The third super-root

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1 Introduction

The nth tetrate (also known as tetration or iterated exponentials) is a unary function of x defined by

$$y = {}^{n}x = \exp_x^n(1) = \underbrace{x^{x^{\cdot^{\cdot^{x}}}}}_{n} = \sum_{k=0}^{\infty} p_{nk} \log(x)^k \tag{1}$$

which has formal power series coefficients described by

$$p_{nk} = \begin{cases} 1 & \text{if } n = 0 \text{ and } k = 0 \\ 0 & \text{if } n = 0 \text{ and } k > 0 \\ 1 & \text{if } n > 0 \text{ and } k = 0 \\ \frac{1}{k}\sum_{j=1}^{k} j p_{n(k-j)} p_{(n-1)(j-1)} & \text{otherwise} \end{cases} \tag{2}$$

We know that $p_{1k} = 1/k!$ because this corresponds to the power series for $\exp(x)$; the inverse of $\log(x)$. We also know

$$p_{2k} = \frac{1}{k!}\sum_{j=1}^{k} \binom{k}{j} j^{(k-j)} \tag{3}$$

because this corresponds to $y = x^x$. We also know

$$p_{\infty k} = \frac{1}{(k+1)!}\sum_{j=0}^{k} \binom{k}{j} k^{(k-j)} = \frac{1}{k!}(k+1)^{(k-1)} \tag{4}$$

because this corresponds to $y = {}^{\infty}x$; the inverse of $y^{1/y} = x$. It also describes the triangle $p_{nk} = p_{\infty k}$ for all $k \le n$. Combining these, we get the recursive form

$$p_{3k} = \begin{cases} p_{\infty k} & \text{if } k \le 3 \\ \frac{1}{k}\sum_{j=1}^{k} j p_{3(k-j)} p_{2(j-1)} & \text{otherwise} \end{cases} \tag{5}$$

which expands to

$$p_{3k} = \begin{cases} \frac{1}{k!}(k+1)^{(k-1)} & \text{if } k \le 3 \\ \frac{1}{k}\sum_{j=1}^{k} \frac{j p_{3(k-j)}}{(j-1)!} \sum_{i=1}^{j-1} \binom{j-1}{i} i^{(j-1-i)} & \text{otherwise} \end{cases} \tag{6}$$

which is sort of a closed-ish recursive form.

2 Super-root

The nth super-root is a unary function of y and is defined as a solution $x = \sqrt[n]{y}_s$ to the equation $y = {}^n x$.

... and that's all we know ...

3 Conclusion