On representations of the Weyl differintegral

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Abstract

We examine the Weyl differintegral by first giving a representation of it using contour integration. This contour integral allows us to talk about all the complex fractional derivatives of some functions $f(w)$ for $w \in \mathbb{R}^+$. We also further expand the Weyl differintegral of a smaller class of these functions into a Taylor series. This gives an entire representation of $\frac{d^z}{dw^z}f(w)$ as a Taylor series in w with coefficients that depend holomorphically on z in a right half plane. We prove a theorem on uniform convergence of sequences of differintegrals and we arrive at a parallel to Weierstrass' theorem on the uniform convergence of holomorphic functions and their derivatives. We give an application in the field of superfunctions or complex iteration theory. In which we can generate a holomorphic and entire superfunction of ψ satisfying certain restrictions. We also create a method of analytically continuing a recursive sequence satisfying some holomorphic recurrence relation. Insofar as taking a_n to $A(z)$ where $A(n) = a_n$; and where both satisfy the same recursive relation.

Keywords: Complex Analysis, Fractional Calculus, Weyl Differintegral, Gamma function, Recursion, Superfunctions, Mellin Transform.

1 Introduction

Fractional calculus has been a mathematical interest ever since Leibniz first posed the question: what is $\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}x$? Or if such a thing even is possible. Contemporaries of Leibniz considered it meaningless. Leibniz said, it was a paradox that would be resolved with time. He had the foresight to think this and say such a thing and he was quite correct. Euler showed interest in it and said (once he discovered his famous Gamma function) that it should be $\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}x = \frac{2}{\sqrt{\pi}}\sqrt{x}$. He also posed a definition for the fractional derivative of polynomials but lacked an analytic expression for the operator. These results were later investigated further by Liouville and Riemann where both came to the operator that is now called the Riemann-Liouville differintegral. They succeeded in doing what Euler couldn't. An analytic integral expression expressing the fractional derivative. We point the reader to ([6], [11], [10]) for a strong resource on the Riemann-Liouville differintegral. Liouville and Riemann however, also investigated what is now known as the Weyl differintegral which is a specific case of the Riemann-Liouville differintegral. Weyl however performed the most research into this form of the operator and it has become named after him for such.

The Weyl differintegral was first applied on holomorphic sums of exponentials by Liouville. Wherein it made sense to call the Weyl operator the exponential differintegral. Weyl investigated slightly different objects as he was interested in periodic functions $p(x+T) = p(x)$ satisfying $\int_0^T p(t)\,dt = 0$ ([8]). His closed form expression applies on more functions than these periodic functions alone and so it has many uses. We note that literature on the Weyl differintegral is sparse and difficult to find. However we have found enough to confidently explore the subject. For more of the history of Fractional Calculus please turn to [10].

In this paper we explore the Weyl differintegral in the framework of a modified inverse Mellin transform, which allows us to get the Weyl differintegral of suitable decaying functions. We retrieve two representations of the exponential differintegral. Each has its advantages and its flaws, but both methods retrieve a holomorphic function $\phi(z) = \frac{d^{-z}}{dw^{-z}}\Big|_{w=0} f(w)$, which allows us to regenerate $\frac{d^z}{dw^z} f(w)$ for w belonging to certain sets dependent on each representation we use. ($\frac{d^z}{dw^z}\Big|_{w=0} f(w) = \Big[\frac{d^z}{dw^z} f(w)\Big]_{w=0}$)

One of the advantages we get from looking at the Weyl differintegral is a method of performing fractional iterations of recurrence relations belonging to a certain class. Insofar as we can take a sequence with some holomorphic recurrence relation (i.e: a sequence $\{a_n\}_{n=0}^{\infty}$ such that $a_{n+1} = F(a_{n-k}, a_{n-k+1}, ..., a_n)$ where F is holomorphic function) and turn it into a holomorphic function such that it satisfies the same holomorphic recursion ($f(n) = a_n$ and $f(s+1) = F(f(s-k), f(s-k+1), ..., f(s))$). This technique allows us to find indefinite products, indefinite sums, and superfunctions. All concepts of complex iteration. The result is not apparent at first but once we prove enough theorems we will familiarize ourselves with the concepts. From there we will see with the theorems we present provide a competent manner of performing iteration. To begin we bring about some necessary results we will use in our investigation.

Definition 1. *The Gamma function Γ is a meromorphic function defined on the entire complex plane with poles at the nonpositive integers. The representation we present is not the usual form, but is still well known.*

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)} + \int_1^{\infty} e^{-t} t^{z-1}\,dt \tag{1}$$

The Gamma function satisfies the functional recurrence relationship, $z\Gamma(z) = \Gamma(z+1)$ and converges to the factorial for natural values, i.e $\Gamma(n+1) = n!$. We are given imaginary asymptotics of the Gamma function, which are:

$$|\Gamma(\sigma+iy)| \sim \sqrt{2\pi} e^{-\sigma-\pi|y|/2} |y|^{\sigma+1/2} \quad y \to \infty \tag{2}$$

We point the reader to [1] for a substantial overview of the Gamma function in fractional calculus. We also point to [2] which has some more results on the asymptotics of the Gamma function as well. The Gamma function serves as a catalyst for fractional calculus. The only ingredient that is just as necessary is the Mellin Transform. This leads us to the Mellin transform, and our necessary introductions revolving around it.

Definition 2. *The Mellin transform $\mathcal{M}$ is a transform that acts on functions g satisfying $\int_0^\infty |g(t)|t^{\sigma-1}\,dt < \infty$ for some $a, b \in \mathbb{R} \cup \{\pm\infty\}$ dependent on g such that $a < \sigma < b$, then the Mellin transform of g is, for $\Re(z) = \sigma$*

$$(\mathcal{M}g)(z) = \int_0^\infty g(t)t^{z-1}\,dt$$

We make the condition of absolute convergence on g so that we may apply the inverse operator $\mathcal{M}^{-1}$ discovered by Mellin which brings us back to the original function g.

$$g(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} (\mathcal{M}g)(z)t^{-z}\,dz$$

And as well, conversely, if we start with the function $g(t)$ defined by the integral on the right, and if it converges absolutely, then

$$\int_0^\infty g(t)t^{z-1}\,dt = (\mathcal{M}g)(z)$$

for all $a < \sigma < b$, as before. We point the reader to ([3],[7]) for a more comprehensive treatment on how the Mellin transform behaves. From this we modify the Mellin transform to give us the Weyl differintegral.

