])^\sim(b \setminus^\circ [b_1]), \\ a_2{}^\sim b_2 &= (a \setminus^\circ [a_2])^\sim(b \setminus^\circ [b_2]).\end{aligned}$$

Now it would be nice to know that (a_2, b_2) is also an added pair in $(a \setminus^\circ [a_1])^\sim(b \setminus^\circ [b_1])$ and (a_1, b_1) an added pair in $(a \setminus^\circ [a_2])^\sim(b \setminus^\circ [b_2])$. So that we can remove both pairs regardless of the starting pair and have $a_1{}^\sim b_1 = a_2{}^\sim b_2 = (a \setminus^\circ [a_1, a_2])^\sim(b \setminus^\circ [b_1, b_2])$. Because — the other way around — if we had that

$$a_1{}^\sim b_1 = a_2{}^\sim b_2 = (a \setminus^\circ [a_1, a_2])^\sim(b \setminus^\circ [b_1, b_2])$$

then already (a_1, b_1) and (a_2, b_2) would be added pairs in $[a_*] \setminus [b_*]$ (see proposition 70).

This possibility to remove added pairs of t in any order from t could be achieved by the opposite direction of the fraction expansion (which we thats why call *cancellativeness*):

$$v^{\sim}w = [v, a_*]^{\sim}[w, b_*] \Longrightarrow v^{\sim}w = [a_*]^{\sim}[b_*]. \tag{13}$$

As property of a left-commutative coppice it is by the way equivalent to the injectivity of $1 \oplus x$ and the left-cancellativeness of the addition (see proposition 84). So the name "cancellativeness" has by incidence a twofold meaning here.

Proposition 69 (Remark). *For each fraction t of $\mathrm{F}(\mathbb{F})$ every removable pair is also an added pair.*

Proof. Consider a removable pair (v, w) of $t = [v, a_*]^{\backprime}[w, b_*]$. We have the following equivalences, where the first line depicts the pair as removable and the last line as added.

$$\begin{aligned}
[v, a_*]^{\sim}[w, b_*] &= [a_*]^{\sim}[b_*] \\
a^{\sim}[va^{\sim}]^{\sim}[wb^{\sim}]b &= a^{\sim}b \\
[va^{\sim}]^{\sim}[wb^{\sim}] &= 1 \\
[va^{\sim}] &= [wb^{\sim}] \\
va^{\sim} &= wb^{\sim} && \text{(by proposition 67)} \\
a^{\sim}b &= v^{\sim}w
\end{aligned}$$

□

Proposition 70. *If $r \mathbin{\backslash^{\circ}} s \rhd \breve{s}$ via δ then all $(v, w) \in \delta$ are added pairs in r.*

Proof. Consecutively adding pairs of δ to s keeps the value of s unchanged (by fraction expansion proposition 68). For a fixed pair $(v, w) \in \delta$ we simply consecutively add all the other pairs of δ to s:

$$v^{\sim}w = \breve{s} = \varphi(s \uplus (\delta \mathbin{\backslash^{\circ}} \{(v, w)\})) = \varphi(r \setminus [v]^{\backprime}[w]).$$

□

The following definition and intermediate properties are slightly customised for use in our main theorem proposition 75, that $\mathbb{F}$ is cancellative.

Definition 33 (cancellative). We call a pair $(v, w) \in X \times X$ of an lc-coppice X *cancellative* iff it satisfies (13) for all $a_*, b_* \in X$. We say $r \in \mathrm{F}(X)$ *is cancellative* iff every pair $(r{\downarrow}, r{\uparrow})$ of r is cancellative. We say the lc-coppice X *is cancellative* iff every pair of X is cancellative.

Proposition 71. *If $r \mathbin{\backslash^{\circ}} s \rhd \breve{r}$ via δ and all pairs of δ are cancellative then $\breve{s} = \breve{r}$, or in other words: δ can be removed from r (yielding s).*

Proof. Let $\delta := [(a_1, b_1), \ldots, (a_n, b_n)]$, by induction the pairs $(a_1, b_1), \ldots, (a_k, b_k) \in \delta$ $(k < n)$ are removable from r, then for the next pair (a_{k+1}, b_{k+1})

$$a_{k+1}{}^{\sim}b_{k+1} = \breve{r} = \left(r \mathbin{\backslash^{\circ}} \frac{[b_1, \ldots, b_k]}{[a_1, \ldots, a_k]} \right)^{\varphi}$$

By cancellativeness (a_{k+1}, b_{k+1}) can be removed from the right side fraction, which means that $[(a_1, b_1), \ldots, (a_{k+1}, b_{k+1})]$ can be removed from r. □

Proposition 72. *If $r \simeq s$ for two reduced cancellative fractions $r, s \in \mathrm{cF}(X)$ then $r = s$.*

Proof. Suppose $r \neq s$. Then $r \setminus s \neq 1$ or $s \setminus r \neq 1$. Without restriction assume $r \setminus s \neq 1$. Because $r \setminus s \rhd \breve{r}$ there is some pair $(v, w) \in r \setminus s$ with $v^{\sim} w = \breve{r}$. The pair (v, w) is added in r because (v, w) is cancellative. This contradicts r being reduced. $\square$

The next proposition looks a bit technical but has a simple motivation. For $r, s \in \mathrm{F}(X)$ we know that

$$r * s = \frac{[r\restriction \breve{s}, s\restriction]}{[s\lfloor \breve{r}^{\sim}, r\lfloor]}.$$

The next proposition states that we can remove any set of pairs $(s\lfloor_i \breve{r}^{\sim}, r\restriction_j \breve{s})$ from $r * s$ whenever $s\lfloor_i = r\restriction_j$. So the cancellativeness is already supplied for those pairs, i.e. let $v = s\lfloor_i \breve{r}^{\sim}$, $w = r\restriction_j \breve{s}$ in definition 33 and notice that $v^{\sim} w = (r * s)^{\varphi}$. For simpler later use we regard $r^{\sim} * s$ instead of $r * s$.

Proposition 73 (Lemma).

$$\textit{If} \quad r = \frac{[r'\restriction]}{[a_*, r'\lfloor]} \quad \textit{and} \quad s = \frac{[s'\restriction]}{[a_*, s'\lfloor]} \quad \textit{then} \quad \breve{r}^{\sim}\breve{s} = \left((r^{\sim} * s) \setminus^{\circ} \frac{[a_* \breve{s}]}{[a_* \breve{r}]} \right)^{\varphi}$$

for all $r', s' \in \mathrm{F}(X)$ and $a_* \in X$.

Proof.

$$\begin{aligned}
\breve{r}^{\sim}\breve{s} &= [r\restriction]^{\sim}[r\lfloor][s\lfloor]^{\sim}[s\restriction] \\
&= [r\restriction]^{\sim}[r'\lfloor a^{\sim}]aa^{\sim}[s'\lfloor a^{\sim}]^{\sim}[s\restriction] \\
&= [r\restriction]^{\sim}[r'\lfloor a^{\sim}][s'\lfloor a^{\sim}]^{\sim}[s\restriction] \\
&= [r\restriction]^{\sim} \left(\frac{[r'\lfloor a^{\sim}]}{[]} * \frac{[]}{[s'\lfloor a^{\sim}]} \right)^{\varphi} [s\restriction] \\
&= [r\restriction]^{\sim} \left(\frac{[r'\lfloor a^{\sim}[s'\lfloor a^{\sim}]^{\sim}]}{[s'\lfloor a^{\sim}[r'\lfloor a^{\sim}]^{\sim}]} \right)^{\varphi} [s\restriction] \\
&= [r\restriction]^{\sim} \left(\frac{[r'\lfloor ([s'\lfloor a^{\sim}]a)^{\sim}]}{[s'\lfloor ([r'\lfloor a^{\sim}]a)^{\sim}]} \right)^{\varphi} [s\restriction] \\
&= [r\restriction]^{\sim} \left(\frac{[r'\lfloor [s\lfloor]^{\sim}]}{[s'\lfloor [r\lfloor]^{\sim}]} \right)^{\varphi} [s\restriction] \\
&= \varphi \frac{[r'\lfloor [s\lfloor]^{\sim}[s\restriction], s\restriction]}{[s'\lfloor [r\lfloor]^{\sim}[r\restriction], r\restriction]} \\
&= \varphi \frac{[r'\lfloor \breve{s}, s\restriction]}{[s'\lfloor \breve{r}, r\restriction]} \\
&= \left((r^{\sim} * s) \setminus^{\circ} \frac{[a_* \breve{s}]}{[a_* \breve{r}]} \right)^{\varphi}
\end{aligned}$$

$\square$

Definition 34 (tree fractions F^*, recursively reduced, cF^*, φ, φ', n). Let F^* be the smallest set that for each $a_1, \dots, a_k$, $b_1, \dots, b_l \in F^*$ $(k, l \in \mathbb{N}_0)$ contains also the multiset fraction $[a_1, \dots, a_k] \setminus [b_1, \dots, b_l]$. Call its elements tree fractions. We define recursively the following (coppice similar) operations.

$$1 := \frac{[]}{[]} \qquad a^\sim := \frac{[a\rfloor]}{[a\lceil]} \qquad a * b := \frac{[a\lceil * b, a\lceil]}{[b\rfloor * a^\sim, a\rfloor]} \qquad a \boxplus b := [a * b^\sim] * b$$

Define the value functions $\varphi\colon F^* \to \mathbb{F}$ and $\varphi'\colon F^* \to \mathrm{F}(\mathbb{F})$ recursively by

$$\varphi \frac{[r\lceil]}{[r\rfloor]} := [\varphi(r\rfloor)]^\sim [\varphi(r\lceil)], \qquad \varphi' \frac{[r\lceil]}{[r\rfloor]} := \frac{[\varphi(r\lceil)]}{[\varphi(r\rfloor)]}.$$

Call $r \in F^*$ *recursively reduced* iff $\varphi'(r)$ is reduced and all $r\rfloor$ and $r\lceil$ are recursively reduced. Let cF^* be the recursively reduced elements r of F^*. For recursion define the following natural value function $\mathrm{n}\colon F^* \to \mathbb{N}$ recursively by

$$\mathrm{n}(r) := 1 + \sum \mathrm{n}(r\rfloor) + \sum \mathrm{n}(r\lceil).$$

Proposition 74. *$\varphi\colon F^* \to \mathbb{F}$ is surjective.*

Proof. Let $G \subset F^*$ be the subset which is generated from $\emptyset$ by the operations $1, \mathtt{x} \boxplus \mathtt{y}, \mathtt{x} * \mathtt{y}, \mathtt{x}^\sim$ of F^*. Then $\varphi\colon G \to \mathbb{F}$ is surjective because $\mathbb{F}$ is also only generated by $\emptyset$. And so $\varphi\colon F^* \to \mathbb{F}$ is surjective. □

Proposition 75 (Theorem). *$\mathbb{F}$ is cancellative.*

Proof. We prove the proposition by recursion over $r, s \in F^*$ that for $v := \varphi(r)$ and $w := \varphi(s)$

$$v^\sim w = [v, t\rfloor]^\sim [w, t\lceil] \Longrightarrow v^\sim w = [t\rfloor]^\sim [t\lceil]$$

for every $t \in \mathrm{F}(\mathbb{F})$, i.e. that (v, w) is cancellative. Because $\varphi\colon F^* \to \mathbb{F}$ is surjective it is then proven for all elements of $\mathbb{F}$. For convenience write $v\rfloor$ for $\varphi(r\rfloor)$ and $\grave{v}$ for $[\varphi(r\rfloor)] \setminus [\varphi(r\lceil)]$ and correspondingly for w and s. We already know that $\mathbb{F} \equiv \mathrm{F}(\mathbb{F})/_\simeq$ (was proposition 66), so the following statement is equivalent to the above.

$$\grave{v}^\sim * \grave{w} \simeq \frac{[w, t\lceil]}{[v, t\rfloor]} \Longrightarrow \grave{v}^\sim * \grave{w} \simeq \frac{[t\lceil]}{[t\rfloor]}$$

We assume that the claim was already shown for all elements $r_2, s_2 \in F^*$ with $\mathrm{n}(r_2) + \mathrm{n}(s_2) < \mathrm{n}(r) + \mathrm{n}(s)$. We can further assume that r and s are recursively reduced. Otherwise we simply remove an added pair from (a subelement of) r yielding r_2 with $\varphi(r) = \varphi(r_2)$ and $\mathrm{n}(r_2) < \mathrm{n}(r)$. By r.h. then the claim is already shown (s respectively). Let us start by expanding the precondition:

$$\frac{[v\rfloor w, w\lceil]}{[w\rfloor v, v\lceil]} \simeq \frac{[w, t\lceil]}{[v, t\rfloor]}.$$

For each lower fraction element x on the left side that is not equal to a lower fraction element on the right side, there must be an upper element y on the left side such that $x^\sim y = \varphi(\grave{v}^\sim * \grave{w}) = v^\sim w$ (and similar for an upper fraction element y on the left side) by definition 31. There are 4 cases of such x and y:

1. $x = v\restriction_i$ and $y = w\restriction_j$, then by r.h. $(v\restriction_i, w\restriction_j)$ is cancellative.
2. $x = v\restriction_i$ and $y = v\lfloor_j w$, then
$$\begin{aligned} v^\sim w = x^\sim y &= v\restriction_i{}^\sim v\lfloor_j w, \\ v^\sim &= v\restriction_i{}^\sim v\lfloor_j, \\ v &= v\lfloor_j{}^\sim v\restriction_i. \end{aligned}$$
And by r.h. this means that $\grave{v}$ was not reduced (in contradiction to our assumption).
3. $x = w\lfloor_i v$ and $y = w\restriction_j$, similar to the previous case.
4. $x = w\lfloor_i v$ and $y = v\lfloor_j w$, then:
$$\begin{aligned} v^\sim w = x^\sim y &= v^\sim w\lfloor_i{}^\sim v\lfloor_j w, \\ 1 &= w\lfloor_i{}^\sim v\lfloor_j, \\ w\lfloor_i &= v\lfloor_j. \end{aligned}$$
Some such $(w\lfloor_i v, v\lfloor_j w)$ can then be removed from $\grave{v}^\sim * \grave{w}$ by proposition 73.

