Definition: The exponential space E

$$\forall \lambda \in \mathbb{C} \;\; f(s) = e^{\lambda s} \in \mathbb{E} \;\; ; \;\; f : \mathbb{C} \to \mathbb{C}$$

$$\begin{aligned}
\forall f, g \in \mathbb{E} \quad &\Rightarrow \quad \alpha f + \beta g \in \mathbb{E} \\
&\Rightarrow \quad f \cdot g \in \mathbb{E} \\
&\Rightarrow \quad f(g) \in \mathbb{E} \\
&\Rightarrow \quad \frac{df}{ds} \in \mathbb{E} \\
\lim_{n\to\infty} f_n = f, \; f_n \in \mathbb{E} \quad &\Rightarrow \quad f \in \mathbb{E}
\end{aligned}$$

Definition: The iterated derivative $\mathcal{E} : \mathbb{E} \to \mathbb{E}$ that sends functions from $f(s) \to \frac{d^s f}{dt^s}\Big|_{t=0}$ This global linear operator is given by the integral transformation:

$$\mathcal{E}_s f(s) = \frac{1}{\Gamma(-s)} \int_0^\infty f(-y) y^{-s-1} dy$$

Where $\Gamma(s) = \int_0^\infty e^{-y} y^{s-1} dy$ and interpolates the factorial shifted back one. This is shown by:

$$\begin{aligned}
\mathcal{E}_s(e^{\lambda s}) &= \frac{1}{\Gamma(-s)} \int_0^\infty e^{-\lambda y} y^{-s-1} dy \\
&= \frac{1}{\Gamma(-s)} \int_0^\infty e^{-u} \left(\frac{u}{\lambda}\right)^{-s-1} \lambda du \\
&= \lambda^s \frac{\Gamma(-s)}{\Gamma(-s)} \\
&= \frac{d^s e^{\lambda t}}{dt^s}\Big|_{t=0}
\end{aligned}$$

Where the integral is conditionally convergent but represents a function globally defined by the analytic continuation of the Gamma function. It is clear that to invert the operation we take the Taylor series and arrive at a series. By this I mean:

$$\mathcal{E} f = g \;\Leftrightarrow\; f = \sum_{j=0}^{\infty} g(j) \frac{s^j}{j!}$$

This is a valid representation since all elements of $\mathbb{E}$ are holomorphic functions. However, we arrive at a more conclusively powerful result by defining a new linear operator–the continuous Taylor transformation:

Definition: The continuous Taylor transformation is given by:

$$\mathcal{Y} f(s) = \int_{-\infty}^{\infty} f(y) \frac{s^y}{\Gamma(y+1)} dy$$

Theorem 1. *If $\mathcal{Y}\mathcal{E}f(s)$ converges it converges to $\alpha f(s)$ for some constant $\alpha \in \mathbb{R}$*

We start by defining the linear operators:

$$Qf(s) = sf(s) \;\; ; \;\; Tf(s) = f(s+1)$$

We show that:

$$\begin{aligned}
\mathcal{E}Qf(s) &= \frac{1}{\Gamma(-s)}\int_0^\infty (-y)f(-y)y^{-s-1}dy \\
&= \frac{(-1)}{\Gamma(-s)}\int_0^\infty f(-y)y^{-s}dy \\
&= \frac{s}{(-s)\Gamma(-s)}\int_0^\infty f(-y)y^{-s}dy \\
&= \frac{s}{\Gamma(1-s)}\int_0^\infty f(-y)y^{-s}dy \\
&= QT^{-1}\mathcal{E}
\end{aligned}$$

This gives the quick result $Q^{-1}\mathcal{E}Q = T^{-1}\mathcal{E}$ Going on to $\mathcal{Y}$

$$\begin{aligned}
\mathcal{Y}Qf(s) &= \int_{-\infty}^\infty yf(y)\frac{s^y}{\Gamma(y+1)}dy \\
&= \int_{-\infty}^\infty yf(y)\frac{s^y}{y\Gamma(y)}dy \\
&= \int_{-\infty}^\infty f(y)\frac{s^y}{\Gamma(y)}dy \\
&= \int_{-\infty}^\infty f(y+1)\frac{s^{y+1}}{\Gamma(y+1)}dy \\
&= s\int_{-\infty}^\infty f(y+1)\frac{s^y}{\Gamma(y+1)}dy \\
&= Q\mathcal{Y}Tf(s)
\end{aligned}$$