Definition 3. *The Weyl differintegral (exponential differintegral) $\frac{d^z}{dw^z}$ is a linear operator that interpolates the natural iterates of the derivative to complex values. For our purposes (on entire functions) it is given by the modified Mellin transform for some $\theta \in \mathbb{R}$ $\theta \in (-\pi, \pi]$, on entire f satisfying ($x \in \Omega$ for some open connected $\Omega \subset \mathbb{C}$) $\int_0^\infty |f(x - e^{i\theta}t)|t^{\sigma-1}\,dt < \infty$ with $a < \sigma < b$, where $a, b \in \mathbb{R} \cup \{\pm\infty\}$. Then $q \in \mathbb{C}$ for $\Re(q) = \sigma$*

$$\frac{d^{-q}}{dx^{-q}}f(x) = \frac{e^{i\theta q}}{\Gamma(q)}\int_0^\infty f(x - e^{i\theta}t)t^{q-1}\,dt$$

Literature on the Weyl differintegral is sparse and tends to be difficult to find however the author notes the following brief papers ([1], [4], [8], [12])–we note our definition varies slightly but we show equivalence in a lemma. Moreover it can be shown that if the following expressions are converging for some $q, p \in \Omega \subset \mathbb{C}$ that $\frac{d^{-q}}{dw^{-q}}\frac{d^{-p}}{dw^{-p}}f(w) = \frac{d^{-q-p}}{dw^{-q-p}}f(w)$ ([4]). We also note that $\frac{d^q}{d(w+t)^q}f(w+$

$t) = \frac{d^q}{dw^q} f(w+t)$ which implies that the Weyl differintegral commutes with the operator T_z such that $T_z f(w) = f(w+z)$ for $z \in \mathbb{C}$. The Weyl differintegral satisfies the exponential as its eigenfunctions, i.e: $\frac{d^{-q}}{dx^{-q}} e^{\lambda x} = \lambda^{-q} e^{\lambda x}$. It is important to note the Weyl differintegral diverges for monomials and we get that $\frac{d^{-q}}{dx^{-q}} x^w \neq \frac{\Gamma(w+1)}{\Gamma(w+q+1)} x^{w+q}$ which is the opposite of what happens when we use the Riemann-Liouville differintegral [6]. Since we do not reference one of the results we will need, we prove them in a quick lemma:

Lemma 1. *If f is analytic in some region including the sector $\{z \in \mathbb{C} : -\pi < \theta \leq \arg(-z) \leq \xi < \pi\}$ and $\int_0^\infty |f(w - e^{i\alpha} x)| x^{\sigma - 1} < \infty$ for $w \in \Omega$ where Ω is open, and for all α such that $\theta \leq \alpha \leq \xi$ then $\theta \leq \arg(\beta) \leq \xi$ and for $\beta w \in \Omega$.*

$$\frac{d^z}{dw^z} f(\beta w) = \beta^z \frac{d^z}{d(\beta w)^z} f(\beta w) \tag{3}$$

Proof. Let us take the line $[0, Re^{i\theta}]$ followed by the arc A_R connecting to $[Re^{i\xi}, 0]$. Denote this sector by S_R. Then if $\Re(q) > 0$ we know $\int_{S_R} f(w - z) z^{q-1}\, dz = 0$. If $\Re(q) < 0$ then the integral is still zero because there has to be a zero of order $-a$ in f to cancel out the pole z^{q-1} contributes otherwise the integral diverges and we arrive at a contradiction. Therefore $\int_{S_R} f(w - z) z^{q-1}\, dz = 0$. We also note that $\lim_{R\to\infty} f(w - e^{i\alpha} R) R^{b-\epsilon} = 0$ because $f = O(R^{-b})$. (These bounds follow by absolute convergence of the integral.) Therefore for $a < \Re(q) < b$, $\int_{A_R} f(w - z) z^{q-1}\, dx \to 0$ $R \to \infty$. This gives that $\lim_{R\to\infty} \int_0^{e^{i\theta} R} f(w - z) z^{q-1}\, dz = \lim_{R\to\infty} \int_0^{e^{i\xi} R} f(w - z) z^{q-1}\, dz$. Implying $e^{i\theta q} \int_0^\infty f(w - e^{i\theta} x) x^{q-1}\, dx = e^{i\xi q} \int_0^\infty f(w - e^{i\xi} x)\, dx$. Therefore it is not hard to see that integrating along a different line only brings about an exponential factor. Therefore $\frac{d^z}{dw^z} f(\beta w) = \beta^z \frac{d^z}{d(\beta w)^z} f(\beta w)$ if $|\beta| = 1$ $\theta < \arg(\beta) < \xi$. Assume now that $\beta \in \mathbb{R}^+$ then: make the substitution $\beta x = u$ to get $\int_0^\infty f(\beta w - \beta x) x^{q-1}\, dx = \beta^{-q} \int_0^\infty f(\beta w - u) u^{q-1}\, dx$. This shows the result. □

Since the Weyl differintegral is a modified Mellin transform, this will be the impetus for how we develop our new representation. It is also motivation for introducing the inverse Mellin transform. If $\frac{d^{-q}}{dx^{-q}} f(x)$ is analytic in the strip $a < \Re(q) < b$ and the integral in $\frac{d^{-q}}{dx^{-q}} f(x)$ is absolutely convergent then we have the following result:

$$f(x - e^{i\theta} t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma(q) e^{-i\theta q} \frac{d^{-q}}{dx^{-q}} f(x) t^{-q}\, dq$$

Now formally take the derivative with respect to t and we can see a very convenient relationship:

$$\frac{d}{dt}\big[f(x - e^{i\theta} t)\big] = -\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma(q+1) e^{-i\theta q} \frac{d^{-q}}{dx^{-q}} f(x) t^{-q-1}\, dq$$

This formal manipulation gives motivation as to why we will talk about the transform on the right. To begin we start from this transform rather than the differintegral expression and build some small results around it.