Let now $q := [v, t\lfloor] \setminus [w, t\restriction]$ be the right side and let $(\grave{v}^\sim * \grave{w}) \setminus q \rhd v^\sim w$ via δ. We have shown that δ can be split into δ_1 (according to case 1) and δ_4 (according to case 4). And that $v^\sim w = (\grave{v}^\sim * \grave{w} \setminus^\circ \delta_4)^\varphi$ by proposition 73. By cancellativeness of δ_1 and proposition 71 we can remove the pairs of δ_1 from $(\grave{v}^\sim * \grave{w}) \setminus \delta_4$, i.e. $v^\sim w = \breve{p}$ where $p := \grave{v}^\sim * \grave{w} \setminus^\circ \delta$. p is the common part of $\grave{v}^\sim * \grave{w}$ and q.
$$q \setminus^\circ p \rhd v^\sim w = \breve{p}$$
Therefore if $[v] \setminus [w] \cap (q \setminus^\circ p) \neq 1$ then already (by proposition 62) $[v] \setminus [w] \subseteq q \setminus^\circ p$, resulting (by proposition 70) in (v, w) being an added and so a removable pair in q (that was to be proved).

If otherwise
$$\frac{[w]}{[v]} \subseteq p \subseteq \frac{[v\lfloor w, w\restriction]}{[w\lfloor v, v\restriction]}$$
then for w there are two possibilities, either $w = v\lfloor w$ or $w = w\restriction$ (for some $v\lfloor$ or $w\restriction$, respectively). If $w = w\restriction_i$ then $\grave{w} \simeq \grave{w}\restriction_i$. Because $\grave{w}\restriction_i$ and $\grave{w}$ are reduced and since by r.h. they are both cancellative we get by proposition 72 that $\grave{w} = \grave{w}\restriction_i$ and by repetition even $s = s\restriction_i$. But $s \in F^*$ can not be an element of itself. So $w \neq w\restriction$ and similarly $v \neq v\restriction$.

It remains that $v = w\lfloor_i v$ and $w = v\lfloor_j w$ for some i and j. But then $w\lfloor_i = 1 = v\lfloor_j$ already and again with the previous arguments
$$\begin{gathered} p' := p \setminus^\circ \frac{[w]}{[v]} = (\grave{v}^\sim * \grave{w}) \setminus^\circ (\delta_4 \uplus [(v, w)]) \setminus^\circ \delta_1 \simeq q, \\ q \setminus^\circ p' \rhd (p')^\varphi, \\ [v] \setminus [w] \subseteq q \setminus^\circ p' \end{gathered}$$
(v, w) is removable in q. □

Proposition 76 (Corollary). *If $\varphi(r) = \varphi(s)$ and $r, s \in F^*$ are recursively reduced then by proposition 72 already $r = s$.*

5.3.1 Deciding Equality in the Fractional Trees

We now construct an algorithm that decides whether $\varphi(r) = \varphi(s)$ for $r, s \in F^*$. To do this we first define computable coppice operations on cF^*. For elements $r, s \in cF^*$ we already know that $\varphi(r) = \varphi(s) \iff r = s$. Though we have no algorithm yet to get the reduced fraction from an arbitrary recursive multiset fraction.

For the sake of precision we first introduce a computable linear order $\ll$ on F^* that we initially need to define $\mathtt{xy}$. The order merely has to be linear and computable, so we can choose a kind of lexicographic order, or whatever the reader fancies.

Definition 35 ($\mathtt{x} * \mathtt{y}$, $\mathtt{x} \star \mathtt{y}$ and $\mathtt{xy}$ on F^*). Define the *reducing multiplication* $\mathtt{xy}$, the *semi-reducing multiplication* $\mathtt{x} \star \mathtt{y}$ and the *non-reducing multiplication* $\mathtt{x} * \mathtt{y}$ on F^* recursively by

$$\begin{aligned} r * s &:= \frac{[r\restriction s, s\restriction]}{[s\downharpoonleft r^\sim, r\downharpoonleft]}, \\ r \star s &:= r * s \setminus^\circ \frac{[h_* s]}{[h_* r^\sim]}, \quad \text{where} \quad h := [s\downharpoonleft] \cap [r\restriction], \\ rs &:= (r \star s) \setminus t, \end{aligned}$$

where $t \subseteq [r\downharpoonleft] \setminus [s\restriction]$ is the $\ll$-smallest $\subseteq$-maximal subfraction such that there is a multiset bijection $\delta\colon [t\downharpoonleft] \to [t\restriction]$ with $x^\sim y = (r \star s) \setminus^\circ t$ for all $(x, y) \in \delta$.

The definition is valid because the used terms $r\restriction s$, $s\downharpoonleft r^\sim$ and $r\downharpoonleft^\sim s\restriction$ (in $x^\sim y$) are all defined by r.h. We will see later (in the proof of proposition 78) that there is anyway at most *one* such non-empty t, so we can drop the usage of $\ll$ then.

Proposition 77. *$\varphi(r \star s) = \varphi(r)\varphi(s)$ and $\varphi(rs) = \varphi(r)\varphi(s)$ for all $r, s \in F^*$.*

Proof. As usual we prove the proposition by recursion over r and s. Let $c := [r\restriction] \cap [s\downharpoonleft]$, $a := [r\restriction] \setminus^\circ c$ and $b := [s\downharpoonleft] \setminus^\circ c$ then

$$\begin{aligned} \breve{r}\breve{s} &= \left(\frac{[r\restriction^\varphi]}{[r\downharpoonleft^\varphi]} * \frac{[s\restriction^\varphi]}{[s\downharpoonleft^\varphi]} \right)^\varphi \\ &= [s\downharpoonleft^\varphi \breve{r}^\sim, r\downharpoonleft^\varphi]^\sim [r\restriction^\varphi \breve{s}, s\restriction^\varphi] \\ &= [b_*{}^\varphi \breve{r}^\sim, c_*{}^\varphi \breve{r}^\sim, r\downharpoonleft^\varphi]^\sim [a_*{}^\varphi \breve{s}, c_*{}^\varphi \breve{s}, s\restriction^\varphi] \\ \text{by proposition 73} \quad &= [b_*{}^\varphi \breve{r}^\sim, r\downharpoonleft^\varphi]^\sim [a_*{}^\varphi \breve{s}, s\restriction^\varphi] \\ \text{by r.h.} \quad &= [(b_* r^\sim)^\varphi, r\downharpoonleft^\varphi]^\sim [(a_* s)^\varphi, s\restriction^\varphi] \\ &= \varphi(r \star s). \end{aligned}$$

Now consider t and δ in the definition of the reducing multiplication. $\varphi'(t)$ is equidistant to $\varphi(r \star s \setminus^\circ t)$ because by r.h. $\varphi(x^\sim y) = \breve{x}^\sim \breve{y}$.

$$\begin{aligned} \varphi'(r \star s \setminus^\circ t) &= \varphi'(r \star s) \setminus^\circ \varphi'(t) \\ \varphi'(t) &\rhd \varphi(\varphi'(r \star s) \setminus^\circ \varphi'(t)) \end{aligned}$$

By expansion (proposition 68) we can add the valued pairs of δ consecutively to $\varphi'(r \star s) \setminus^\circ \varphi'(t)$, then

$$\begin{aligned}\varphi(\varphi'(r \star s)) &= \varphi(\varphi'(r \star s) \setminus^\circ \varphi'(t)),\\ \breve{r}\breve{s} = \varphi(r \star s) &= \varphi(\varphi'(r \star s \setminus^\circ t)) = \varphi(r \star s \setminus^\circ t) = \varphi(rs).\end{aligned}$$

□

Proposition 78. *If $r, s \in F^*$ are recursively reduced then rs is also recursively reduced.*

Proof. The elements $r\restriction s$ and $s\lfloor r^\sim$ of rs are by r.h. recursively reduced, so all elements of rs are recursively reduced and by proposition 76 and proposition 77 we do not need to care about the application of the value function and can use `xy` of $\mathbb{F}$ instead of `xy` of F^* on them (which we will do). Further we are trained enough in the correct application of the value function so for convenience we omit it also on r or s.

We have to show that $\varphi'(rs)$ is reduced. So let us look out for possibly added pairs in $\varphi'(rs)$ (that would invalidate it being reduced). By expansion (proposition 68) also $x^\sim y = rs$ for an added pair (x, y) in $\varphi'(rs)$. There are again four cases for added pairs in $\varphi'(rs)$:

1. $(s\lfloor_i r^\sim, r\restriction_j s)$. Then conclude

$$\begin{aligned}(s\lfloor_i r^\sim)^\sim r\restriction_j s &= rs,\\ rs\lfloor_i{}^\sim r\restriction_j s &= rs,\\ s\lfloor_i{}^\sim r\restriction_j &= 1,\\ r\restriction_j &= s\lfloor_i.\end{aligned}$$

   But all those elements were already removed in $\mathtt{x} \star \mathtt{y}$.

2. $(s\lfloor_i r^\sim, s\restriction_j)$. Then conclude

$$\begin{aligned}(s\lfloor_i r^\sim)^\sim s\restriction_j &= rs,\\ rs\lfloor_i{}^\sim s\restriction_j &= rs,\\ s\lfloor_i{}^\sim s\restriction_j &= s.\end{aligned}$$

   By cancellativeness of $\mathbb{F}$ (proposition 75) the pair $(s\lfloor_i, s\restriction_j)$ would be an added pair of s.

3. $(r\lfloor_i, r\restriction_j s)$. Similar to the previous case.

4. $(r\lfloor_i, s\restriction_j)$. Then conclude

$$r\lfloor_i{}^\sim s\restriction_j = rs = \varphi(\varphi'(r \star s) \setminus^\circ \varphi'(t)).$$

   If t was not empty, i.e. there exist $x \in [r\lfloor]$ and $y \in [s\restriction]$ with

$$x^\sim y = \varphi'(r \star s) \setminus^\circ \varphi'(t),$$

then by r.h. the left side is reduced and so must be the right side. But by cancellativeness of $\mathbb{F}$ (proposition 75) $(r\lfloor_i, s\lceil_j)$ would be an added pair of the right side. If otherwise t was the empty fraction,

$$r\lfloor_i{}^{\sim} s\lceil_j = \varphi(\varphi'(r \star s)) \tag{14}$$

consider all added pairs in $\varphi'(r\star s)$, i.e. the pairs that satisfy (14), and put them in δ' (we know that two of those pairs are either completely equal or both pair elements differ, so δ' is a multiset bijection too) and t' respectively. By cancellativeness

$$x^{\sim}y = \varphi(\varphi'(r \star s) \setminus^{\circ} \varphi'(t'))$$

for all $(x, y) \in \delta'$. The term $\varphi'(r \star s) \setminus^{\circ} \varphi'(t')$ is then reduced (the added pairs of both possible cases are removed), so

$$x^{\sim}y = \varphi'(r \star s) \setminus^{\circ} \varphi'(t')$$

for all $(x, y) \in \delta' \neq \emptyset$. The empty t was not $\subseteq$-maximal.

□

Proposition 79 (Theorem). *The reducing multiplication* $\mathtt{xy}$ *is an operation on* cF^* *and*

$$(cF^*, 1, [\mathtt{x}], \mathtt{xy}, \mathtt{x}^{\sim}) \equiv \mathbb{F}.$$

Proposition 80 (Corollary). *The outcome of the reducing multiplication* $\mathtt{xy}$ *does not depend on the initially chosen linear order* $\ll$.

With this knowledge we can define a cancel function $c\colon F^* \to cF^*$

$$c(r) := \frac{[]}{[c(r\lfloor)]}\frac{[c(r\lceil)]}{[]}.$$

We can see by simple recursion with the previous propositions, that it keeps the values (as expected of a cancel function).

Proposition 81 (Remark). *There are elements* $v, w \in \mathbb{F}$ *such that* $[v]^{\sim}[w] \neq [a_*][b_*]^{\sim}$ *for any* $a_*, b_* \in \mathbb{F}$. *Moreover any elements* v, w *with* $|v| \neq |w|$ *and* $|v|\,|w| \geq 1$ *satisfy this property.*

Proof. Assume there are $b_*, a_* \in \mathbb{F}$ with $\breve{r} = ab^{\sim}$, by proposition 66:

$$\frac{[w]}{[v]} \simeq \frac{[a_*b^{\sim}]}{[b_*a^{\sim}]}.$$

We first see that $v^{\sim}w \neq [v]^{\sim}[w]$ because otherwise $|w|\,/\,|v| = (|w| + 1)/(|v| + 1)$ and so $|v| = |w|$. Without restriction suppose $a_1b^{\sim} = w$, $b_1a^{\sim} = v$. The remaining right fraction must be equidistant to $ab^{\sim}$ what in turn means that $a \setminus^{\circ} [a_1] = b \setminus^{\circ} [b_1] =: c$ by the definition of reducing multiplication.

$$\begin{aligned} b &= [b_1, c_*] = [va, c_*] = [v[wb, c_*], c_*] \\ |b| &= |v|\,(|w|\,|b| + |c|) + |c| = |v|\,|w|\,|b| + |v|\,|c| + |c| \\ (1 - |v|\,|w|)\,|b| &= |v|\,|c| + |c| \end{aligned}$$

Because $|b|\,, |v|$ and $|c|$ are always positive this is a contradiction. □

Proposition 82 (Embeddability). *We can embed $\mathbb{P}$ into $\mathbb{F}$.*

Proof. We simply can embed $\mathbb{P}_R$ into cF^* via the homomorphism $\iota([a_*]) := [] \setminus [\iota(a_*)]$. Because there is no denominator the fraction is always reduced and so ι is injective.

