Incomplete towers and tetrational representation of numbers

In the thread submitted by KAR and myself to the NKS Forum (News and Announcements), see in the Annex, section 8, "*The Tower Extension*"(4-12-2005):

http://forum.wolframscience.com/showthread.php?s=e6b02c07c27408884ed7d282fb4693c5&threadid=956.

we observed that any real number > 0 can be represented as:

(1) $$y = b \wedge (b \wedge (b \wedge (b \wedge ...(b \wedge x)...))) = b^{b^{\cdot^{\cdot^{x}}}}$$

At that moment, also Rubtsov and myself, we thought to introduce an alternative definition of tower, by using an operation that we called "anti-power" and given by:

(2) $$b^x = {}_b x$$

It seemed to us a simple change of convention, giving more evidence to the exponent, instead of to the base. The advantage was that it allowed us to put the same evidence in the last "floor" of the towers, as the most important "ground floor", as follows:

(3) $$y = b \wedge (b \wedge (b \wedge (b \wedge ...(b \wedge x)...))) = {}_{b^{b^{\cdot^{\cdot}}}} x$$

This was implementing a kind of "push-down" representation of tetration, with a clear evidence put on the "modifier" variable *x*. This representation had clear advantages, but also the disadvantage to be another new logotype, requiring new rules and new efforts of assimilation, in this already complicated domain. So, we decided to adopt another convention, inspired by the study [1] of Clenshow, Olver and Turner, concerning the representation of large numbers (SLI, the Symmetric Level Index number notation).

In fact, still keeping the "classical" definition of what we called an inhomogeneous or *incomplete tower* as (1), we observed that it was always possible to find the "last floor" *x*, by systematically applying the $\log_b$ operator, as follows:

(4) $$x = \log_b^n y$$ *n*: number of iterations of the *b*^ operations in (1)

Then, we thought of the possibility to re-write (1) as follows:

(5) $$y = x * {}^n b$$

where operator '*', a new "asterisk" operator, would indicate the inhomogeneous "extension" of the homogeneous tower ${}^n b$ by means of exponent *x*. We had some doubts concerning the possibility of choosing between the following two expressions:

$$y = x * {}^n b \qquad \text{or} \qquad y = {}^n b * x$$

keeping always in mind that, if we stipulate that the asterisk operator is strictly linked to the base *b* of the $\log_b$ operator, the two expressions are equivalent. Finally, we preferred notation (5). In this respect, it was interesting to observe that it is always possible to write expression (1) by extending the number of log iterations up to the stage when $1 < x < b$, by observing that:

(6) $$1 * {}^n b = {}^n b \qquad \text{and} \qquad b * {}^n b = {}^{n+1} b$$

In the situations where $1 < x < b$, we shall say that (5) is in the *standard canonical format* and we shall indicate by *p* the "canonical" power (and tower) extension. We shall than have:

(7) $$y = p * {}^n b$$ b : hyperbase, n : hyperexponent, p : tower extension

with: $1 < p < b$

Keeping in mind (6), in the intermediate cases ($p \neq 1$ or $p \neq b$), we supposed that we could put:

(8) $$\boxed{y = p * {}^{n}b = {}^{n+q}b}$$

$1 < p < b$, and $0 < q < 1$

p: tower extension, q: super-mantissa.

Now, the main problem is to find the relation between p and q. In fact, we may observe that, by applying the n-iterated $\log_b^n$ to (8), recalling the meaning of (4), we have:

(9) $${}^{q}b = \log_b^n\left(p * {}^{n}b\right) = p$$

Therefore, in conclusion, the relationship between the tower extension and the super-mantissa is:

(10) $$\boxed{q = \text{slog}_b\, p} \qquad \text{super-mantissa } 0 < q < 1$$

Expression (10), in case of "natural" base $b = e$, can be calculated by using the best available approximation of the natural slog function. Therefore, for instance, we could write:

(11) $$y = p * {}^{n}e = {}^{n+q}e$$

$$1 < p < e \qquad \rightarrow \qquad q = \text{slog}_e\, p$$

As an example, we may take the following large number, written both in the scientific notation and by using the above-mentioned conventions:

$$y = 1.5 \times 10^{500000} = 2.635937 * {}^{3}e$$

The precision of the conversion is given by the precision of the "slog" calculations. Nevertheless any loss of precision is compensated by the possibility of defining a tetrational order of magnitude, instead of a general indication of "computing overflow".

In the mentioned NKS Forum thread, the authors (KAR and GFR) described a general hyper-format for number notation based on three components (p, b, k), interlinked by two operators (generally indicated as $\oplus, \otimes$) for representing and noting a real number x, as follows:

(12) $$x = p \oplus (b \otimes k)$$

The three numerical components are: p (the significant figures, or the significance), b (the base of the notation), k (the order of magnitude). This scheme, inspired by the hyper-operation hierarchy (see NKS Forum http://www.rotarysaluzzo.it/filePDF/Iperoperazioni%20(1).pdf), can be implemented at three hyper-operation ranks, as follows:

(13)
$$y = p + (b \times n) = b \times \left(\frac{p}{b} + n\right) = b\left(\frac{p}{b} + n\right) \qquad 0 < p < b$$

$$y = p \times (b \wedge n) = b \wedge \left(\log_b p + n\right) = b^{(\log_b p + n)} \qquad 1 < p < b$$

$$y = p * (b \# n) = b \# \left(\text{slog}_b\, p + n\right) = {}^{(\text{slog}_b p + n)}b \qquad 1 < p < b$$

The first line of (13) describes the representation of a number y as a "p-modulo-b" congruence (clock arithmetic). The second line is a version of the scientific "floating point" notation. The third line is the newly proposed and extremely compact tetrational notation (11).

G. F. Romerio – 4th November 2007

[1] - Clenshaw, C. W.; Olver, F. W. J.; Turner, P. R. - *Level-index arithmetic, An introductory survey*, Numerical Analysis and Parallel Processing (P. R. Turner, ed.), Lecture Notes in Math., vol. 1397, Springer-Verlag, New York, 1989.