2 The analytic continuation of the Weyl differintegral of entire functions

Lemma 2. *If ϕ is holomorphic in the strip $a < \sigma < b$ and satisfies the bounds: $|\phi(q)| < Ce^{\alpha|\Im(q)|}$ for some $0 \leq \alpha < \pi/2$ and $C \in \mathbb{R}^+$ then*

$$\psi(z, x) = \frac{x^{-z}}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma(q + z)\phi(q)x^{-q}\, dq \tag{4}$$

is meromorphic over the whole complex plane in z and continuous in x on the positive real line with countable discontinuities.

Proof. Take the sequence of functions ψ_n for $n \in \mathbb{N}$

$$\psi_n(z, x) = \frac{1}{2\pi i} \int_{-n}^{n} \Gamma(\sigma + iy + z)\phi(\sigma + iy)x^{-\sigma - iy}\, dy$$

And fix z momentarily. By Mellin's inversion theorem [3], if this integral converges for a fixed z as n goes to infinity, it will be continuous in x with countable discontinuities.–We show that it converges now. By the asymptotics of the Gamma function (2). $|\Gamma(\sigma \pm iy)| \leq Ce^{-\sigma - \pi/2|y|}|y|^{\sigma - 1/2}$. Therefore the terms in the integral are bounded by a decaying exponential as n grows. To show this, take $n > N$ such that the function under the integral satisfies this bound then:

$$\int_{N}^{n} |\Gamma(\sigma \pm iy + z)\phi(\sigma \pm iy)|\, dy \leq C \int_{N}^{n} e^{\alpha|y| - \pi/2|y + \Im(z)|}|y + \Im(z)|^{\sigma + \Re(z) - 1/2}\, dy$$

The right side converges as n goes to infinity, which is a simple calculation. It follows because the exponential decays faster than the monomial can grow since $0 \leq \alpha < \pi/2$. We note now, that when the limit is taken at infinity we can shift the line of integration because σ lives in an open set and σ is arbitrary by Cauchy's theorem; this ensures that we are never integrating over a pole of the Gamma function. We conclude that the function is meromorphic because the bounds are uniformly convergent on compact subsets Ω of $\mathbb{C}$. To show this we can find constants M_i, M_s, K on each compact such that when we take $y \in [N, \infty)$ we get the inequality

$$e^{\alpha|y| - \pi/2|y + \Im(z)|}|y + \Im(z)|^{\sigma + \Re(z) - 1/2} \leq e^{\alpha|y| - \pi/2|y + M_i|}|y + M_s|^{K + \sigma - 1/2}$$

This right side no longer depends on z, and in the integral converges just as well as n grows, and so ψ_n converges to ψ uniformly. ψ is therefore at best meromorphic in z since the Gamma function is meromorphic. Continuity in x with countable discontinuities follows again by Mellin's inversion theorem. $\square$

Theorem 1. *If $\phi(q)$ is holomorphic in the strip $a < \Re(q) = \sigma < b$ and $|\phi(q)| < Ce^{\alpha|\Im(q)|}$ for $0 \leq \alpha < \pi/2$ and $C \in \mathbb{R}^+$ then if $\psi(z,x)$ is defined as in (4) with $\Re(z) > -a$. We then get:*

$$\frac{1}{\Gamma(q)} \int_0^\infty \psi(z,x) x^{q-1}\, dx = \phi(q-z)$$

for all $a < \Re(q-z) < b$

Proof. We know, $|\Gamma(q+z)\phi(q)| < Ce^{(\alpha-\pi/2)|y|}|y|^{\sigma+\Re(z)-\frac{1}{2}}$ for all $a < \Re(q) < b$ as $|\Im(q)| \to \infty$ and so we can apply Mellin's inversion theorem on

$$p(z,x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(q+z)\phi(q) x^{-q}\, dq$$

in the strip $a < \Re(q) < b$ because $p(z,x)$ is defined by an absolutely converging integral, however, $\psi(z,x) = x^{-z} p(z,x)$. And so the inversion applies on ψ when $a < \Re(q-z) < b$. We can evaluate this transform and see we get:

$$\frac{1}{\Gamma(q)} \int_0^\infty p(z,x) x^{q-z-1}\, dx = \frac{\Gamma(q+z-z)}{\Gamma(q)} \phi(q-z)$$

This only works when $\Re(z) > -a$ because otherwise the line of integration passes over a pole of the Gamma function and the strip of convergence of the Mellin transform changes. This is enough to show the result. $\square$

The following corollary is very important and represents the statement that the exponential differintegral can be represented with ψ. It follows rather swiftly and shows we have uncovered a new representation of Weyl differintegral.

Corollary 1. *If ψ is defined by (4) then $\psi(z,x) = \frac{d^z}{d(-x)^z}\psi(0,x)$*

Proof. We will fix z again and worry only about the second variable of ψ. Firstly we observe that for $\Re(z) > -a$ and $a < \Re(q-z) < b$ we have $\frac{d^{-q}}{d(-x)^{-q}}\psi(z,x)\Big|_{x=0} = \phi(q-z) = \frac{d^{z-q}}{d(-x)^{z-q}}\psi(0,x)\Big|_{x=0}$ because $\psi(0,x) = f(-e^{i\theta}x)$. We also observe that $\frac{d^{-q}}{d(-x)^{-q}}\psi(z,x+t)\Big|_{x=0} = \psi(z-q,t)$. Which is the statement that the Weyl differintegral is transfer invariant. Therefore for $\Re(z) > -a$:

$$\psi(z,t) = \frac{d^z}{d(-x)^z}\psi(0,x+t)\Big|_{x=0} = \frac{d^z}{d(-t)^z}\psi(0,t)$$

$\square$

With these results we have analytically continued the Weyl differintegrals of certain functions (on any line in the complex plane starting at the origin and going to infinity) to the entire complex plane. We would like to extend these results further so that we have a function that is entire and has complex derivatives of all order and a subset of all complex integrals. In order to do this we add more restrictions on ϕ and find a representation involving Taylor series. We now give a Lemma analytically continuing the differintegral of f for certain functions which allows us to talk about the function and its differintegral much more.