On the other hand we could have shown the embeddability long before the construction of cF^* by using the power-inverse-iterated functions $\mathbb{P}_I^\circ$. We already know that $\mathbb{P}$ can be embedded into $\mathbb{P}_I^\circ$, say via ι, (though we still not know whether $\mathbb{P}_I^\circ \equiv \mathbb{F}$), we know that $\mathbb{P}_I^\circ$ is a $\emptyset$-generated coppice, and that we have a homomorphism $\alpha\colon \mathbb{P} \to \mathbb{F}$.

$$\begin{tikzcd} & \mathbb{F} \arrow[d, "\varphi"] \\ \mathbb{P} \arrow[ur, "\alpha"] \arrow[r, "\iota"] & \mathbb{P}_I^\circ \end{tikzcd}$$

Because $\varphi \circ \alpha = \iota$ is injective, also α is injective. □

Conjecture 4 (difficult). *The function $x \mapsto x^n$ on $\mathbb{F}$ is injective for each $n \in \mathbb{N}$.*

5.4 Miscellaneous Observations on Coppices

Proposition 83 (Left-Commutativity). *For a coppice* **X** *the respectively all-quantified equations*

$$\begin{aligned} x \oplus (y \oplus z) &= y \oplus (x \oplus z), \\ [x][y]^\sim &= [y[x]^\sim]^\sim[x[y]^\sim] \end{aligned}$$

are equivalent (where $[\mathtt{x}] := \mathtt{x} \oplus 1$ *denotes the corresponding increment function). It shows that equation* (9) *is also sufficient for left-commutativity.*

Proof.

$$x \oplus (y \oplus z) = [x(y \oplus z)^\sim](y \oplus z) = [xz^\sim[yz^\sim]^\sim][yz^\sim]z$$

and so left-commutativity will be expressed in terms of $[\mathtt{x}]$ by the following equivalences.

$$\begin{aligned} [xz^\sim[yz^\sim]^\sim][yz^\sim]z &= [yz^\sim[xz^\sim]^\sim][xz^\sim]z \\ [xz^\sim[yz^\sim]^\sim][yz^\sim] &= [yz^\sim[xz^\sim]^\sim][xz^\sim] \\ [x[y]^\sim][y] &= [y[x]^\sim][x] \\ [y][x]^\sim &= [x[y]^\sim]^\sim[y[x]^\sim] \end{aligned}$$

□

Proposition 84 (Cancellativeness). *For an lc-coppice having one of the following (respectively all-quantified) properties implies having all of the properties.*

$$r \oplus x = r \oplus y \Longrightarrow x = y \tag{15}$$

$$1 \oplus x = 1 \oplus y \Longrightarrow x = y \tag{16}$$

$$x^\sim y = (x \oplus r)^\sim (y \oplus s) \Longrightarrow x^\sim y = r^\sim s \tag{17}$$

$$x^\sim y = [x, a_*]^\sim [y, b_*] \Longrightarrow x^\sim y = a^\sim b \tag{18}$$

An lc-coppice having (one of) these properties is called cancellative.

Proof. (15)$\Longrightarrow$(16) is a specialisation. For (16)$\Longrightarrow$(17) we make the following implications.

$$\begin{aligned} x^{\sim}y &= (x \oplus r)^{\sim}(y \oplus s) \\ (x \oplus r)x^{\sim} &= (y \oplus s)y^{\sim} \\ 1 \oplus rx^{\sim} &= 1 \oplus sy^{\sim} \\ rx^{\sim} &= sy^{\sim} \\ x^{\sim}y &= r^{\sim}s \end{aligned}$$

The implication (17)$\Longrightarrow$(18) is again a specialisation. For (18)$\Longrightarrow$(15) we make the following conclusions.

$$\begin{aligned} r \oplus x &= r \oplus y \\ [rx^{\sim}]x &= [ry^{\sim}]y \\ xy^{\sim} &= [rx^{\sim}]^{\sim}[ry^{\sim}] \\ (rx^{\sim})^{\sim}(ry^{\sim}) &= [rx^{\sim}]^{\sim}[ry^{\sim}] \\ (rx^{\sim})^{\sim}(ry^{\sim}) &= 1 \\ x &= y \end{aligned}$$

□

Proposition 85 (Corollary). $\mathbb{P}_I^{\circ}$ *is cancellative.*

Proof. We simply verify equation (16). For $f, g \in \mathbb{P}_I^{\circ}$

$$f^{\mathrm{id}} = g^{\mathrm{id}} \qquad \overset{g>0}{\Longrightarrow} \qquad \left(\frac{f}{g}\right)^{\mathrm{id}} = 1 \qquad \overset{\mathrm{id}>0}{\Longrightarrow} \qquad \frac{f}{g} = 1 \qquad \Longrightarrow \qquad f = g.$$

□

Another example for a cancellative lc-coppice is $\mathbb{Q}_+$ and the trivial coppice.

Conjecture 5. *All finite lc-coppices are cancellative.*

Conjecture 6 (straight). *There are non-cancellative lc-coppices.*

Proposition 86. $(y \oplus r)x^{\sim} = (x \oplus r)y^{\sim} \Longrightarrow y = x$ *for all elements* x, y, r *of a left-commutative coppice.*

Proof. We make the following conclusions.

$$\begin{aligned} (y \oplus r)x^{\sim} &= (x \oplus r)y^{\sim} \\ (y \oplus r)x^{\sim} &= xy^{\sim} \oplus ry^{\sim} \\ 1 \oplus (y \oplus r)x^{\sim} &= 1 \oplus (xy^{\sim} \oplus ry^{\sim}) \\ 1 \oplus (y \oplus r)x^{\sim} &= xy^{\sim} \oplus (1 \oplus ry^{\sim}) \\ 1 \oplus (y \oplus r)x^{\sim} &= (x \oplus (y \oplus r))y^{\sim} \\ x \oplus (y \oplus r) &= (x \oplus (y \oplus r))y^{\sim}x \\ 1 &= y^{\sim}x \\ y &= x \end{aligned}$$

□

Proposition 87 (General Expansion). *$x^\sim y = r^\sim s \Longrightarrow x^\sim y = (x \oplus r)^\sim (y \oplus s)$ for all elements x, y, r, s of an lc-coppice.*

Proof.

$$\begin{aligned} x^\sim y &= r^\sim s \\ rx^\sim y &= s \\ y \oplus rx^\sim y &= y \oplus s \\ (1 \oplus rx^\sim)y &= (y \oplus s) \\ (x \oplus r)x^\sim y &= (y \oplus s) \\ x^\sim y &= (x \oplus r)^\sim (y \oplus s) \end{aligned}$$

□

5.4.1 Prime Factorisation with Non-Reducing Multiplication

This lengthy section is somewhat unconnected to the rest of this work, but at least it presents some nontrivial results.

Definition 36 (F^+, prime, prime factorisation in F^*). Let F^+ be the set of non-empty nominators of recursive fractions, i.e. F^+ consists of all non-empty elements $[a_*]$ where $a_* \in F^*$. We call $c \in F^+$ *prime* iff there are no $a, b \in F^+$ such that $c = a * b$. A prime factorisation of an $r \in F^*$ is a sequence of primes $p_1, \ldots, p_k \in F^+$ ($k \geq 0$) with a signature $\varepsilon\colon \{1, \ldots, k\} \to \{+1, -1\}$ such that $r = p_1^{\varepsilon(1)} * \ldots * p_k^{\varepsilon(k)}$.

Proposition 88 (Side Cancel).

$$\begin{aligned} r * t = s * t &\Longrightarrow r = s \\ t * r = t * s &\Longrightarrow r = s \end{aligned}$$

for all $r, s, t \in F^*$.

Proof. From the first equation we get:

$$\begin{aligned} \frac{[r\restriction * t, t\restriction]}{[t\lfloor * r^\sim, r\lfloor]} &= \frac{[s\restriction * t, t\restriction]}{[t\lfloor * s^\sim, s\lfloor]}, \\ \frac{[r\restriction * t]}{[t\lfloor * r^\sim, r\lfloor]} &= \frac{[s\restriction * t]}{[t\lfloor * s^\sim, s\lfloor]}. \end{aligned}$$

So by r.h. $[r\restriction] = [s\restriction] =: a$ (by $r\restriction_i * t = s\restriction_j * t \Longrightarrow r\restriction_i = s\restriction_j$). So we obtain

$$[t\lfloor * a^\sim * [r\lfloor], r\lfloor] = [t\lfloor * a^\sim * [s\lfloor], s\lfloor]. \tag{19}$$

If $[r\lfloor] \neq [s\lfloor]$ then there are i, j such that $r\lfloor_i = t\lfloor_j * a^\sim * [s\lfloor]$ (or $t\lfloor_j * a^\sim * [r\lfloor] = s\lfloor_i$). Because of symmetry we only regard the first case. Then

$$|t\lfloor * a^\sim * [r\lfloor]| \geq |[r\lfloor]| > |r\lfloor_i| = |t\lfloor_j * a^\sim * [s\lfloor]| \geq |[s\lfloor]| > |s\lfloor|\,,$$

hence $t\lfloor * a^\sim * [r\lfloor] \neq s\lfloor$, and so $[t\lfloor * a^\sim * [r\lfloor]] \subseteq [t\lfloor * a^\sim * [s\lfloor]]$. Because they have the same number of elements we get even $[t\lfloor * a^\sim * [r\lfloor]] = [t\lfloor * a^\sim * [s\lfloor]]$. Removing the equal elements of both sides in (19) yields $[r\lfloor] = [s\lfloor]$.

The second equation follows from the first by application of $(r * t)^\sim = t^\sim * r^\sim$ and $(r^\sim)^\sim = r$. □

Proposition 89 (Lemma). *For $r, s \in F^*$ and primes $p, q \in F^+$ from each of the following conditions follows $p = q$ (and $r = s$ by proposition 88).*

$$r * p = s * q \tag{20}$$
$$r * p^\sim = s * q^\sim \tag{21}$$
$$p * r = q * s \tag{22}$$
$$p^\sim * r = q^\sim * s \tag{23}$$

Proof. Let us begin with the positive case (20):

$$\frac{[r\restriction * p, p_*]}{[r\lfloor]} = \frac{[s\restriction * q, q_*]}{[s\lfloor]}.$$

If $[p_*] \neq [q_*]$ then there are i, j such that $p_i = s\restriction_j * q$. Then

$$|r\restriction * p| \geq |p| > |p_i| = |s\restriction_j * q| \geq |q| > |q_*|\,,$$

therefore $r\restriction * p \neq q_*$ and so $[r\restriction * p] \subseteq [s\restriction * q]$. If $[r\restriction] \neq \emptyset$ then there are k, l such that $r\restriction_k * q = s\restriction_l * p$ and by r.h. $p = q$. But if $[r\restriction] = \emptyset$ then $p = [p_*] = [s\restriction * q, q_*] = [s\restriction] * q$. Primality of p infers $[s\restriction] = 1$ and then $p = q$.

Then continue with the negative case (21):

$$\frac{[r\restriction * p^\sim]}{[p_* * r^\sim, r\lfloor]} = \frac{[s\restriction * q^\sim]}{[q_* * s^\sim, s\lfloor]}.$$

If $[r\restriction] \neq \emptyset$ and $[s\restriction] \neq \emptyset$ then there are i, j with $r\restriction_i * p^\sim = s\restriction * q^\sim$ and by r.h. $p = q$. Say $[r\restriction] = \emptyset$ then $[s\restriction] = \emptyset$ too and we have the equation

$$[p_* * [r\lfloor], r\lfloor] = [q_* * [s\lfloor], s\lfloor].$$

By the previous absolute value argument we can assume $[p_* * [r\lfloor]] \subseteq [q_* * [s\lfloor]]$. Split q into q^A and q^B such that:

$$[q_*] = [q_*^A, q_*^B], \tag{24}$$
$$[p_* * [r\lfloor]] = [q_*^A * [s\lfloor]], \tag{25}$$
$$[r\lfloor] = [q_*^B * [s\lfloor], s\lfloor]. \tag{26}$$

We make the following conclusions.

by (26)	$[r\lfloor] = q^B * [s\lfloor]$
into (25)	$[p_* * q^B * [s\lfloor]] = [q_*^A * [s\lfloor]]$
by proposition 88	$[p_* * q^B] = [q_*^A]$
$\uplus q^B$	$[p_* * q^B, q_*^B] = [q_*]$
	$p * q^B = q$
by primality of q	$p = q$

The remaining cases (22) and (23) can be derived by the laws $(r * s)^\sim = s^\sim * r^\sim$ and $(r^\sim)^\sim = r$. □

There is no unique prime factorisation in F^*, for example:

$$[a] * [b]^\sim = [b * [a]^\sim]^\sim * [a * [b]^\sim].$$

(Note that $[x]$ and $[y]^\sim$ are prime for each $x \in F^*$.) But we will see that the prime factorisations are unique up to this rule. Call this rule *pm-swap*.

Proposition 90 (Unique Prime Factorisation for F^+). *Each $a \in F^+ \cup \{1\}$ has a unique prime factorisation.*

Proof. Existence: for $a = 1$ we have the empty sequence, for a being prime we have the one-element sequence consisting of a, otherwise $a = b * c$ for some $b, c \in F^+$. By r.h. b and c have such a prime factorisation so a has the concatenation.

