Generalized superfunction trick

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Teaser

What is the general argument behind the "superfunction trick"?

It seems to me that it is possible to study the general machinery lurking behind this superfunction trick. As JmsNxn correctly notice, he is using a more complex "mutation" of the mentioned trick, but if I can make some progress understanding the old trick, I guess I know how to easily expand the construction category-theoretically to include his iterated composition. Let's begin by setting some notation and by stating a "fake theorem" that illustrates my sentiment about this matter better than words.

Disclaimer on notation. From now on I will call "a function" only a binary relation that is everywhere defined and single-valued, i.e. for every element in the domain exists a unique element in the codomain and so on...; I'll denote functional composition by juxtaposition $gf = g \circ f$ and integer iteration by f^n. With s i vaguely mean the successor endomap and with mul_a the endomap that scales by $x \mapsto a \cdot x$.

For two given endofunctions $f : X \to X$ and $g : Y \to Y$ define the set $[f, g] \subseteq Y^X$ as the solution set of the equation

$$\chi f = g\chi$$

we define two subsets of sequences $\phi_n \in (Y^X)^{\mathbb{N}}$ as follows

$$[f,g]_\Delta := \{\phi_n : \phi_{n+1} f = g\phi_n\} \quad \text{and} \quad [f,g]^{op}_\Delta := \{\phi_n : \phi_n f = g\phi_{n+1}\}$$

Ok! We are now ready for the...

SuperLazy Prototheorem. *Given functions $f : X \to X$ and $g : Y \to Y$ and a sequence of functions $\phi_{-} : \mathbb{N} \to Y^X$ if the conditions*

1. *for every natural number, $\phi_n f = g\phi_{n+1}$ or $\phi_{n+1} f = g\phi_n$;*

2. *ϕ_0 is "appropriate";*

are met then the "limit" of the sequence $\phi_n \to \phi$ exists and lands in the subset $[f, g] \subseteq Y^X$, i.e.

$$\phi f = g\phi$$

This "fake theorem" depends fundamentally on the existence and on our ability to build sequences of maps satisfying (1). Luckily this is not a problem! Such kinds of sequences exist, are definable by recursion and are abundant: in fact we can prove these

Easy Lemmas. *Given functions $f : X \to X$ and $g : Y \to Y$. For every function $\phi : X \to Y$ we prove that:*

1. *f is split-mono $\Rightarrow$ exists a sequence α_n s.t. $\alpha_0 = \phi$ and $\alpha_{n+1} f = g\alpha_n$;*
   f is split-epi $\Rightarrow$ given $\alpha_n, \alpha'_n \in [f, g]_\Delta$ if $\alpha_0 = \alpha'_0$ then $\alpha_n = \alpha'_n$;
   if f is iso $Y^X \simeq [f, g]_\Delta$

2. *g is split-epi $\Rightarrow$ exists a sequence β_n s.t. $\beta_0 = \phi$ and $\beta_n f = g\beta_{n+1}$;*
   g is split-mono $\Rightarrow$ given $\alpha_n, \alpha'_n \in [f, g]^{op}_\Delta$ if $\alpha_0 = \alpha'_0$ then $\alpha_n = \alpha'_n$;
   if g is iso $Y^X \simeq [f, g]^{op}_\Delta$

3. *if $\phi \in [f, g]$ the constant sequence $\phi!$ is in both $[f, g]_\Delta$ and $[f, g]^{op}_\Delta$;*

4. *for every $\gamma \in [f, f]_\Delta$ and $\delta \in [g, g]_\Delta$, if $\phi_n \in [f, g]_\Delta$ then $\delta\phi\gamma \in [f, g]_\Delta$;*

Where split-mono (split-epi) means that the function has a retract (a section): in the context of arbitrary set functions this is equivalent to being injective (surjective but requires axiom of choice).

For the proof of the former I'll have to work by a sequence of, hopefully finite and convergent, approximated attempts, but the latter result is pretty trivial to prove and generalize because it's pure algebra (I'll soon add the proof as an appendix in the pdf).

Some context

But let's start from scratches and provide some context. I'm sure I'm not providing the oldest references on this site but in

[TF] 2008 jul, Trappmann, Robbins - Tetration Reference

the principal Abel and Schroeder functions are defined as follow

$$\mathcal{A}[f] = \lim_{n\to\infty} (s^{-1})^n \circ \log_a \circ f^n$$

$$\mathcal{S}[f] = \lim_{n\to\infty} (\mathrm{mul}_a^{-1})^n \circ f^n$$

for $a = f'(0)$ s is the successor and mul_a is multiplication by a. While in the thread

[KSuLog] 2008 nov, bo198214 - Kneser's Super Logarithm:

bo198214 writes that one way to construct the principal Schroeder function is given by the limit

$$\chi = \lim_{n\to\infty} (\mathrm{mul}_c^{-1})^n \circ (s^{-1})^c \circ f^n$$

where c is a fixed point, which Kneser (1949) composes with an Abel function of multiplication, log, to obtain an Abel function and solve the half-iterate of exp.