Lemma 3. *If $f(w) = \sum_{n=0}^{\infty} a_n \frac{w^n}{n!}$ is an entire function and $\phi(z) = \frac{d^{-z}}{dw^{-z}}\Big|_{w=0} f(w) = \frac{1}{\Gamma(z)} \int_0^{\infty} f(-w) w^{z-1}\, dw$ converges absolutely for $a < \Re(z) < b$. Then ϕ can be analytically continued to $\Re(z) < b$ by:*

$$\phi(z) = \frac{1}{\Gamma(z)} \Big(\sum_{n=0}^{\infty} a_n \frac{(-1)^n}{n!(z+n)} + \int_1^{\infty} f(-w) w^{z-1}\, dw \Big)$$

Proof. Take $\int_0^{\infty} = \int_0^1 + \int_1^{\infty}$. Then we know that–if $\Re(z) > 0$:

$$\begin{aligned} \int_0^1 f(-w) w^{z-1}\, dw &= \sum_{n=0}^{\infty} a_n \frac{(-1)^n}{n!} \int_0^1 w^{n+z-1}\, dw \\ &= \sum_{n=0}^{\infty} a_n \frac{(-1)^n}{n!(z+n)} \end{aligned}$$

Where these steps are justified because f's Taylor series has uniform convergence on all of $\mathbb{C}$. We observe that this expression is entire when we multiply it by the inverse Γ function. This is because the simple zeroes of $1/\Gamma$ occur when the simple poles of $\int_0^1 f(-w) w^{z-1}\, dw$ occur.

For $\Re(z) < 0$ we note that $-k-1 < \Re(z) < -k$ implies that the first k terms of our Taylor series for f are zero. This implies that $\int_0^1 f(-w) w^{z-1}\, dw = \sum_{n=0}^{\infty} a_n \frac{(-1)^n}{n!(z+n)}$ holds for all $\Re(z) < b$

Observing $\int_1^{\infty}$ we see that, quite apparently for $\epsilon > 0$ small:

$$\int_1^N |f(-w)| w^{\sigma-1}\, dw \le \int_1^N |f(-w)| w^{b-\epsilon-1}\, dw < \infty$$

for any $\sigma < b - \epsilon < b$. Therefore the limit as $N \to \infty$ converges for all $\sigma < b$. The result follows. □

Theorem 2. *If $\phi(q)$ is a holomorphic function satisfying the bounds $|\phi(q)| < Ce^{\alpha|\Im(q)| - \theta\Im(q)} e^{\rho|\Re(q)|}$ for some $\theta \in (-\pi, \pi]$ and $\rho \ge 0$, $0 \le \alpha < \frac{\pi}{2}$ as $\Re(q) < b$. Then if for $0 < a < \sigma < b$ we have $f(-e^{i\theta}x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(q) e^{-i\theta q} \phi(q) x^{-q}\, dq$*

then in $z, w \in \mathbb{C}$ with $\Re(z) > -b$ the following series converges uniformly on all compact subsets.

$$\frac{d^z}{dw^z} f(w) = \sum_{n=0}^{\infty} \phi(-n-z) \frac{w^n}{n!}$$

Proof. In order to show this let us take our new expression for the Weyl differintegral of f:

$$\frac{d^z}{d(-x)^z} f(-e^{i\theta}x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma(q+z) e^{-i\theta q} \phi(q) x^{-q-z} \, dq$$

$|e^{-i\theta q}\phi(q)| < Ce^{\alpha|\Im(z)|+\rho|\Re(z)|}$ and so therefore satisfies the conditions of Theorem 1, and so $\frac{d^z}{d(-x)^z} f(-e^{i\theta}x)$ is a holomorphic function in z for $\Re(z) > -b$. Now let us create the contour C_R, which is a semicircle to the left of the line $[\sigma - iR, \sigma + iR]$ taken in the positive orientation. Denote the arc by A_R. We show that as $R \to \infty$ $\int_{A_R} \Gamma(q+z)e^{-i\theta q}\phi(q)x^{-q-z}\, dq \to 0$. Take $0 < x < e^{-1-\rho}$; then by the asymptotics of $e^{i\theta q}\phi(q)$ and the asymptotics of the Gamma function (2):

$$|\Gamma(q+z)e^{-i\theta q}\phi(q)x^{-q}| < Ce^{(\alpha - \frac{\pi}{2})|\Im(q)| - \xi|\Re(q)|} |\Im(q) + \Im(z)|^{\Re(z)+\Re(q)-1/2}$$

for some $\xi = -\rho - \ln(x) > 0$. These asymptotics hold as $\Re(q) \to -\infty$ by observing Stirling's formula which implies $|\Gamma(q)| \to 0$ uniformly as $\Re(q) \to -\infty$. (We point the reader to [1] and [2] for a statement of Stirling's formula)

$$\begin{aligned}
T &= |\int_{A_R} \Gamma(q+z)e^{-i\theta q}\phi(q)x^{-q}\, dq| \\
&\leq \int_0^{\pi} |\Gamma(\sigma + Rie^{it} + z)e^{-i\theta(\sigma + iRe^{it})}\phi(\sigma + iRe^{it})x^{-\sigma - iRe^{it}}(-Re^{it})|\, dt \\
&\leq CR\int_0^{\pi} e^{(\alpha - \pi/2)|R\cos(t) + \Im(z)| - \xi|R\sin(t) + \Re(z)|}|R\cos(t) + \Im(z)|^{\Re(z)+\sigma - \sin(t)|R| - 1/2}\, dt \\
&\leq (\pi K_z R)e^{(\alpha - \pi/2)|Rcos(\gamma)| - \xi|R\sin(\gamma)|}|Rcos(\gamma) + \Im(z)|^{\Re(z)+\sigma - \sin(\gamma)|R| - 1/2}
\end{aligned}$$