This shows: $Q^{-1}\mathcal{Y}Q = \mathcal{Y}T$
This allows to write

$$Q^{-1}\mathcal{E}Q = T^{-1}\mathcal{E}$$
$$Q^{-1}\mathcal{Y}Q = \mathcal{Y}T$$

This gives by simple manipulation:

$$\mathcal{Y}\mathcal{E}Q = Q\mathcal{Y}\mathcal{E}$$

Therefore $\mathcal{YE}$ commutes with Q which is multiplication by the independent variable s. Iterating this

$$s^y \mathcal{YE} f(s) = \mathcal{YE} s^y f(s)$$

Therefore if we take an analytic function $p(s)$ and take the partial sums of its Taylor series:

$$p_N(s) = \sum_{j=0}^{N} a_j s^j$$

It is very clear that

$$\mathcal{YE} p_N(s) f(s) = p_N(s) \mathcal{YE} f(s)$$

by the linearity of $\mathcal{YE}$ and then since $\mathcal{YE}$ is a continuous operator we can take the limit of the partial series from the outside:

$$\begin{aligned} p(s)\mathcal{YE}f(s) &= \lim_{N\to\infty} p_N(s)\mathcal{YE}f(s) \\ &= \lim_{N\to\infty} \mathcal{YE}p_N(s)f(s) \\ &= \mathcal{YE}\lim_{N\to\infty} p_N(s)f(s) \\ &= \mathcal{YE}p(s)f(s) \end{aligned}$$

Therefore $\mathcal{YE}$ commutes with multiplication by every analytic function, and hence $\mathcal{YE}f = u(s)f(s)$ for some analytic function u, we show that u is constant. Take $\frac{d}{ds}e^{\lambda s} = \lambda e^{\lambda s}$ to give $\mathcal{E}\frac{d}{ds}e^{\lambda s} = \lambda^{s+1}$, this result holds for linear combinations and hence we arrive at the identity $\mathcal{E}\frac{d}{ds}f = T\mathcal{E}f$. Now taking the integral of $\mathcal{Y}f$ we get:

$$\begin{aligned} \int \mathcal{Y}f(s)ds &= \int\int_{-\infty}^{\infty} f(y)\frac{s^y}{\Gamma(y+1)}dyds \\ &= \int_{-\infty}^{\infty} f(y)\frac{\int s^y ds}{\Gamma(y+1)}dy \\ &= \int_{-\infty}^{\infty} f(y)\frac{s^{y+1}}{\Gamma(y+2)}dy \\ &= \int_{-\infty}^{\infty} f(y-1)\frac{s^y}{\Gamma(y+1)}dy \end{aligned}$$

Therefore $\frac{d^{-1}}{ds^{-1}}\mathcal{Y}f = \mathcal{Y}T^{-1}f$, combining this with the last identity we get $\frac{d^{-1}}{ds^{-1}}\mathcal{YE}\frac{df}{ds} = \mathcal{YE}f$ and therefore $\mathcal{YE}$ commutes with the derivative and we have:

$$\frac{d}{ds}\mathcal{YE}f(s) = \mathcal{YE}\frac{df}{ds}$$

$$\frac{d}{ds}u(s)f(s) = u(s)\frac{df}{ds}$$

and therefore $u(s)$ is the constant map and the result follows. $\square$

Example: Take $0 < b < 1 \in \mathbb{R}$. Using our new formulas we can show that:

$$b^{b^s} = \sum_{n=0}^{\infty} \frac{\ln(b)^n b^{ns}}{n!}$$

$$\begin{aligned} \mathcal{Y} b^{b^s} &= \mathcal{E}^{-1} b^{b^s} \\ &= \sum_{n=0}^{\infty} \mathcal{E}^{-1} \frac{\ln(b)^n b^{ns}}{n!} \\ \int_{-\infty}^{\infty} b^{b^y} \frac{s^y}{\Gamma(y+1)} dy &= \sum_{n=0}^{\infty} \frac{\ln(b)^n b^{sb^n}}{n!} \end{aligned}$$

This integral on the left handside converges since $b^{b^y} \to 1$ as $y \to \infty$ and the Gamma function on the bottom over powers the exponential. Also $b^{b^y} \to 0$ as $y \to -\infty$ and $\lim_{X \to -\infty} \frac{\Gamma(1+X)}{b^{b^X}} = 0$. The series converges just as well. Numerical evaluation has shown this result to be true.