Uniqueness: All nonempty products can not be 1. So the only prime factorisation for $a = 1$ is the empty sequence. Every prime factorisation can not contain a negative element otherwise the product would have a denominator. So we know that the signature of each prime factorisation is purely positive. If $p_1 * \cdots * p_k = q_1 * \cdots * q_m$ for prime factorisations $p_1, \ldots, p_k$ and $q_1, \ldots, q_m$ then by repeated application of (20) we get either $p_1 * \cdots * p_{k-m} = 1$ or $1 = q_1 * \cdots * q_{m-k}$, hence $k = m$ and $p_i = q_i|_{1 \le i \le k}$. □

Proposition 91 (Unique Prime Bifactorization for F^*). *For each $r \in F^*$ there are unique prime sequences $p_1, \ldots, p_k, q_1, \ldots, q_m \in F^+$ such that $r = {q_1}^\sim * \cdots * {q_m}^\sim * p_1 * \cdots * p_k$.*

Proof. For existence glue together the prime factorisations for $[r\downharpoonleft]$ inverted and $[r\upharpoonleft]$. For uniqueness let

$${q_1}^\sim * \cdots * {q_m}^\sim * p_1 * \cdots * p_k = {q'_1}^\sim * \cdots * {q'_{m'}}^\sim * p'_1 * \cdots * p'_{k'}$$

then repeated application of (20) and (23) leads to either

$$\begin{aligned}
&{q_1}^\sim * \cdots * {q_{m-m'}}^\sim * p_1 * \cdots * p_{k-k'} = 1 && \text{if} \quad m \ge m', k \ge k',\\
&{q'_1}^\sim * \cdots * {q'_{m'-m}}^\sim * p'_1 * \cdots * p'_{k'-k} = 1 && \text{if} \quad m' \ge m, k' \ge k,\\
&{q_1}^\sim * \cdots * {q_{m-m'}}^\sim = p'_1 * \cdots * p'_{k'-k} && \text{if} \quad m \ge m', k' \ge k,\\
&{q'_1}^\sim * \cdots * {q'_{m'-m}}^\sim = p_1 * \cdots * p_{k-k'} && \text{if} \quad m' \ge m, k \ge k'.
\end{aligned}$$

We easily verify that in each case already $k = k'$ and $m = m'$ have to be satisfied and so $(p_1, \ldots, p_k) = (p'_1, \ldots, p'_{k'})$ and $(q_1, \ldots, q_m) = (q'_1, \ldots, q'_{m'})$. □

Proposition 92. *Let $p, a \in F^+$, if p is prime then $[p_* * a^\sim]$ is prime too.*

Proof. Otherwise would $[p_* * a^\sim] = b * q = [b_* * q, q_*]$ for some $a, q \in F^+$ and prime q (denominators in multiplicands would lead to a denominator in the product). So we split p into p^A and p^B with

$$p^A \uplus p^B = p, \tag{27}$$

$$[p^A_* * a^\sim] = [b_* * q], \tag{28}$$

$$[p^B_* * a^\sim] = q. \tag{29}$$

Every p_*^A cannot be purely negative because each $b_* * q$ has a nominator (because $q \in F^+$). Then let d_i be the last (positive) prime of (the prime bifactorization of) p_*^A and make the following conclusions.

$$
\begin{aligned}
& & p_i^A &= c_i * d_i && (30)\\
&\text{into (28)} & c_i * d_i * a^\sim &= b_i * q && (31)\\
&\text{definition of } * & c_i * [a_* * d_i{}^\sim]^\sim * [d_{i*} * a^\sim] &= b_i * q && (32)\\
&\text{r.h. and (20)} & [d_{i*} * a^\sim] &= q && (33)\\
& & d_1 * a^\sim &= \cdots = d_k * a^\sim && (34)\\
&\text{proposition 88} & d_1 &= \cdots = d_k =: q' && (35)\\
&\text{(35) into (33)} & [p_*^B * a^\sim] &= q = [q'_* * a^\sim] && (36)\\
&\text{proposition 88} & p^B &= q' && (37)\\
&\text{(35) into (30)} & p^A &= [c_* * q'] && (38)\\
& & p &= [c_* * q', q'_*] = c * q' && (39)
\end{aligned}
$$

The last line means that p is not prime in contradiction to the assumption. □

Proposition 93. *Each prime factorisation of an $r \in F^*$ has the same number of negative and the same number of positive elements.*

Proof. Application of $p * q^\sim = [q_* * p^\sim]^\sim * [p_* * q^\sim]$ for primes p and q in a prime factorisation of r leads to a prime factorisation with the same number of positive and the same number of negative elements (by proposition 92). Repeated application transfers each prime factorisation into the unique prime bifactorization of r. □

Proposition 94. *Let $p, a \in F^+$, if $[p_* * a^\sim]$ is prime then p is prime too.*

Proof. Let $b = [a_* * p^\sim] \in F^+$. The prime factorisations for a and b only contain positive elements. Then we know that $[p_* * a^\sim]^\sim * b = a * p^\sim$. The prime factorisation of the left side has exactly one negative element, so the prime factorisation of the right side must have exactly one negative element. So p must be prime. □

Proposition 95. *Each prime factorisation of r is already determined by its signature. So we will also call a signature a prime factorisation.*

Proof. This a direct consequence of proposition 89. The last prime elements of the same sign have to be equal. □

Proposition 96 (Unique Maximal Prime Factorisation for F^*). *Order the signatures by lexicographic order. Then each $r \in F^*$ has a unique signature-maximal prime factorisation.*

Proof. By proposition 95. □

The minimal prime factorisation is where all negative signs are to the left, it is the same as the unique prime bifactorisation.

If we are now looking for a similar prime factorisation for cF^* with the cancelled multiplication `xy`, then we first observe that the prime elements are singletons, because

every $[a,b] = [a[b]^\sim][b]$ and so every $[a_1,\dots,a_n]$ is a product of singletons and not even a unique product because also $[a,b] = [b[a]^\sim][a]$. Till now there has not much light been shone onto the singleton representation of $\mathbb{F}$, i.e. less is known about the congruence classes of cancelled words representing $\mathbb{F}$. But we know already that there is no unique maximal prime factorisation by the above said (there are up to $n!$ different singleton factorisations for $[a_1,\dots,a_n]$).

We are again interested in swapping negative singletons to the right (which is not always possible by proposition 81). The question is about swapping a negative singleton to the right through different presentations (of some $[c_1,\dots,c_n]$) as positive singleton factorisations.

Conjecture 7 (straight). *Given the following equations (all variables from $\mathbb{F}$)*

$$[x]^\sim[a_1]\cdots[a_n] = [a_1'][x_1]^\sim[a_2]\cdots[a_n] = \ldots = [a_1']\cdots[a_{n-1}'][x_{n-1}]^\sim[a_n] = [a_1']\cdots[a_n'][x_n]^\sim$$

and some $[b_1]\cdots[b_n] = [a_1]\cdots[a_n]$ then there exist $b_1',\dots,b_n'$ satisfying

$$[x]^\sim[b_1]\cdots[b_n] = [b_1'][y_1]^\sim[b_2]\cdots[b_n] = \ldots = [b_1']\cdots[b_{n-1}'][y_{n-1}]^\sim[b_n] = [b_1']\cdots[b_n'][y_n]^\sim$$

and $x_n = y_n$ (and hence $[b_1']\cdots[b_n'] = [a_1']\cdots[a_n']$).

A more general conjecture would arise if we allowed multiple negative singletons that all become swapped to the right. Whether this is independent of different presentations (of the negative and of the positive elements) and swapping orders.

6 Order and Topology

6.1 Ordered Trees

The main result of this section is that there are partial orders on $\mathbb{B}$ and $\mathbb{P}$ that are compatible with all higher operations (see proposition 106 and proposition 108), but no such linear orders (proposition 109). We start with considering a natural partial order on $\mathbb{B}$.

Definition 37 ($\leq$ on $\mathbb{B}$). We define the relation $\leq$ on $\mathbb{B}$ recursively by

$$\begin{aligned} 1 &\leq 1, \\ 1 &\leq a_L \oplus a_R, \\ a_L \oplus a_R &\not\leq 1, \\ a_L \oplus a_R \leq b_L \oplus b_R &\iff a_L \leq b_L \wedge a_R \leq b_R \end{aligned}$$

for all $a_L, a_R \in \mathbb{B}$.

Proposition 97. *$a \leq b$ in $\mathbb{B}_R$ iff $a_i \leq b_i$ for each index i.*

Proposition 98 (Remark). *For each $a \in \mathbb{B}$ there exists $n \in \mathbb{N}$, such that $a \leq 2^n$. Particularly $(\mathbb{B}, \leq)$ is directed.*

Proof. $a_L \leq 2^{n_L}$, $a_R \leq 2^{n_R}$, then $a_L \oplus a_R \leq 2^{n_L} \oplus 2^{n_R} \leq 2^m \oplus 2^m = 22^m = 2^{m+1}$ □

Proposition 99 (Remark). $(\mathbb{B}, \leq)$ *is a distributive lattice.*

Proposition 100 (Remark). $\leq$ *is the smallest* $\mathtt{x} \oplus \mathtt{y}$*-compatible partial order on* $\mathbb{B}$ *for which holds:* $1 \leq 2$.

Proposition 101 (Remark). *The order* $\leq$ *is left-compatible with the multiplication on* $\mathbb{B}$*, i.e* $ca \leq cb$ *follows from* $a \leq b$ *for each* $a, b, c \in \mathbb{B}$.

But it is not right-compatible! Not even $c \leq bc$ and $c \leq 2c$ are true. As counterexample consider the following. Let $c = ((1,2),1)$, then $2c = (((1,2),1),((1,2),1))$. For $c \leq 2c$ must hold $(1,2) \leq c$, hence $2 \leq 1$ which is false.

The question arises whether one can extend the relation $\leq$ on $\mathbb{B}$, such that it is also right-compatible with $\mathtt{xy}$.

Conjecture 8 (difficult). *There is no distributive lattice on* $\mathbb{B}$*, which is both side compatible with* $\mathtt{x} \oplus \mathtt{y}$ *and* $\mathtt{xy}$.

Definition 38 ($\leq_{LR}$, $\leq_L$, $\leq_R$, $\leq_X$). For $a, b \in \mathbb{B}$ define (recursively) $a \leq_{LR} b$ as: (at least) one of the following conditions is valid.

$$a = 1 \quad \text{and} \quad b = 1 \tag{40}$$

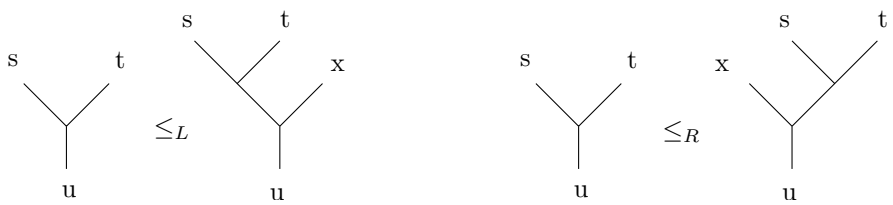
$$a \leq_{LR} b_L \tag{41}$$

$$a \leq_{LR} b_R \tag{42}$$

$$a_L \leq_{LR} b_L \quad \text{and} \quad a_R \leq_{LR} b_R \tag{43}$$

Dropping case (41) (and replacing $\leq_{LR}$ by $\leq_R$) we define $\leq_R$ and dropping case (42) (and replacing $\leq_{LR}$ by $\leq_L$) we define $\leq_L$. We write $\leq_X$ if the surrounding proposition is valid for $\leq_R$, $\leq_L$ and $\leq_{LR}$.

Visually you can imagine $a \leq_L b$, $a \leq_R b$, $a \leq_{LR} b$ as that you can repeatedly contract left/right/left or right edges with attached subtree x in b resulting in a.



Proposition 102 (Remark). $\leq_X$ *is a partial order.*

Proof left to the reader.

Proposition 103. $\langle a_1, \ldots, a_k \rangle \leq_R \langle b_1, \ldots, b_l \rangle$ *for* $a, b \in \mathbb{B}_R$ *holds exactly if there exists an injective monotonous* $\alpha \colon \{1, \ldots, k\} \to \{1, \ldots, l\}$ *with* $a_i \leq_R b_{\alpha(i)}$*. Call such an* α *order assignment.*

Proof. In the following we use the abbreviation $K := \{1, \dots, k\}$ and $L := \{1, \dots, l\}$. Let us first show the "then", i.e. in each case of definition 38 there exists this order assignment α. Let it already be shown for $a \leq_R b_R$, $a_L \leq_R b_L$ and $a_R \leq_R b_R$. In case (40) α is the trivial order assignment $\emptyset \to \emptyset$. In case (42) by recursion we have an order assignment $\beta\colon K \to L \setminus \{1\}$ and hence an order assignment $\alpha\colon K \to L$. In case (43) $a_1 = a_L \leq_R b_L = b_1$ and by recursion there is an order assignment $\beta\colon K \setminus \{1\} \to L \setminus \{1\}$. Then $\alpha = \{(a_1, b_1)\} \cup \beta$ is also an order assignment.

Now the direction "if", i.e. if there exists an order assignment $\alpha\colon L \to K$ then one of the cases of definition 38 must occur. If $a = 1$ then occurs case (42) or (40). Otherwise $\alpha(1)$ must be defined. If $\alpha(1) = 1$ then $\beta := \alpha \setminus \{(1,1)\}$ is an order assignment for $a_R \leq_R b_R$, by r.h. then case (43) applies. If otherwise $\alpha(1) > 1$ then by monotony $\beta := \alpha|K \to L \setminus \{1\}$ is an order assignment and by r.h. case (42) applies. □

Proposition 104. *$x \leq_X a \ltimes_n x$ for all $x, a \in \mathbb{B}$, $n \geq 2$.*

Proof. We prove the proposition by induction over n:

1. $n = 2$: Do recursion over a.
   (a) $a = 1$: $x \leq_X x = 1x$.
   (b) $a \neq 1$: Let it be shown for $a := a_L$ and $a := a_R$. Then $x \leq_X a_{\{L,R\}}x$ and by (41) or (42) $x \leq_X (a_L x) \oplus (a_R x) = ax$.
2. $n > 2$: Let it be shown for $n := n - 1$, do recursion over a.
   (a) $a = 1$: $x \leq_X x = 1 \ltimes_n x$.
   (b) $a \neq 1$: Let it be shown for $a := a_L$ and $a := a_R$, then $x \leq_X a_R \ltimes_n x \leq_X (a_L \ltimes_n x) \ltimes_{n-1} (a_R \ltimes_n x) = a \ltimes_n x$.