Where $0 \leq \gamma \leq \pi$ is the point where the function in the integral achieves its supremum and K_z is some constant depending on z. As R grows, despite the value of γ or z, these bounds approach zero. Furthermore as the semicircle grows it encapsulates more of the poles of the Gamma function. From our expression of the Gamma function (1) $\Gamma(q) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(q+n)} + \int_1^{\infty} e^{-t}t^{q-1}\, dt$. Where the rightmost term is entire in q. The residues of our integral expression therefore come out to, using Cauchy's Theorem ($\frac{1}{2\pi i}\int_{\partial\Delta, z\in\Delta} \frac{g(\zeta)}{\zeta - z}\, d\zeta = g(z)$):

$$\frac{1}{2\pi i}\int_{C_R}\Gamma(q+z)e^{-i\theta q}\phi(q)x^{-q-z}\,dq = \sum_{n=0}^{\lfloor R/2-\sigma\rfloor} e^{i\theta(n+z)}\phi(-n-z)\frac{(-x)^n}{n!}$$

Where $\lfloor R/2-\sigma\rfloor$ is the floor function. Letting R tend to infinity we find this series converges, since along any line parallel to the real axis ϕ is bounded by an exponential. Therefore we obtain:

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(q+z)e^{-i\theta q}\phi(q)x^{-q-z}\,dq = \sum_{n=0}^{\infty}\phi(-n-z)e^{i\theta(q+n)}\frac{(-x)^n}{n!}$$

However this series on the right converges for all $x \in \mathbb{C}$ and for all $\Re(z) > -b$. Therefore:

$$\frac{d^z}{d(-w)^z}f(-e^{i\theta}w) = \sum_{n=0}^{\infty}\phi(-z-n)e^{i\theta(z+n)}\frac{(-w)^n}{n!}$$

Therefore, since from (3) we have $\frac{d^z}{dw^z}f(w) = \xi^z\frac{d^z}{d(\xi w)^z}f(w)$. Our f satisfies the conditions of Lemma 1 so that we get the final result:

$$\frac{d^z}{dw^z}f(w) = \sum_{n=0}^{\infty}\phi(-n-z)\frac{w^n}{n!}$$

□

2.1 Examples

We give a few small examples of differintegrals of certain functions.

1. If we take the function $f(w) = e^{\lambda w}$ and we take $\theta = \arg(\lambda)$ then $\frac{e^{-i\theta q}}{\Gamma(z)}\int_0^\infty f(-e^{-i\theta}w)w^{z-1}\,dw = \lambda^{-z}$. This shows that $\frac{d^z}{dw^z}e^{\lambda w} = \lambda^z e^{\lambda w} = \sum_{n=0}^{\infty}\frac{\lambda^z+nw^n}{n!}$
2. Let us take the function $f(w) = \frac{e^w-1}{w}$. We know that in the strip $0 < \Re(z) < 1$ we get $\frac{1}{\Gamma(z)}\int_0^\infty f(-w)w^{z-1}\,dw = \frac{1}{1-z}$. This implies for $\Re(z) > -1$ we get $\frac{d^z}{dw^z}\frac{e^w-1}{w} = \sum_{n=0}^{\infty}\frac{w^n}{n!(z+n+1)}$
3. If we take some polynomial $p(w) = \sum_{n=0}^{N}a_n w^n$ then $\frac{1}{\Gamma(z)}\int_0^\infty p(-w)e^{-w}w^{z-1}\,dw = \sum_{n=0}^{N}a_n(-1)^n z(z+1)(z+2)\cdots(z+n-1)$ for all $\Re(z) > 0$. Therefore $\forall z \in \mathbb{C}$: $\frac{d^z}{dw^z}p(w)e^w = \sum_{k=0}^{\infty}\left(\sum_{n=0}^{N}a_n(z+k)(z+k-1)\cdots(z+k-n+1)\right)\frac{w^k}{k!}$

3 Sequences of differintegrals

Theorem 3. *If $\{\phi_n\}_{n=0}^{\infty}$ is a holomorphic sequence of functions that converge uniformly to ϕ on all compact subsets of $\mathbb{H}$, where $\mathbb{H} = \{z \in \mathbb{C}; \Re(z) > -b\}$. Which on top of $|\phi_n(q)| \leq Ce^{\rho|\Re(q)|+\alpha|\Im(q)|}$ and $|\phi(q)| \leq Ce^{\rho|\Re(q)|+\alpha|\Im(q)|}$ for some $\rho \geq 0$ and $0 \leq \alpha < \frac{\pi}{2}$, then the dual holomorphic function in z and w $\frac{d^z}{dw^z} f_n(w)$ such that $\phi_n(-z) = \frac{d^z}{dw^z}\Big|_{w=0} f_n(w)$ converges uniformly to $\frac{d^z}{dw^z} f$ in w on compact subsets of $\mathbb{C}$ and in z on compact subsets of $\mathbb{H}$*

Proof. Firstly let us take the compact set K where w lives. Take the sequence of partial sums

$$\left(\frac{d^z}{dw^z}\right)_m f_n(w) = \sum_{k=0}^{m} \phi_n(-z-k)\frac{w^k}{k!}$$