Few posts later Sheldonison defines an Abel function of exp (slog developed for the fixed point c) with this limit

$$\psi^{-1} = \lim_{n\to\infty} \exp^n \circ s^c \circ \exp_c \circ (s^{-1})^n$$

$$\psi = \lim_{n\to\infty} s^n \circ \log_c \circ (s^{-1})^c \circ \ln^n$$

Question!What is the general pattern here?! Let's ignore now the convergence issue of the limit for a moment, and I'll try later to, at least, blackbox it and return to it when the underlying algebraic argument is clear to me.

The general scheme here seems to comprise the following:

1. They select a pair of functions $f : X \to X$ and $g : Y \to Y$ with some properties, i.e. continuous, analytic, linear, or no property at all, e.g. (f, s) in the case of principal Abel; (f, mul_a) in the case of Principal

Schroeder; $(s, \exp)$ in the case of tetration; $(\exp, s)$ in the case of super-logarithm;

2. they go on drawing from their magician's hat a function $\phi : X \to Y$, a kind of first approximation chosen so as to obtain some desired properties related to the fixed points, to the behavior or to the very success of the limiting construction, e.g. $\log_{f'(0)}$ in the principal Abel function; the identity in the principal Schroeder function defined in [TF]; subtraction by the fixed-point s^{-c} in [KSuLog] and, if I'm not mistaken $2\sinh$ in the Tommy-method;

3. in all the cases shown, by inverting f or g, they define recursively from a "broken conjugation" a sequence of functions ϕ_n with base of the recursion $\phi_0 = \phi$ such that $\phi_n \in [f,g]_\Delta$. In other words, if the definition of $[f,g]_\Delta$ wasn't very attractive, this means that the sequence ϕ_n behaves imperfectly almost like a solution of $\chi f = g\chi$;

4. they take the limit of the sequence ϕ_n and get the desired function.

$$\lim_{n\to\infty} \phi_n = \chi$$

   Well, probably for every $x \in X$ they evaluate the sequence $\phi_n(x) \in Y$ defining for every x a sequence over Y and then they evaluate that limit in Y

$$\begin{aligned} \phi_n(x_0) &\to y_0 \\ \phi_n(x_1) &\to y_1 \\ \phi_n(x_2) &\to y_2 \\ \vdots \quad &\to \vdots \end{aligned}$$

   studying the subset $R \subseteq X$ for which the sequence $\phi_n(x)$ converges[1].

5. what we obtain at the end should be one of the desired inaccessible elements of $[f,g]$, e.g. $[f,s]$ is the solution set of the Abel equation on f; $[f, \mathrm{mul}_a]$ is the solution set of the Schroeder equation on f; $[\exp, s]$ contains the slog; $[s, \exp]$ contains tetration; in general $[s,f]$ is the set of superfunctions of f;

[1]How is this set called in dynamics? Seems some kind of convergence "disk".

The general trick

Let's do an attempt at proving the Lazy Theorem, really a conjecture, but first let me rewrite the theorem:

Lazy Theorem. *Given functions $f : X \to X$ and $g : Y \to Y$, a sequence $\phi \in [f,g]_\Delta$ (or $\in [f,g]^{op}_\Delta$), i.e $\phi_{n+1} f = g\phi_n$ (or $\phi_n f = g\phi_{n+1}$). Given a "**convergence notion**" over X^Y, if $\phi_n \to \phi$ for this notion then $\phi \in [f,g]$, i.e.*

$$\phi f = g\phi$$

Proof attempt: to prove this we must (A1) assume to have a **convergence notion** (a metric or a topology) for functions in Y^X or over a well-behaved subset of it $S \subseteq Y^X$. We next proceed proving (A2) that ϕ_n does in fact converges in S under this notion: we hopefully achieve that by a careful construction of ϕ_n from a chosen "**appropriate**" ϕ_0.