□

Proposition 105. *$x \leq_X xa$ for all $x, a \in \mathbb{B}$.*

Proof. Recurse over x: for $x = 1$, $1 \leq_X a = 1a$. Let $x \neq 1$ then $x = x_L \oplus x_R \leq_X (x_L a) \oplus (x_R a) = xa$. □

Proposition 106. *The partial orders $\leq_{LR}$,$\leq_L$ and $\leq_R$ are compatible to all operations* $\mathtt{x} \ltimes_n \mathtt{y}$ *on $\mathbb{B}$.*

Proof. We prove the proposition by induction over n. For $a \leq_X b$ we show the two equations

$$a \ltimes_n x \leq_X b \ltimes_n x, \tag{44}$$

$$x \ltimes_n a \leq_X x \ltimes_n b. \tag{45}$$

For $n = 1$ (44) and (45) are clear by (43). Now assume (44) and (45) are already shown for $n := n - 1$. We show (44) and then (45). First let us show (45). We recurse over x. For $x = 1$ it is clear. For $x \neq 1$ assume it is already shown for $x := x_L$ and $x := x_R$ then by assumption (45) and (44) for $n := n - 1$:

$$\begin{aligned} x \ltimes_n a = (x_L \ltimes_n a) \ltimes_{n-1} (x_R \ltimes_n a) &\leq_X (x_L \ltimes_n a) \ltimes_{n-1} (x_R \ltimes_n b) \\ &\leq_X (x_L \ltimes_n b) \ltimes_{n-1} (x_R \ltimes_n b) = x \ltimes_n b. \end{aligned}$$

Now let us show (44), we recurse over (a, b).

1. $a = 1, b = 1$: true by reflexivity.
2. $a = 1, b \neq 1$: by proposition 104.
3. $a \neq 1, b = 1$: Because $a \leq_X b$ it must $a = 1$, hence equal to case 1.
4. $a \neq 1, b \neq 1$: Let it already been shown for $(a, b) := (a_L, b_L), (a_R, b_R), (a, b_L), (a, b_R)$. We regard the 4 cases for $a \leq_X b$:
   (a) $a = 1, b = 1$: Handled in case 1.
   (b) $a \leq_X b_L$, we can assume that $\leq_X$ is either $\leq_L$ or $\leq_{LR}$:
      i. $n = 2$:
      $$ax \leq_X b_L x \leq_L b_L x \oplus b_R x = bx$$
      ii. $n \geq 3$:
      $$\begin{aligned} & a \ltimes_n x \leq_X b_L \ltimes_n x \\ \text{by proposition 14} \quad & = \Pi(b_{Li} \ltimes_n x) \ltimes_{n-1} x \\ \text{by proposition 105 and assumption} \quad & \leq_X (\Pi(b_{Li} \ltimes_n x)\Pi(b_{Ri} \ltimes_n x)) \ltimes_{n-1} x \\ & = b \ltimes_n x \end{aligned}$$
   (c) $a \leq_X b_R$:
   $$a \ltimes_n x \leq_X b_R \ltimes_n x \leq_X (b_L \ltimes_n x) \ltimes_{n-1} (b_R \ltimes_n x) = b \ltimes_n x$$
   by (43) and (42).
   (d) $a_L \leq_X b_L$ and $a_R \leq_X b_R$:
   $$a \ltimes_n x = (a_L \ltimes_n x) \ltimes_{n-1} (a_R \ltimes_n x) \leq_X (b_L \ltimes_n x) \ltimes_{n-1} (b_R \ltimes_n x) = b \ltimes_n x$$

□

Now let us look at a natural order of $\mathbb{P}$.

Definition 39 ($\leq$ on $\mathbb{P}$). Define $[a_1, \ldots, a_k] \leq [b_1, \ldots, b_l]$ as: there exists an injection $\alpha \colon \{1, \ldots, k\} \to \{1, \ldots, l\}$ so that $a_i \leq b_{\alpha(i)}$ for each $1 \leq i \leq k$.

We easily verify that this is a partial order. This order is clearly the image of the order $\leq_R$ on $\mathbb{B}_R$ under π where the order assignment maps to the α of the definition. Moreover it is the image of the order $\leq$ on $\mathbb{B}$ under π:

Proposition 107. *If $a \leq b$ in $\mathbb{B}$ then $\pi(a) \leq \pi(b)$. If $p \leq q$ in $\mathbb{P}$ then there exist $a, b \in \mathbb{B}$ so that $\pi(a) = p$, $\pi(b) = q$ and $a \leq b$.*

Proof. The first part is easy to see: If $a \leq b$ for $a, b \in \mathbb{B}$ then by definition also $a \leq_X b$, particularly $a \leq_R b$ and so $\pi(a) \leq \pi(b)$.

For the reverse direction simply sort p and q such that $p_i \leq q_i$ for each p_i and regard both sorted multisets as the elements $a, b \in \mathbb{B}_R$ respectively. Then proposition 97 finishes the proof. □

We have the following corollary of $\leq$ on $\mathbb{P}$ being the image of $\leq_R$ on $\mathbb{B}$.

Proposition 108 (Theorem). *All higher operations* $\mathtt{x} \ltimes_n \mathtt{y}$ *on* $\mathbb{P}$ *are* $\leq$*-compatible, i.e. for each* $a, b, c, d \in \mathbb{B}$ *if* $a \leq b$ *and* $c \leq d$ *then*

$$a \ltimes_n c \leq b \ltimes_n d. \tag{46}$$

This can not happen with a linear order as the following theorem shows. Note that a *strict* linear order can only be compatible with injective functions, but $x \mapsto x \ltimes_i a$ is mostly not injective, so the non-strict case is the interesting one here.

Proposition 109. *For* $\mathbb{B}$ *and for* $\mathbb{P}$ *there is no linear order that is right-compatible with each of the operations* $\mathtt{x} \ltimes_2 \mathtt{y}$*,* $\mathtt{x} \ltimes_3 \mathtt{y}$ *and* $\mathtt{x} \ltimes_4 \mathtt{y}$*. (Where the order* $\leq$ *is said to be right-compatible with the operation* $*$ *iff* $a \leq b \Longrightarrow a * c \leq b * c$*).*

Proof. Without restriction assume that $1 \leq 2$. If $\leq$ is right-compatible with $\mathtt{x} \ltimes_2 \mathtt{y}$ — which is associative — then by repeated multiplication of 2 at the right sides we verify that $2^m \leq 2^{m+1}$ for all $m \in \mathbb{N}$. By repetition $m \leq n$ always implies $2^m \leq 2^n$. By linearity we even get the opposite direction

$$2^m \leq 2^n \Longrightarrow m \leq n. \tag{47}$$

Because $\leq$ should be right-compatible with $\mathtt{x} \ltimes_3 \mathtt{y}$ too, we have the following conclusions.

$$\begin{aligned} a &\leq b \\ a \ltimes_3 2 &\leq b \ltimes_3 2 \\ 2^{|a|} &\leq 2^{|b|} \\ |a| &\leq |b| \end{aligned}$$

Now consider the elements $a = (1, (1, (1, 1))) \in \mathbb{B}$ ($a = [1, 1, 1] \in \mathbb{P}$) and $b = ((1, 1), 1) \in \mathbb{B}$ ($b = [[1]] \in \mathbb{P}$). If $a \leq b$ then by right-compatibility with $\mathtt{x} \ltimes_4 \mathtt{y}$ it must $a \ltimes_4 c \leq b \ltimes_4 c$ for all $c \in \mathbb{B}$ ($c \in \mathbb{P}$).

$$\begin{aligned} a \ltimes_4 c &\leq b \ltimes_4 c \\ |a \ltimes_4 c| &\leq |b \ltimes_4 c| \\ a \curlywedge' |c| &\leq b \curlywedge' |c| \\ n^{n^3} &\leq n^{n^n} \quad \text{for all} \quad n \in \mathbb{N} \end{aligned}$$

But we know that $2^{2^3} \geq 2^{2^2}$ and $4^{4^3} \leq 4^{4^4}$ (same for case $b \leq a$). □

This result (conferred to coppices) may sound quite unpromising with regard to completion. But a look at the complex numbers, that are also not multiplicative linearly orderable, but are complete, relativises this. Though it is not possible to (right) order, it might be possible that there is a linear left-compatible order on $\mathbb{P}$. This thought is also supported by the injectivity of the functions $x \mapsto a \ltimes_i x$. Unfortunately the only mentioned linear candidate $<_M$ does not meet.

Problem 9 (medium). *Is there a linear order on* $\mathbb{P}$ *(or* $\mathbb{B}$*) such that the functions* $x \mapsto a \ltimes_i x$ *are strictly increasing for each* $a \in \mathbb{P}$ *and* $i \in \mathbb{N}$*?*

Proposition 110. *$<_M$ on $\mathbb{P}$ is both side compatible with* $\mathtt{xy}$ *but it is not left-compatible with* $\mathtt{x} \ltimes_4 \mathtt{y}$.

Proof. Both side compatibility follows already from the both side compatibility of $<_\uparrow$ with $\mathtt{x} \circ \mathtt{y}$. We know that $2^n = [2^{n-1}, \ldots, 2, 1]$. It is easy to see that for each $a \in \mathbb{P}$ there is an $n \in \mathbb{N}$ such that $a \leq_M 2^n$. So for $a = 1$ it is trivial and by induction there is an $n' \in \mathbb{N}$ such that $\max\{a_*\} \leq 2^{n'}$ and so $a \leq_M 2^{n'+1}$. Let us use the following notation.

$$n \times a := \underbrace{a, \ldots, a}_{n\times}$$

Then $2^n \leq_M [n \times 1]^n$ for $n \geq 1$ by multiplicative compatibility, particularly also $[n \times 1]^n$ surpasses every $a \in \mathbb{P}$ for $n \to \infty$. But then by left-compatibility of $\mathtt{x} \ltimes_4 \mathtt{y}$ we would conclude:

$$\begin{aligned} [n \times 1] &\leq_M [2], \\ 2 \ltimes_4 [n \times 1] &\leq_M 2 \ltimes_4 [2], \\ [n \times 1]^{n+1} &\leq_M [2]^3. \end{aligned}$$

□

6.2 Order and Topology on Coppices

Definition 40 (coppice order, ordered coppice). A *coppice order* is a strict (unless stated otherwise) order on a coppice that is compatible with all its operations except inversion. An *ordered coppice* is a coppice with a coppice order on it.

It is necessary to indicate whether the incremental or the additive coppice is used. If no indication is given it is assumed to be an additive coppice.

For example a coppice with $[a] = [b]$ for some $a < b$ can not be strictly ordered but maybe non-strictly ordered. Conversely every strictly ordered coppice can also be non-strictly ordered. Let us write down the defining set of conditions for an ordered coppice (a strict order only needs to satisfy irreflexivity and transitivity):

Proposition 111. **C** *is an ordered* incremental *coppice exactly if there exists a set $P \subseteq C$ such that for all $a, b \in P$ and all $x \in C$:*

$$a^\sim \notin P \quad \textit{(irreflexive)} \tag{48}$$

$$ab \in P \quad \textit{(transitive)} \tag{49}$$

$$[x]^\sim[ax] \in P \quad \textit{(incremental compatible)} \tag{50}$$

$$x^\sim ax \in P \quad \textit{(multiplicative compatible)} \tag{51}$$

If additionally either $x = 1$, $x^\sim \in P$ or $x \in P$, it describes a linearly ordered coppice.

Proof. The set P describes the set of all elements $x > 1$. The compatibility conditions have all the form "for all $x, y \in C$ if $x < y$ then $T(x) < T(y)$" where the term $T(x)$ is either $c \oplus x$, $x \oplus c$, cx or xc. This is directly translated into "for all $x \in C$ and $a \in P$ is $T(x)^\sim T(ax) \in P$". To write out the proof is neither thrilling nor difficult, so we leave it. □

Proposition 112. **C** *is an ordered (additive) coppice exactly if there exists a set $P \subseteq C$ such that for all $a, b \in P$ and all $x \in C$:*

$$a^\sim \notin P \quad \textit{(irreflexive)} \tag{52}$$
$$ab \in P \quad \textit{(transitive)} \tag{53}$$
$$[x]^\sim[ax] \in P \quad \textit{(right additive compatible)} \tag{54}$$
$$[ax]^\sim[x]x^\sim ax \in P \quad \textit{(left additive and multiplicative compat.)} \tag{55}$$

where $[\mathtt{x}] := \mathtt{x} \oplus 1$ *denotes the coppice's increment.*

Proof. The schema is the same as in the previous proof. Right additive compatibility is equivalent to incremental compatibility by the following equivalences.

$$\begin{aligned}
\forall x, y, r \quad & x < y \Longrightarrow x \oplus r < y \oplus r \\
\forall x, y, r \quad & x < y \Longrightarrow [xr^\sim]r < [yr^\sim]r \\
\forall x, y, r \quad & x < y \Longrightarrow [xr^\sim] < [yr^\sim] \\
\forall x, y \quad & x < y \Longrightarrow [x] < [y]
\end{aligned}$$

And left additive compatibility has the following equivalences.

$$\begin{aligned}
\forall x, y, r \quad & x < y \Longrightarrow r \oplus x < r \oplus y \\
\forall x, y, r \quad & x < y \Longrightarrow [rx^\sim]x < [ry^\sim]y \\
\forall x, r, a > 1 \quad & [rx^\sim]x < [r(ax)^\sim]ax \\
\forall x, r, a > 1 \quad & [rx^\sim] < [rx^\sim a^\sim]a \\
\forall x, a > 1 \quad & [x] < [xa^\sim]a \\
\forall x, a > 1 \quad & 1 < [x]^\sim[xa^\sim]a \\
\forall x, a > 1 \quad & 1 < [xa]^\sim[x]a
\end{aligned}$$

Property (55) has the following equivalences (if assuming the other conditions).

$$x^\sim ax \in P \quad \text{(multiplicative compatible)} \tag{56}$$
$$[xa]^\sim[x]a \in P \quad \text{(left additive compatible)} \tag{57}$$

We first get (56) by applying (53) on (54) and (55), and then (57) by substituting $a := xax^\sim$ in (55) where the left hand a is in P whenever the right hand a is in P by (56). Back substitution implies (55). The other details are left to the reader. □

We see that a (additive) coppice order is a specialisation of an incremental coppice order.