Then take $n > N$ such that $|\phi(-z) - \phi_n(-z)| < \epsilon$ for all $z \in \Omega_m$ Where $\Omega_m \subset \Omega_{m+1}$ and $z, z+1, z+2, ..., z+m \in \Omega_m$, Then, if $|w| < R$ for all $w \in K$:

$$|\left(\frac{d^z}{dw^z}\right)_m f - \left(\frac{d^z}{dw^z}\right)_m f_n| \leq \sum_{k=0}^{m} |\phi(-z-k) - \phi_n(-z-k)|\frac{R^k}{k!} < \epsilon \sum_{k=0}^{m} \frac{R^k}{k!}$$

Then, by the bounds on ϕ, we know that $\left(\frac{d^z}{dw^z}\right)_m f(w) \to \frac{d^z}{dw^z} f(w)$ uniformly. Take $m > M$ such that $|\frac{d^z}{dw^z} f(w) - \left(\frac{d^z}{dw^z}\right)_m f(w)| < \epsilon$. Furthermore by the bounds on ϕ_n we can take m big enough so that: $|\frac{d^z}{dw^z} f_n - \left(\frac{d^z}{dw^z}\right)_m f_n| < \epsilon$ Then by the triangle inequality

$$\begin{aligned}
|\frac{d^z}{dw^z} f(w) - \frac{d^z}{dw^z} f_n(w)| &< |\frac{d^z}{dw^z} f(w) - \left(\frac{d^z}{dw^z}\right)_m f(w)| + |\left(\frac{d^z}{dw^z}\right)_m f(w) - \left(\frac{d^z}{dw^z}\right)_m f_n(w)| \\
&\quad + |\frac{d^z}{dw^z} f_n(w) - \left(\frac{d^z}{dw^z}\right)_m f_n(w)| \\
&< \epsilon(\sum_{k=0}^{m} \frac{R^k}{k!} + 2) < \epsilon(e^R + 2)
\end{aligned}$$

Since ϵ is arbitrary we see uniform convergence in z and w, with this the result follows. □

Corollary 2. *If $g(q, w)$ is holomorphic and entire in w and and holomorphic in q on $[a, b] \subset \Omega \subset \mathbb{C}$, and $|\frac{d^{-z}}{dw^{-z}}\Big|_{w=0} g(q, w)| < Ce^{\alpha|\Im(z)|+\rho|\Re(z)|}$ and $\frac{d^z}{dw^z}\Big|_{w=0} g(q, w)$ is holomorphic on $\Re(z) > -b$ then:*

$$\frac{d^z}{dw^z} \int_a^b g(q, w)\, dq = \int_a^b \frac{d^z}{dw^z} g(q, w)\, dq$$

Proof. The expression

$$I_n(w) = \sum_{j=0}^{n} g(a + j\frac{b-a}{n}, w)\frac{b-a}{n}$$

converges uniformly to $\int_a^b g(q,w)\,dq$

Take

$$\phi_n = \frac{d^z}{dw^z} I_n(w) = \sum_{j=0}^{n} \frac{d^z}{dw^z} g(a + j\frac{b-a}{n}, w)\frac{b-a}{n}$$

Setting $w = 0$ we see $\phi_n \to \phi$ uniformly, as it converges to $\int_a^b \frac{d^z}{dw^z} g(q,0)dq$. $|\phi|, |\phi_n| < Ce^{\alpha|\Im(z)|+\rho|\Re(z)|}$. Therefore the previous theorem applies and the result follows. □

Corollary 3. *If* $\Phi_n = \sum_{k=0}^{n} \phi_k$ *converges to* $\Phi = \sum_{k=0}^{\infty} \phi_k$ *uniformly and* $|\Phi_n(z)|, |\Phi(z)| < Ce^{\alpha|\Im(z)|+\rho|\Re(z)}$ *for* $0 \leq \alpha < \pi/2$ *and* $\rho \geq 0$ *and* $C]in\mathbb{R}^+$*. Then if* $\frac{d^z}{dw^z} F_n(w)\Big|_{w=0} = \Phi_n$*, then* $\frac{d^z}{dw^z} F_n(w) \to \frac{d^z}{dw^z} F(w)$ *uniformly on compacts in* w *for each* z *with* $\Re(z) > -b$*, where* $\Phi = \frac{d^z}{dw^z} F(w)\Big|_{w=0}$ *And furthermore if* $\frac{d^z}{dw^z} f_k(w)\Big|_{w=0} = \phi_k$ *then*

$$\frac{d^z}{dw^z} F(w) = \sum_{k=0}^{\infty} \frac{d^z}{dw^z} f_k(w)$$

Proof. This proof is an immediate consequence of uniform convergence. The author leaves it as a simple calculation. □

3.1 Examples

1. As a sequence of sums, $f(w) = e^{e^w} - 1 = \sum_{n=1}^{\infty} \frac{e^{nw}}{n!}$ so that $\frac{1}{\Gamma(z)} \int_0^{\infty} f(-w) w^{z-1}\,dw = \sum_{n=1}^{\infty} \frac{n^{-z}}{n!}$. This is clearly a uniformly convergent sum and we leave the boundedness condition as a calculation so it is not difficult to see that $\frac{d^z}{dw^z} e^{e^w} - 1 = \sum_{k=0}^{\infty} \frac{w^k}{k!} \sum_{n=0}^{\infty} \frac{n^{k+z}}{n!}$

2. As a second case, consider a function defined by a fourier transform $f(\xi) = \int_{-\infty}^{\infty} g(\zeta) e^{-2\pi\xi\zeta i}\,d\zeta$. Then if $\int_{-\infty}^{\infty} |g(\zeta)||\zeta|^{\Re(z)}\,d\zeta < Ce^{\alpha|\Im(z)|+\rho|\Re(z)|}$ for $\Re(z) > -b$ then:

$$\frac{d^\xi}{dz^\xi} f(\xi) = (-2\pi i)^z \int_{-\infty}^{\infty} g(\zeta)\zeta^z e^{-2\pi\zeta\xi i}\,d\zeta$$