Proposition 113. $\mathbb{B}^\circ$ *and* $\mathbb{F}$ *can be endowed with coppice orders that extend the orders of* $\mathbb{B}$ *and* $\mathbb{P}$ *respectively.*

Proof. Set simply $a < b :\Longleftrightarrow |a| < |b|$, where $|\mathtt{x}| : \mathbb{B}^\circ \to \mathbb{Q}_+$ is the universal coppice epimorphism, and $\mathbb{Q}_+$ is also a factor of $\mathbb{F}$ so we also have a unique coppice homomorphism $|\mathtt{x}| : \mathbb{F} \to \mathbb{Q}_+$. And the orders $\leq$ and $\leq_X$ on $\mathbb{B}$ and/or $\mathbb{P}$ are all compatible with the order on $\mathbb{N}$, i.e. $|\mathtt{x}| : \mathbb{B} \to \mathbb{N}$ and $|\mathtt{x}| : \mathbb{P} \to \mathbb{N}$ are order preserving homomorphisms (that are extended by $|\mathtt{x}| : \mathbb{B} \to \mathbb{Q}_+$ and $|\mathtt{x}| : \mathbb{F} \to \mathbb{Q}_+$ respectively). □

Though what really interests us are orders that induce (non-trivial) Hausdorff topologies (see proposition 133). For example the above given orders are completely useless for convergence because $\varepsilon^\sim < a < \varepsilon$ for all $\varepsilon > 1$ would only imply $|a| = 1$ and not the needed $a = 1$. Elements a with $\varepsilon^\sim < a < \varepsilon$ for all $\varepsilon > 1$ are called *pseudo units*. So for convergence concerns we would at least demand that the coppice order does not have pseudo units. We can render this more precisely in the remaining chapter.

Definition 41 (coppice topology, topological coppice). A *coppice topology* is a topology on a coppice such that all its operations are continuous. A *topological coppice* is a coppice with a coppice topology on it. Or in other words it is a topological group with continuous increment. A *Hausdorff coppice* is a topological coppice with a Hausdorff coppice topology on it.

Note, that the definition is independent of using the incremental or additive coppice notation (definition 24), because continuous operations of continuous operations are again continuous and $x \oplus y = (xy^\sim \oplus 1)y$. Interestingly this is different for an *ordered* coppice.

Let us write down the complete set of requirements for a Hausdorff coppice:

Proposition 114. **C** *is a Hausdorff coppice iff there is a set $\mathfrak{E}$ of sets of C such that for each following condition for each $U, U' \in \mathfrak{E}$ and each $x \in C$ there exist $V \in \mathfrak{E}$ that satisfies the condition.*

$$V \subseteq U \cap U' \tag{58}$$

$$V^\sim V \subseteq U \tag{59}$$

$$[xV] \subseteq [x]U \tag{60}$$

$$x^\sim V x \subseteq U \tag{61}$$

$$\bigcap \mathfrak{E} = \{1\} \tag{62}$$

Proof. The main idea is that $\mathfrak{E}$ describes a neighbourhood base of 1 and by continuity of the multiplication neighbourhoods U of x translate into $x^\sim U$ being neighbourhood of 1. For the details regarding a group see [7], Chapter III, §1, no. 2. The condition (60) is a direct translation of the continuity of the increment. □

There is a natural way to achieve a topology from an order. This is by taking the open intervals around an element p, i.e. the sets $\{x \colon a < x < b\}$ for all a, b with $a < p < b$, as a neighbourhood base of each point p. Though this topology only exists if the intersection of two open intervals around p contains again an open interval around p, we can generally define:

Definition 42 (interval topology, topological order). Call the coarsest topology on an ordered space, in which each point has at least the it containing open intervals as neighbourhoods, its *interval topology*. Call an order *topological* if its interval topology is Hausdorff and the open intervals containing a point form a neighbourhood base of that point.

Proposition 115. *For a coppice order the following conditions are sufficient and necessary to be topological. For each following condition for each $\varepsilon, \varepsilon' > 1$ and each coppice*

element x there exists $\delta > 1$ that satisfies the condition.

$$\delta \leq \varepsilon, \varepsilon' \tag{63}$$

$$\delta^2 \leq \varepsilon \tag{64}$$

$$\# \qquad [x\delta] \leq [x]\varepsilon \tag{65}$$

$$\# \qquad x^{\sim}\delta x \leq \varepsilon \tag{66}$$

$$\{e \colon \forall x > 1 \quad x^{\sim} \leq e \leq x\} = \{1\} \quad \textit{(contains no pseudo units)} \tag{67}$$

The # marked items are consequences of being a coppice order, i.e. cancelled.

Proof. We first show that under condition (63) the intervals $(\delta^{\sim}r, \delta r)$, $\delta > 1$, form a neighbourhood base for each point r in the interval topology. We show that 1. some interval of the above form is contained in each interval around r and that 2. these intervals indeed form a neighbourhood base. 1. Let (a, b) an open interval around r, take now $\delta_a := ra^{\sim} > 1$ and $\delta_b := br^{\sim}$ and let $\delta > 1$ with $\delta \leq \delta_a, \delta_b$ by (63). Then $(\delta^{\sim}r, \delta r) \subseteq (a, b)$. 2. For each two intervals $({\delta_1}^{\sim}r, \delta_1 r)$ and $({\delta_2}^{\sim}r, \delta_2 r)$, again the interval $(\delta^{\sim}r, \delta r)$ with δ of (63) is contained in the intersection. Conversely (63) follows from the demand that the intervals containing the point form a neighbourhood base.

The other conditions are simply a translation of the conditions for an Hausdorff topological coppice (proposition 114) which is given by a neighbourhood base around 1. The marked lines however do not contribute. For $x^{\sim}\delta x \leq \varepsilon$ we simply choose $\delta := x\varepsilon x^{\sim} > 1$ by (56) and for $[x\delta] \leq [x]\varepsilon$ we simply choose $\delta := \varepsilon$ by (57).

Compare also [3], 3. for a similar set of conditions for ordered groups. □

Now there is a way to generate coppice orders by proposition 112, i.e. we start with a set P_0. And then we add successively the elements mentioned in the conditions of proposition 112: ab, $[x]^{\sim}[ax]$, $[ax]^{\sim}[x]x^{\sim}ax$. Now a first question is what set P_0 actually generates an (strict) order, i.e. such that never x together with $x^{\sim}$ is in P.

Definition 43 (positive closure $P(\mathtt{x})$). For a coppice $\mathbf{C}$ and a set $P_0 \subseteq C$ define the *positive closure* $P(P_0)$ to be the smallest set S such that $P_0 \subseteq S$ and that for each $a, b \in S$ and each $x \in C$ also $ab \in S$ and $[x]^{\sim}[ax] \in S$ and $[ax]^{\sim}[x]x^{\sim}ax \in S$.

Proposition 116. *Let $\mathbf{C}$ a coppice with factor $\mathbb{Q}_+$ via a homomorphism $|\mathtt{x}| : \mathbf{C} \to \mathbb{Q}_+$ and let $P_0 \subseteq C$. If $|x| > 1$ for each $x \in P_0$, then $P(P_0)$ is (the set of positive elements of) a coppice order.*

Proof. If $|a|, |b| > 1$ and $|x|$ arbitrary then it is clear that

$$|ab| = |a|\,|b| > 1,$$

$$|[x]^{\sim}[ax]| = \frac{|a|\,|x| + 1}{|x| + 1} > 1,$$

$$|[ax]^{\sim}[x]x^{\sim}ax| = \frac{|a|\,(|x| + 1)}{|a|\,|x| + 1} > 1.$$

Suppose now $x, x^{\sim} \in P(P_0)$ then $|x|, |x|^{-1} > 1$ which is a contradiction. □

Proposition 117 (Corollary). *$P(\mathbb{P} \setminus \{1\})$ is a coppice order of $\mathbb{F}$ and $P(\mathbb{B} \setminus \{1\})$ is a coppice order of $\mathbb{B}^{\circ}$.*

Problem 10. *Is the coppice order $P(\mathbb{P} \setminus \{1\})$ of $\mathbb{F}$ topological? Is the coppice order $P(\mathbb{B} \setminus \{1\})$ of $\mathbb{B}^\circ$ topological? For what initial sets P_0 is $P(P_0)$ a topological coppice order?*

Proposition 118. *$P(\mathbb{B} \setminus \{1\}) = P(\{2\})$ in $\mathbb{B}^\circ$ and $P(\mathbb{P} \setminus \{1\}) = P(\{2\})$ in $\mathbb{F}$.*

Proof. One applies right additive and left additive rule to recursively show that $\mathbb{B} \subseteq P(\{2\}) \cup \{1\}$ in $\mathbb{B}^\circ$ and $\mathbb{P} \subseteq P(\{2\}) \cup \{1\}$ in $\mathbb{F}$. □

Problem 11. *Is the coppice order $P(\{2\})$ decidable on $\mathbb{F}$?*

Proposition 119. *Every coppice order is dense.*

Proof. The idea is to take the arithmetic mean as in-between-element, so for all coppice elements a, b we make the following two lines of conclusions.

$$\begin{aligned} a &< b & \qquad a &< b \\ a \oplus a &< a \oplus b & \qquad a \oplus b &< b \oplus b \\ 2a &< a \oplus b & \qquad a \oplus b &< 2b \\ a &< 2^\sim(a \oplus b) & \qquad 2^\sim(a \oplus b) &< b \end{aligned}$$

□

Proposition 120. *A linear coppice order is topological.*

Proof. (63) is trivially satisfied by letting δ be the smaller element of ε and ε'. There can also be no pseudo units (67) because a pseudo unit e would be comparable with 1 but by denseness (proposition 119) there exists x between e and 1 and so e can no more be a pseudo unit. For a linear coppice order, (67) implies (64): Assume there would be no δ with $\delta^2 \leq \varepsilon$ then $\varepsilon < \delta^2$ for all $\delta > 1$. Fix a δ', $1 < \delta' < \varepsilon$, then $\varepsilon < \delta\delta \leq \delta'\delta$ for all $1 < \delta \leq \delta'$. Now $1 < \delta'^\sim\varepsilon < \delta$ implies $\delta'^\sim\varepsilon = 1$ by (67), but this is a contradiction. □

So to create Hausdorff coppice topologies we have two choices, either we generate some partial order on the coppice and show that the conditions of proposition 115 are satisfied, or we extend a partial order to a linear coppice order. The last way seems much more promising. Though we can only presently show that $\mathbb{B}^\circ$ allows a linear *incremental* coppice order (see proposition 123).

Problem 12 (difficult). *Does $\mathbb{B}^\circ$ allow a linear coppice order?*

Conjecture 13 (difficult). *$\mathbb{F}$ allows a linear coppice order.*

There may be even the question whether there is only one (up to swapping direction) linear coppice order on $\mathbb{B}^\circ$ and $\mathbb{F}$ possible. At least this is true for $\mathbb{Q}_+$:

Proposition 121 (Remark). *Up to swapping order direction there is only one linear coppice order possible on $\mathbb{Q}_+$.*

Proof. Because we consider up to swapping order direction, we assume $1 < 2$. By additive compatibility $2 < 3 < 4 < \ldots$, i.e. on $\mathbb{N}$ the order is already determined as the usual order of $\mathbb{N}$. But then $p_1/q_1 < p_2/q_2$ is equivalent to $p_1q_2 < p_2q_1$ and so the order is determined on whole $\mathbb{Q}_+$. □

Problem 14 (difficult). *How many linear coppice orders with* $1 < 2$ *are available on* $\mathbb{B}^\circ$ *and on* $\mathbb{F}$*?*

So these are all problems and conjectures, but what we can at least show is that $\mathbb{B}^\circ_w$ has a linear *incremental* coppice order. It is already known (first proofs [26] and [17] in 1948) that each free group can be equipped with a linear group ordering. An elegant way is to do so with the Magnus expansion, which we will use in the next proof.

Proposition 122 (Lemma, $<^\mu$). *To each linear order* $<$ *on a set* X *we can assign a linear order* $<^\mu$ *on* $\mathrm{cW}(X)$ *such that*

1. $a < b \iff [a] <^\mu [b]$ *for all* $a, b \in X$
2. $<^\mu$ *is a linear group ordering on* $\mathrm{cW}(X)$
3. *Let* $\ll$ *be the restriction of* $<$ *to a subset* $Y \subseteq X$*. Then* $\ll^\mu$ *is the restriction of* $<^\mu$ *to* $\mathrm{cW}(Y) \subseteq \mathrm{cW}(X)$*.*

Proof. Let $\mathbb{Z}[[X]]$ be the formal power series with non-commuting variables in X. To be more precise $\mathbb{Z}[[X]]$ is the set of functions, mapping each monomial $x_1 \cdots x_n$, $n \in \mathbb{N}_0$, $x_1, \ldots, x_n \in X$ to its integer coefficient. It is equipped with coefficient wise addition, and multiplication by distribution, i.e. defined by

$$\begin{aligned}(\alpha + \beta)(x_1 \cdots x_n) &:= \alpha(x_1 \cdots x_n) + \beta(x_1 \cdots x_n),\\ (\alpha\beta)(x_1 \cdots x_n) &:= \sum_{0 \le k < n} \alpha(x_1 \cdots x_k)\beta(x_{k+1} \cdots x_n).\end{aligned}$$

$\mathbb{Z}[[X]]$ is known to be a ring. Its units are are all α with $\alpha(1) \neq 0$ (where 1 denotes the empty monomial). We have then a homomorphism $\mu\colon \mathrm{cW}(X) \to \mathbb{Z}[[X]]$ (the Magnus expansion) defined by

$$\begin{aligned}\mu([x_1]^{\varepsilon_1} \cdots [x_n]^{\varepsilon_n}) &:= (1 + x_1)^{\varepsilon_1} \cdots (1 + x_n)^{\varepsilon_n},\\ \frac{1}{1+x} &= 1 - x + x^2 - x^3 + x^4 \pm \cdots .\end{aligned}$$

We denote the image of μ by $\Gamma(X)$ which is a multiplicative subgroup of $\mathbb{Z}[[X]]$. It is well-known that μ is injective and hence a (multiplicative) group isomorphism between $\mathrm{cW}(X)$ and $\Gamma(X)$.

Further $\Gamma(X)$ can be linearly ordered in the following way. We first order the monomials by $x_1 \cdots x_m < y_1 \cdots x_n$ iff either $m < n$ or for $m = n$ that $x_1 \cdots x_m$ is lexicographic *greater* than $y_1 \cdots y_n$ (using the linear order on X). This twist in the second part assures condition 1 in the end. Then we know that each power series of $\Gamma(X)$ contains only a finite number of variables, so the order on the monomials is a well-order and we can define $\alpha < \beta$ as $\alpha(M) < \beta(M)$ for the smallest monomial M with $\alpha(M) \neq \beta(M)$, i.e. it is the lexicographic order on the functions $\Gamma(X)$. To show that this order is compatible with the power series multiplication is left to the reader.

Thus we have assigned a linear group order on $\mathrm{cW}(X) \equiv \Gamma(X)$ to each linear order on X. This satisfies condition 2. Also condition 1 is satisfied: if $x_1 < x_2$ in X then the monomial x_1 is greater than the monomial x_2 so the minimum of both monomials is x_2 and $\mu(x_1)(x_2) = 0 < 1 = \mu(x_2)(x_2)$ in $\mu(x_1) = 1 + x_1$ and $\mu(x_2) = 1 + x_2$. Condition 3 is satisfied because also $\Gamma(X) \subseteq \Gamma(Y)$. $\square$

Proposition 123. *$\mathbb{B}^\circ$ has a linear incremental coppice order.*

Proof. Define the W_i as in definition 27. We equip each W_i with the order $<_i$, where $<_0$ is defined as being the empty order and $<_{i+1}$ is defined as being ${<_i}^\mu$. By proposition 122 is $<_i$ a restriction of $<_k$ to W_i for all $i < k$. Hence we have a linear order on $\mathbb{B}_w^\circ = \cup_{i=1}^\infty W_i$ defined by $a < b \iff a <_i b$ for i being the minimal index such that $a, b \in W_i$. The reader may verify that this is a linear group ordering on $\mathbb{B}_w^\circ$ and that $[a] < [b]$ for $a < b$, $a, b \in \mathbb{B}_w^\circ$. □

7 Power-Inverse-Iterated Functions

There are two major open questions regarding the power-inverse-iterated functions. We dedicate this chapter to approach solutions. These questions are

Conjecture 15 (medium). *The order $<_\uparrow$ is linear on $\mathbb{P}_I^\circ$.*

Conjecture 16 (difficult). *$\mathbb{P}_I^\circ$ is isomorphic to $\mathbb{F}$.*

In the following we assume $f \in \mathbb{P}_I^\circ$ to be given by a fractional tree and will write that in a mixed notation:

$$f = \left(\mathrm{id}^{f\downharpoonleft}\right)^{-1} \circ \mathrm{id}^{f\upharpoonleft} = \pi \frac{[f\upharpoonleft]}{[f\downharpoonleft]}.$$

Where the $f\upharpoonleft$ and $f\downharpoonleft$ are always regarded to be in a product context, i.e. $(f\downharpoonleft)$ is short for $\Pi(f\downharpoonleft)$, and where $\pi\colon \mathbb{F} \to \mathbb{P}_I^\circ$ is the universal homomorphism. We only have to be conscious about that this may not be a unique representation for $f \in \mathbb{P}_I^\circ$ even if it is cancelled.

7.1 Regarding Conjecture 15

Proposition 124. *The order $<_\uparrow$ on $\mathbb{P}_I^\circ$ is linear if and only if the fixed points of every function $f \in \mathbb{P}_I^\circ$ have an upper bound.*

Proof. $f <_\uparrow g \iff \mathrm{id} <_\uparrow f^{-1} \circ g =: h$. Because h is continuous it crosses id every time it changes from above id to below id and back. So if it does not ultimately decide being above or below id it has above unbounded fixed points. Otherwise it has ultimately no more fixed points. □

As a specialisation of the more general conjecture 3 we claim

Conjecture 17 (difficult). *Every function of $\mathbb{P}_I^\circ$ has only a finite number of fixed points.*

The way we proved the linearity on $\mathbb{P}_I$ does not work for $\mathbb{P}_I^\circ$. The lexicographic order on $\mathbb{P}$ that was the major tool to prove $\mathbb{P}_I$ linearly ordered, is not valid in $\mathbb{P}_I^\circ$ for example consider the power-inverse-iterated function

$$f := \pi \frac{[1,1]}{[1]} = \left(\mathrm{id}^{\mathrm{id}}\right)^{-1} \circ \mathrm{id}^{\mathrm{id}^2}.$$

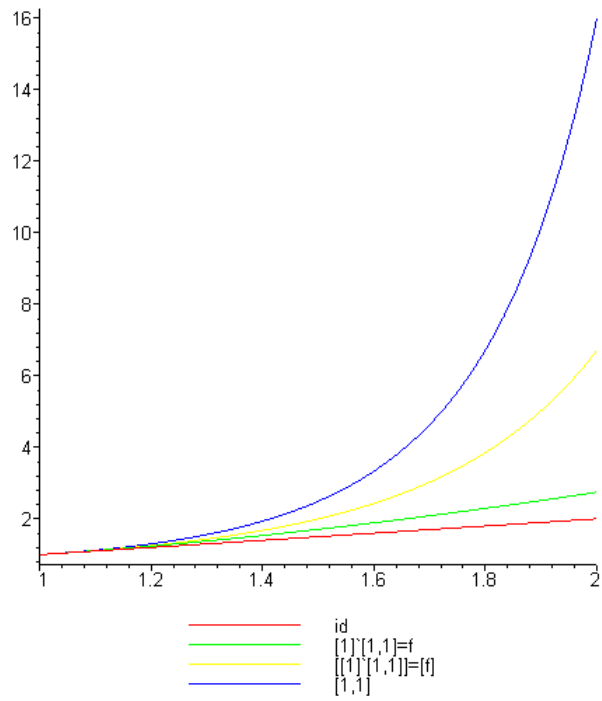
We first recognise that $\mathrm{id} < f$ (where $f < g$ means $f(x) < g(x)$ for all x of their domain) because we have the equivalence

$$\mathrm{id}^{\mathrm{id}} < \mathrm{id}^{\mathrm{id}^2} \iff \mathrm{id} < \left(\mathrm{id}^{\mathrm{id}}\right)^{-1} \circ \mathrm{id}^{\mathrm{id}^2}.$$

So f is the maximum element if we compare $[1,1]$ and $[f]$ element wise. In $\mathbb{P}_I$ this would already mean that $\overline{[1,1]} <_\uparrow \overline{[f]}$. Not so in $\mathbb{P}_I^\circ$, as we see by the following equivalences.

$$\begin{aligned} \pi([f]) &< \pi([1,1]) = \mathrm{id}^{\mathrm{id}^2} \\ \text{apply } \frac{\ln}{\ln(x)} \qquad f &< \mathrm{id}^2 \\ \left(\mathrm{id}^{\mathrm{id}}\right)^{-1} \circ \mathrm{id}^{\mathrm{id}^2} &< \mathrm{id}^2 \\ \mathrm{id}^{\mathrm{id}^2} &< \mathrm{id}^{\mathrm{id}} \circ \mathrm{id}^2 = \mathrm{id}^{2\,\mathrm{id}^2} \end{aligned}$$

If we however consider the order $<_\uparrow$ not heading to ∞ but heading to 1 we succeed in



proving it a linear coppice order.

Definition 44 ($<_{a\downarrow}$). For functions $f, g\colon (a,\infty) \to \mathbb{R}$, $a \in \mathbb{R}$, define the relation $f <_{a\downarrow} g$ as: there exists a $\delta > 0$ such that $f(x) < g(x)$ for all $x \in (a, a+\delta)$.

Proposition 125. *For $f, g\colon (a-\varepsilon,\infty) \to \mathbb{R}$ being real analytic, $f <_{a\downarrow} g$ is equivalent to $f^{(n)}(a) < g^{(n)}(a)$ for (existing) n being the smallest $i \in \mathbb{N}_0$ such that $f^{(i)}(a) \neq g^{(i)}(a)$. Or, in other words, the sequence of derivatives at a of f is lexicographically smaller than that of g. Hence either $f <_{a\downarrow} g$, $f = g$ or $g <_{a\downarrow} f$.*

Proof. Because the n-th derivative is continuous, $f^{(n)}(a) < g^{(n)}(a)$ implies that there exists an $\delta > 0$ such that $f^{(n)}(x) < g^{(n)}(x)$ for $x \in [a, a+\delta)$. But by integrating from a

to x we then get that $f^{(n-1)}(x) < g^{(n-1)}(x)$ for $x \in (a, a+\delta)$ and so on till $f(x) < g(x)$ for $x \in (a, a+\delta)$. If vice versa $f(x) < g(x)$ already for $x \in (a, a+\delta)$ and there would be no n with $f^{(n)}(a) < g^{(n)}(a)$ then by analyticity $f = g$. If n exists but $f^{(n)}(a) > g^{(n)}(a)$ then by the previous consideration would $f(x) > g(x)$ for $x \in (a, a+\delta_2)$. So it must $f^{(n)}(a) < g^{(n)}(a)$ for some n. □

Proposition 126. *$\mathbb{P}_I$ is a by $<_{a\downarrow}$ linearly ordered coppice for each $a \in \{1\} \cup \mathbb{R}_{>1}$.*

Proof. First we notice that each $f \in \mathbb{P}_I^\circ$ is real analytic (i.e. its Taylor expansion in each point has a non-zero radius of convergence and is equal to the function inside that radius) because it is made up only by composition, inversion and multiplication involving the base functions exp, ln and id (note that $f^g = \exp \circ ((\ln \circ f) \cdot g)$). The base functions exp, ln and id are clearly analytic and each composition, inversion and multiplication of analytic functions is again analytic.

Further note that we can analytically extend the domain of the functions $f \in \mathbb{P}_I^\circ$ a bit below 1. One can recursively show the following properties for each $f \in \mathbb{P}_I^\circ$: There exists $\varepsilon > 0$ and $\hat{f}\colon (1-\varepsilon, 1+\varepsilon) \to \mathbb{R}_+$, with $\hat{f}(x) = f(x)$ for $x \in (1, 1+\varepsilon)$, $\hat{f}$ is real analytic and $\hat{f}'(1) = 1$. These properties convey from f, g to $f \circ g$, to f^{-1} (by implicit function theorem) and to f^g. By analytic continuation each extension is unique on its domain.

Now linearity is provided by proposition 125. Compatibility with composition is clear by the direct definition and strict increase of the functions. For additive coppice compatibility we have to show $h^f <_{a\downarrow} h^g$ and $f^h <_{a\downarrow} g^h$ for $f <_{a\downarrow} g$ and $f, g, h \in \mathbb{P}_I^\circ$. But these are again clear by the direct definition of $<_{a\downarrow}$ and the strict increase of the functions. □

Proposition 127 (Corollary). *For any $f \in \mathbb{P}_I^\circ$ there is no accumulation point of its fixed points except if $f = \mathrm{id}$.*

Proof. Because at that accumulation point a neither $f <_{a\downarrow} \mathrm{id}$ nor $\mathrm{id} <_{a\downarrow} f$. □

7.2 Regarding Conjecture 16

Proposition 128. *$\mathbb{P}_I^\circ \equiv \mathbb{F}$ if and only if for every $f_1, \ldots, f_m, g_1, \ldots, g_n \in \mathbb{P}_I^\circ$ the equation $f_1 \cdots f_m = g_1 \cdots g_n$ implies $m = n$ and a bijection $\alpha\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $g_i = f_{\alpha(i)}$ for $i = 1, \ldots, n$.*

Proof. Let $\pi\colon \mathbb{F} \to \mathbb{P}_I^\circ$ be the universal epimorphism. $\mathbb{P}_I^\circ \equiv \mathbb{F}$ is equivalent to $\pi(r) = \pi(s) \Longrightarrow r = s$ for every $s, t \in \mathbb{F}$, or differently put: $\pi(t) = \mathrm{id} \Longrightarrow t = 1$ for all $t \in \mathbb{F}$.

$$\pi(t) = \mathrm{id} \iff \mathrm{id}^{\Pi\pi(t\downarrow)} = \mathrm{id}^{\Pi\pi(t\uparrow)} \iff \Pi\pi(t\downarrow) = \Pi\pi(t\uparrow)$$

So $\Pi\pi(t\downarrow) = \Pi\pi(t\uparrow)$ must imply $[t\downarrow] = [t\uparrow]$ what was the assertion. □

The only thing that we can really show is that $m = n$ in the last proposition. This heavily supports our conjecture. To show $m = n$ let us strike out a bit more. As we have seen taking the logarithm to base x to show order of elements is a quite natural transformation. Now there are completely surprising laws for taking $\log_{\mathrm{id}}$. Usually these laws are known from the differentiation of functions!