4 Applications in solving for analytic functions that satisfy recursive relationships.

4.1 A result on superfunctions

A superfunction is a nonstandard, fairly new concept to arise in modern mathematics and the word itself has a very recent history. It is a rather intuitive concept to imagine, however it possesses quite a difficulty in being handled. Typically the concept arises in logic, or recursion and is only beginning to arise in complex analysis. We point the reader to the following literature if they wish to further research on the matter [5]. There are a various number of papers here by Dmitrii Kouznetsov all on various subjects involving superfunctions. We give a definition of what a superfunction is and jump straight into the result. We note superfunctions arise more often than not as a pure mathematical vein and therefore their use is usually for math alone. However there have been known to be applications in physics (the growing mass of a rolling snowball) but because of the general difficulty of finding the superfunction of a function f there has been a limited amount of work been put into the field. Even finding the super function of a simple function like $p(x) = x^3 - 1$ proves to be quite intricate. Though the author does not squander the work that already exists–it is simply apparent that the concept exists in a niche.

Definition 4. *If ψ is some holomorphic function, such that for open and connected K, Ω, $\psi : \Omega \to K$, then if the holomorphic function $\Psi : K \to \Omega$ satisfies the relationship $\psi(\Psi(z)) = \Psi(z+1)$ for $z, z+1 \in K$ then Ψ is called a superfunction of ψ.*

Some noteworthy examples of superfunctions are the functions $\psi(z) = ez$ then $\Psi(z) = e^z$. And also $\psi(z) = z + e$ then $\Psi(z) = ez$. Schroder also showed that if $\psi(z) = 2z^2 - 1$ then $\Psi(z) = \cos(2^z \xi)$ where ξ is fixed. Many different superfunctions exist for a single function. We can note that if $p(z)$ is one periodic and $p(0) = 0$ then $\Psi(z+p(z))$ is also a superfunction of ψ. Therefore uniqueness is very often an issue needed to be dealt with when talking about superfunctions. We will be a bit casual in our treatment of our uniqueness conditions but we note that our definition ensures a unique solution we simply avoid proof here. The method by which superfunctions are solved typically involves complicated iteration techniques from advanced complex dynamics. This is not the case for the method we are about to introduce. It is very surprising to see superfunctions arise so naturally and simply using Weyl differintegral calculus.

The reader will take care to note the generality of the methods used and the fact that this technique can be modified to work on different functions as well. We only iterate basic functions because the result follows almost wholly from Theorem 2 and it still displays the proof method very clearly. We make a note on notation: $\psi^0(c) = c$ $\psi^1(c) = \psi(c)$ $\psi^2(c) = \psi(\psi(c))$ $\psi^3(c) = \psi(\psi(\psi(c)))$ ..... So that we are talking about iterates of a function. It is very common to write $\psi^z(c) = \Psi(z)$ for some $c \in \mathbb{C}$. This shall be the notation we adopt.

Theorem 4. *If $\psi^n(\xi) : \mathbb{C} \to \mathbb{C}$ is entire in ξ for all n, and if $\vartheta(w) = \sum_{n=0}^{\infty} \psi^n(\xi) \frac{w^n}{n!}$ is entire in w and $\int_0^\infty |\vartheta(-w)| w^{\sigma-1} \, dw < \infty$ for $a < \sigma < b$ then if $\Re(z) > -b$*

$$\Psi(z) = \frac{1}{\Gamma(-z)} \Big[\sum_{n=0}^{\infty} \psi^n(\xi) \frac{(-1)^n}{n!(n-z)} + \int_1^\infty \vartheta(-w) w^{-z-1} \, dw \Big] \tag{5}$$

and if $|\Psi(z)| < Ce^{\alpha|\Im(z)+\rho|\Re(z)|}$ and if $|\psi(\Psi(z))| < Ce^{\alpha|\Im(z)|+\rho|\Re(z)|}$ for some $0 \leq \alpha < \pi/2$ and $\rho \geq 0$ then we get $\psi(\Psi(z)) = \Psi(z+1)$. So that Ψ is a super function of ψ

Proof. It is not difficult to see that, for $a < \sigma < b$ by Theorem 2 that:

$$\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma(z) \Psi(1-z) w^{-z} \, dz = \vartheta'(-w)$$

And very well from Theorem 2, knowing that $\Psi(-n) = \psi^n(\xi)$:

$$\begin{aligned} \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma(z) \psi(\Psi(-z)) w^{-z} \, dz &= \sum_{n=0}^{\infty} \psi(\Psi(-n)) \frac{(-w)^n}{n!} \\ &= \sum_{n=0}^{\infty} \psi^{n+1}(\xi) \frac{(-w)^n}{n!} \\ &= \vartheta'(-w) \end{aligned}$$

Now the inverse Mellin transform is one to one–since $\vartheta'(e^w)$ is represented by a Fourier transform and the Fourier transform is one to one. This shows that the two functions must be equal. This shows the result.

□

4.2 Examples

1. Take $\psi(z) = dz$ and $c = 1$ then $\psi^n(c) = d^n$ and $f(-w) = \sum_{n=0}^{\infty} d^n \frac{(-w)^n}{n!} = e^{-dw}$ Therefore:

   $$\frac{1}{\Gamma(z)} \int_0^\infty e^{-dw} w^{z-1} \, dw = d^{-z}$$

   And quite so $\Psi(z) = d^z = \psi^z(1)$

2. As another simple example take $\psi(z) = z + d$ and $c = 0$. Then $\psi^n(0) = nd$ and $f(-w) = d \sum_{n=0}^{\infty} n \frac{(-w)^n}{n!} = (-dw)e^{-w}$. And so

   $$\Psi(-z) = \frac{d}{\Gamma(z)} \int_0^\infty (-w) e^{-w} w^{z-1} \, dw = -dz$$

   So that we get the result we wanted.