Proposition 129. *For a function f let $f' := \log_{\mathrm{id}} f = \frac{\ln \circ f}{\ln}$ then for $f, g \in \mathbb{P}_I^\circ$ the following (differentiation type) laws are valid.*

$$\mathrm{id}' = 1 \tag{68}$$

$$(f \circ g)' = (f' \circ g) \cdot g \tag{69}$$

$$(f^{-1})' = \frac{1}{f' \circ f^{-1}} \tag{70}$$

$$(f^{-1} \circ g)' = \frac{g'}{f' \circ (f^{-1} \circ g)} \tag{71}$$

Proof. (68) is trivial. Then for a usual $f \in \mathbb{P}_I^\circ$ we conclude

$$\begin{aligned}
\left(\mathrm{id}^{f\downarrow}\right)^{-1} \circ \mathrm{id}^{f\uparrow} &= f, \\
\mathrm{id}^{f\uparrow} &= \left(\mathrm{id}^{f\downarrow}\right) \circ f = f^{f\downarrow \circ f} = \left(\mathrm{id}^{f'}\right)^{f\downarrow \circ f} = \mathrm{id}^{f' \cdot (f\downarrow \circ f)}, \\
f' &= \frac{f\uparrow}{f\downarrow \circ f}.
\end{aligned}$$

For (71) let us continue by applying the multiplication rule of $\mathbb{F}$.

$$\begin{aligned}
(f^{-1} \circ g)' &= \left(\mathrm{id}^{(g\downarrow \circ f) \cdot f\uparrow}\right)^{-1} \circ \left(\mathrm{id}^{(f\downarrow \circ g) \cdot g\uparrow}\right) \\
&= \frac{(f\downarrow \circ g) \cdot g\uparrow}{((g\downarrow \circ f) f\uparrow) \circ f^{-1} \circ g} \\
&= \frac{(f\downarrow \circ g) \cdot g\uparrow}{(g\downarrow \circ g) \cdot (f\uparrow \circ f^{-1} \circ g)} = \frac{((f\downarrow \circ f) \circ (f^{-1} \circ g)) \cdot g\uparrow}{(g\downarrow \circ g) \cdot (f\uparrow \circ (f^{-1} \circ g))} \\
&= \frac{g'}{f' \circ (f^{-1} \circ g)}
\end{aligned}$$

The rule (70) is a specialisation and the rule (69) can be derived by:

$$\begin{aligned}
(f \circ g)' &= \left(\left(f^{-1}\right)^{-1} \circ g\right)' = \frac{g'}{(f^{-1})' \circ f \circ g}, \\
&= \frac{g'}{\frac{1}{f' \circ f^{-1}} \circ f \circ g}, \\
&= g' \cdot (f' \circ g).
\end{aligned}$$

□

Proposition 130. *For all $f \in \mathbb{P}_I^\circ$ the function f' is $: \mathbb{R}_{>1} \to \mathbb{R}_+$ and $f'(x) \to 1$ for $x \to 1$.*

Proof.

$$f' = \frac{f\uparrow}{f\downarrow \circ f}$$

Now we know that each $f\uparrow(x), f\downarrow(x), f(x) \to 1$ for $x \to 1$ and so does f'. □

Proposition 131 (Corollary). *If $f_1 \cdots f_m = g_1 \cdots g_n$ then already $m = n$ for arbitrary $f_1, \ldots, f_m, g_1, \ldots, g_n \in \mathbb{P}_I$.*

Proof. We apply $\mathtt{x}'$ on both sides and can conclude the following.

$$\begin{aligned} f_1'(x) + \cdots + f_m'(x) &= g_1'(x) + \cdots + g_n'(x) \\ f_1'(1) + \cdots + f_m'(1) &= g_1'(1) + \cdots + g_n'(1) \\ m &= n \end{aligned}$$

□

8 Prospects

Having constructed the free coppices $\mathbb{B}^\circ$, $\mathbb{F}$ and $\mathbb{Q}_+$ the next immediate question is about convergence and completion. There are two candidates for completion, the order completion and the topological completion (which coincide on $\mathbb{R}_+$), see also [3], [27], [20], [11]. Order completion is roughly taking all supremums and infimums, though there is a well known statement in order theory (see [15] 9.1.1)

Proposition 132. *Any order complete directed group is an Archimedian lattice-ordered group and so commutative.*

Without explaining all the details we can say *order completion* is out of question for us because the groups we are interested in are *not* commutative. By the way: an order of a non-commutative groups can not be integrally closed, i.e. there are always $r \not\leq 1$ and s such that $r^n \leq s$ for all $n \in \mathbb{N}$.

The *topological completion* is roughly taking all Cauchy sequences. Uniform structures arise naturally from the topology of a group. There are the left, the right and the bilateral uniformity of a topological group. The left uniformity consists of all the entourages W containing the pairs (x, y) such that $x^\sim y \in U$ for some environment U of 1. Where for the right uniformity must $xy^\sim \in U$ and for the bilateral uniformity must $x^\sim y \in U$ and $xy^\sim \in U$.

The most interesting uniformity for us is the bilateral or also called the LR-uniformity because the following theorem (see [32], theorem 5.9) does neither hold for the left nor for the right uniformity.

Proposition 133. *Each Hausdorff group has a bilateral completion.*

Conjecture 18 (straight). *Each Hausdorff coppice has a bilateral completion.*

We should make a note about the completion of $\mathbb{Q}_+$ to $\mathbb{R}_+$. As the reader might already have detected, usually one would regard the completion of $\mathbb{Q}_+$ being $\mathbb{R}_+ \cup \{0\}$ and not $\mathbb{R}_+$. But the completion depends on the uniformity. Of course we consider $\mathbb{Q}_+$ as being equipped with the interval topology, but we can consider *two* uniformities on $\mathbb{Q}_+$. The *additive* uniformity is generated by the entourages

$$V_\varepsilon = \{(x, y) \colon -\varepsilon < x - y < \varepsilon\} \quad \text{for} \quad \varepsilon > 0$$

and the *multiplicative* uniformity (which we regard for the topological coppice $\mathbb{Q}_+$) is generated by the entourages

$$V_\varepsilon = \{(x,y)\colon 1/\varepsilon < x/y < \varepsilon\}eps\} \quad \text{for} \quad \varepsilon > 1.$$

The Cauchy completion of the latter is $\mathbb{R}_+$, because every sequence converging to zero is no multiplicative Cauchy sequence (x_n/x_m becomes arbitrary big for $m \to \infty$).

If we now have a non-trivial Hausdorff coppice, then we also have a non-trivial completion. But what about the higher operations? Our initial idea was to continue the higher operations.

Definition 45 (n-operations coppice). A multi-sorted algebraic structure $\mathbf{H}$ is called an *n-operations coppice*, where $n \geq 2$ is a natural number or infinity, iff it has the sorts H_{i+1} for $0 \leq i < n$ with $H := H_1 = H_2$ and $H_i \subseteq H_{i-1}$ for $i \geq 3$, and has the operations

$$\begin{aligned} &1 \in H, \\ &\mathtt{x}^\sim \colon H \to H, \\ &\ltimes_i \colon H \times H_i \to H_i \quad \text{for each} \quad i \geq 1, \end{aligned}$$

where we write $\mathtt{x} \oplus \mathtt{y}$ for $\mathtt{x} \ltimes_1 \mathtt{y}$ and $\mathtt{xy}$ for $\mathtt{x} \ltimes_2 \mathtt{y}$, and it satisfies

1. $H_i \neq \{1\}$ for $i \geq 3$ to exclude trivial higher operations.
2. The operations $\mathtt{x} \ltimes_i \mathtt{y}$ are related by

$$(a \oplus b) \ltimes_i x = (a \ltimes_i x) \ltimes_{i-1} (b \ltimes_i x) \tag{72}$$

   for each $a, b \in H$ and $x \in H_i$, $i \geq 2$.
3. $(H, 1, \mathtt{xy}, \mathtt{x}^\sim)$ is a group and operates on H_i via $\mathtt{x} \ltimes_i \mathtt{y}$ for each $i \geq 3$, i.e.

$$1 \ltimes_i x = x \tag{73}$$
$$(ab) \ltimes_i x = a \ltimes_i (b \ltimes_i x) \tag{74}$$

   for all $a, b \in H$ and $x \in H_i$, $i \geq 3$.

An ∞-operations coppice is also called higher operations coppice. If the operation $\mathtt{x} \oplus \mathtt{y}$ has the property *xyz* we call it an *xyz* n-operations coppice. Every coppice can be regarded as 2-operations coppice. We say the n-operations coppice $\mathbf{H}$ *heightens* the m-operations coppice $\mathbf{G}$ iff $m < n$ and $\mathtt{x} \ltimes_i \mathtt{y}$ is identical on $H_i = G_i$ for $i < m$. $\mathbf{H}$ is called a *topological n-operations coppice* iff $\mathbf{H}$ is equipped with a topology such that all $\mathtt{x} \ltimes_i \mathtt{y}$ and $\mathtt{x}^\sim$ are continuous.

The reason why we can not use just one sort in a higher operations coppice is similar to why we define x^r only for positive x and why we define $\mathbb{P}_I^\circ$ on $\mathbb{R}_{>1}$, i.e. because x^2 is not injective on $\mathbb{R}$ but on $\mathbb{R}_+$ and x^x is not injective on $\mathbb{R}_+$ but on $\mathbb{R}_{>1}$. The following proposition shows that this is crucial on higher operations coppices too.

Proposition 134. *There is no 4-operations coppice with $H = H_3 = H_4$.*

Proof. We show that $x \mapsto (1 \oplus 2) \ltimes_4 x$ is not injective over H. It has the same value at $2^\sim$ and $2^\sim \ltimes_3 2^\sim$.

$$\begin{aligned}
(2^\sim \ltimes_3 x)(2^\sim \ltimes_3 x) &= 2 \ltimes_3 (2^\sim \ltimes_3 x) = x \\
(2^\sim \ltimes_3 x) \ltimes_3 ((2^\sim \ltimes_3 x) \ltimes_3 y) &= ((2^\sim \ltimes_3 x)(2^\sim \ltimes_3 x)) \ltimes_3 y = x \ltimes_3 y \\
(1 \oplus 2) \ltimes_4 (2^\sim \ltimes_3 2^\sim) &= ((2^\sim \ltimes_3 2^\sim)(2^\sim \ltimes_3 2^\sim)) \ltimes_3 (2^\sim \ltimes_3 2^\sim) \\
&= 2^\sim \ltimes_3 (2^\sim \ltimes_3 2^\sim) = (1 \oplus 2) \ltimes_4 2^\sim
\end{aligned}$$

A bit more laxly we can write the previous lines as

$$(1 \oplus 2) \ltimes_4 (2^\sim)^{\frac{1}{2}} = (2^\sim)^{\frac{1}{2}(\frac{1}{2})^{\frac{1}{2}}(\frac{1}{2})^{\frac{1}{2}}} = (2^\sim)^{\frac{1}{2}\frac{1}{2}} = (1 \oplus 2) \ltimes_4 2^\sim.$$

But if we now apply $(1 \oplus 2)^\sim \ltimes_4$ on the front of both sides we get $2^\sim \ltimes_3 2^\sim = 2^\sim$ then $2^\sim = 2^\sim 2^\sim$ then $1 = 2^\sim$ then $1 = 2$ then $1 \ltimes_3 x = 2 \ltimes_3 x$ then $x = xx$ then $1 = x$ then $H_4 = \{1\}$. □

Though the question remains whether

Conjecture 19 (medium). *There exists a higher operations coppice with $H_4 = H_i$ for all $i \geq 5$.*

Conjecture 20 (difficult). *$\mathbb{B}^\circ$ and $\mathbb{F}$ can be embedded into a higher operations coppice (with $H_4 = H_i$ for all $i \geq 5$).*

Conjecture 21 (heavy). *There is a topological completion of $\mathbb{F}$, that can be heightened to a topological higher operations coppice.*

Conjecture 22 (straight). *The free mono-sorted ($H = H_3$) associative 3-operations coppice is isomorphic to the subset of the positive real numbers that is generated by starting with 1, and taking all sums, products, reciprocals and powers (equipped with $\mathtt{x} \ltimes_1 \mathtt{y} = \mathtt{x}+\mathtt{y}$, $\mathtt{x} \ltimes_2 \mathtt{y} = \mathtt{xy}$, and $\mathtt{x} \ltimes_3 \mathtt{y} = \mathtt{y}^{\mathtt{x}}$).*

Proposition 135. *The mono-sorted associative 3-operations coppice $(\mathbb{R}_+, 1, \frac{1}{\mathtt{x}}, +, \mathtt{xy}, \mathtt{y}^{\mathtt{x}})$ can not be heightened.*

Proof. Assume we had an element $h \neq 1$ in H_4 then

$$\begin{aligned}
3 \ltimes_4 h &= (2+1) \ltimes_4 h = h^{2 \ltimes_4 h} = h^{h^h}, \\
3 \ltimes_4 h &= (1+2) \ltimes_4 h = (2 \ltimes_4 h)^h = h^{h^2}, \\
h^{h^h} &= h^{h^2} \iff h^h = h^2 \iff h = 2.
\end{aligned}$$

Now consider

$$\begin{aligned}
4 \ltimes_4 2 &= ((2+1)+1) \ltimes_4 2 = 2^{2^{2^2}} = 2^{2^4}, \\
4 \ltimes_4 2 &= (1+(1+2)) \ltimes_4 2 = 2^{2^3}.
\end{aligned}$$

□

We see, the study of arborescent numbers opens a rich field for research. These numbers are always a good deal more complicated than the "associative" numbers, but it sounds promising being able to perform the major number constructions on them too. These are the embedding into a division structure — which we showed in this contribution — and the embedding into topologically complete structures, which awaits investigation.

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