3. We can also solve for $\psi(\xi) = d\xi + r$. It is clear that $\psi^n(\xi) = d^n\xi + r\frac{1-d^n}{1-d}$ Therefore: $\sum_{n=0}^{\infty} \psi^n(\xi)\frac{(-w)^n}{n!} = e^{-dw}(\xi - \frac{r}{1-d}) + \frac{r}{1-d}e^{-w}$ Giving:

$$\psi^z(c) = d^z(\xi - \frac{r}{1-d}) + \frac{r}{1-d}$$

We will expand more on these results in the proceeding paper and examine more complicated functions iterated. We will also show the different ways we can nest coefficients so that we can manipulate these functions more cleverly.

4.3 Analytic recurrence relations

In this section we bring forth a theorem on analytically continuing a sequence satisfying a recursive relationship to a holomorphic function that satisfies the same recursive relationship. We see that this result again plays in the field of pure mathematics and is a mathematical interest more than an applied tool. It shows a strong connection between recursion and the Weyl differintegral and shows a very quick way of analytically continuing recursive sequences.

Theorem 5. *If we have some sequence* $\{a_n\}_{n=0}^{\infty}$ *such that it satisfies a recurrence relation:* $a_{n+k+1} = F(a_n, ..., a_{n+k}, -n)$*. Where* $F(z_1, z_2, ..., z_k, \zeta)$ *is entire in* $z_1, z_2, ..., z_k, \zeta$*. Furthermore, if:* $f(w) = \sum_{k=0}^{\infty} a_n \frac{w^n}{n!}$ *is entire and* $\int_0^{\infty} |f(-w)|w^{\sigma-1}\,dw < \infty$ *for* $a < \sigma < b$ *and* $\phi(z) = \frac{d^{-z}}{dw^{-z}}\Big|_{w=0} f(w)$ *is holomorphic for* $\Re(z) < b$ *and satisfies* $|\phi(z)| < Ce^{\alpha|\Im(z)|+\rho|\Re(z)|}$*; as well* $F(\phi(z-k), \phi(z-k+1), ..., \phi(z), z)$ *is holomorphic on* $\Re(z) < b$ *and satisfies* $|F(\phi(z-k), \phi(z-k+1), ..., \phi(z), z)| < Ce^{\alpha|\Im(z)|+\rho|\Re(z)|}$ *for some* $0 \leq \alpha < \pi/2$ *and* $\rho \geq 0$ *then:*

$$\phi(z-k-1) = F(\phi(z), \phi(z-1), ..., \phi(z-k), z)$$

Proof. The proof of this result follows very similarly to how it followed for Theorem 4 on superfunctions. Plug in the integral expression and we know that:

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(z)\phi(z-k-1)w^{-z}\,dz = f^{(k+1)}(-w)$$

And we know that $\phi(-n) = a_n$ and that, by Theorem 2:

$$\begin{aligned}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(z)F(\phi(z), ..., \phi(z-k), z)w^{-z}\,dw &= \sum_{n=0}^{\infty} F(a_n, a_{n+1}, ..., a_{n+k}, -n)\frac{(-w)^n}{n!} \\ &= \sum_{n=0}^{\infty} a_{n+k+1}\frac{(-w)^n}{n!} \\ &= f^{(k+1)}(-w)\end{aligned}$$

Therefore since the inverse Mellin transform is one to one,

$$\phi(z-k-1) = F(\phi(z-k), \phi(z-k+1), ..., \phi(z), z)$$

□

4.4 Examples

1. Our first example will be recovering the Fibonacci sequence. If $a_0 = 1$ and $a_1 = 1$ and $a_n = a_{n-1} + a_{n-2} = \frac{\phi^n - \psi^n}{\sqrt{5}}$ where $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$. Therefore:

$$f(-w) = \sum_{n=0}^{\infty} a_n \frac{(-w)^n}{n!} = \frac{1}{\sqrt{5}}\left(e^{-\phi w} - e^{-\psi w}\right)$$

   Therefore as we suspect:

$$\frac{d^z}{dw^z}\Big|_{w=0} f(w) = \frac{\phi^z - \psi^z}{\sqrt{5}}$$

   Which is Binet's formula for the analytic continuation of the Fibonacci sequence. This method will apply generally on linear recurrence relations, since their solutions are exponentials. The reader can see the more general result by observing [9].

2. Our second example will be $a_0 = 0$ and $a_n = a_{n-1} + n = \frac{n(n-1)}{2}$. Quite clearly: $f(-w) = \frac{w^2 e^{-w}}{2}$ so that

$$\frac{d^z}{dw^z}\Big|_{w=0} f(w) = \frac{z(z-1)}{2}$$

3. As a result we can pull from a hat that is more complicated. We can take $\frac{d^{-z}}{dw^{-z}}\Big|_{w=0} f(w) = \frac{1}{\sqrt{\Gamma(1-z)}}$ which satisfies the conditions of Theorem 2. So that:

$$f(w) = \sum_{n=0}^{\infty} \frac{w^n}{n!^{3/2}}$$

   is Weyl differintegrable on all of $\mathbb{C}$. And it satisfies:

$$\frac{1}{\sqrt{z+1}} \frac{d^z}{dw^z} f(w)\Big|_{w=0} = \frac{d^{z+1}}{dw^{z+1}} f(w)\Big|_{w=0}$$

   Which is the same recursion that its Taylor coefficients satisfy $\frac{1}{\sqrt{n+1}} a_n = a_{n+1}$. We note we use the principal branch of the square root function.

5 Final remarks

As a final remark we note that we have investigated and shown a useful form of the Weyl differintegral in terms of a contour integral with the Γ function in the kernel. We have only scratched the surface of how this operator shall behave on more intricate functions. Introducing more complicated theorems we are able to analytically continue many recurrence relations, including: superfunctions, continuum sums and continuum products. All of which require a strong intimacy with the Weyl differintegral. We hope to express more in the next paper now that we have laid the first brick of our foundation in the subject of Weyl calculus. We hope the reader has found some tool in the paper and we appreciate the reader for reading it and hope again that it sparked some interest. We close in anticipation.

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