The last thing needed to justify the thesis is the following deduction: in the first place we must assume that left and right composition[2] are at least sequentially continuous (A3), that the shift of a convergent sequence converges and it does to the same limit (A5).

$$\begin{aligned} \lim_{n\to\infty} \phi_n f &= \lim_{n\to\infty} g\phi_{n+1} \\ (\lim_{n\to\infty} \phi_n) f &= g(\lim_{n\to\infty} \phi_{n+1}) && (A3) \\ \phi f &= g\phi && (A4) \end{aligned}$$

we conclude $\phi \in [f,g]$. □

The proof, as we can see, urges us to make clear some assumptions and prove some lemmas: by taking $X = Y = \mathbb{R}$ some of them are basic results taught as undergrad analysis material. But is there more to the *"superfunction trick"*? Am I overlooking something?

Since we are using the limit of the sequence at infinity only because we want to have a fixed point $\phi_{\infty+1} = \phi_\infty$ can we weaken the hypotheses asking only fixed points of the "broken conjugation" function $\gamma_{(g^{-1};f)} : Y^X \to Y^X$ defined as $\gamma_{(g^{-1};f)}(h) = g^{-1}hf$?

[2] In this case the functions that have to be sequentially continuous are $h_f : Y^X \to Y^X$ and $h^g : Y^X \to Y^X$ defined by $h_f : \chi \mapsto \chi \circ f$ and $h^g : \chi \mapsto g \circ \chi$

Appendix A: recursive construction lemma

Let's define two fundamental functions that can make the notation and the statements more succinct. The first is the evaluation of a sequence at zero: given $a \in X^{\mathbb{N}}$ we define $\mathrm{ev}_0(s) \coloneqq s_0 \in X$; the second is iterative recursion: given $t : X \to X$ for every $x \in X$ we define $\mathrm{rec}_t(x) = r \in X^{\mathbb{N}}$ as the sequence r defined by $r_0 \coloneqq x$ and $x_{n+1} \coloneqq t(x_n)$.

$$X \underset{\mathrm{ev}_0}{\overset{\mathrm{rec}_t}{\rightleftarrows}} X^{\mathbb{N}}$$

For every t this pair of functions has the remarkable property that $\mathrm{ev}_0 \circ \mathrm{rec}_t = \mathrm{id}_X$: it's no surprise that evaluation is surjective, but we get for free that recursion is injective. Sure, by recursion theorem we know that, chosen a step function, recursive defined functions are completely determined by their value at zero.

Another remarkable property is that every endomap t defines in this way an idempotent endomap $\mathrm{rec}_t \circ \mathrm{ev}_0$ on $X^{\mathbb{N}}$ that fixes the subset $[s, t]$ where s is the successor.

Now consider the sequences on Y^X: fixed a $t : Y^X \to Y^X$ we get a pair

$$Y^X \underset{\mathrm{ev}_0}{\overset{\mathrm{rec}_t}{\rightleftarrows}} (Y^X)^{\mathbb{N}}$$

and remember that $[f, g] \subseteq Y^X$ and $[f, g]_\Delta, [f, g]_\Delta^{op} \subseteq (Y^X)^{\mathbb{N}}$.

Proposition 1. *Given functions $f : X \to X$ and $g : Y \to Y$:*

1. *if f is split-mono $\mathrm{ev}_0 : [f, g]_\Delta \to Y^X$ is surjective;*
   if f is split-epi $\mathrm{ev}_0 : [f, g]_\Delta \to Y^X$ is injective;
   if f is iso $Y^X \simeq [f, g]_\Delta$
2. *if g is split-epi $\mathrm{ev}_0 : [f, g]_\Delta^{op} \to Y^X$ is surjective;*
   if g is split-mono $\mathrm{ev}_0 : [f, g]_\Delta^{op} \to Y^X$ is injective;
   if g is iso $Y^X \simeq [f, g]_\Delta^{op}$
3. *$[f, g]$ injects in both $[f, g]_\Delta$ and $[f, g]_\Delta^{op}$ via constant sequence;*

4. for every $\gamma \in [f,f]_\Delta$ and $\delta \in [g,g]_\Delta$, if $\chi \in [f,g]_\Delta$ then $\delta\chi\gamma \in [f,g]_\Delta$;

Proof: (1a) for every $\phi : X \to Y$ we build sequence in $\alpha \in [f,g]_\Delta$ s.t. $\alpha_0 = \phi$. We do it recursively: let $r : X \to X$ be a retraction of f then

$$\alpha := \mathrm{rec}_{\gamma_{(g;r)}}(\phi)$$

where $\gamma_{(g;r)} : \phi \mapsto g\phi r$. As we recalled before, recursion by any function is a right inverse to evaluation, hence if we will find a recursion that send us in $[f,g]_\Delta$ the evaluation is surjective: $\gamma_{(g;r)}$ our choice. In symbols, let $\alpha := \mathrm{rec}_{\gamma_{(g;r)}}(\phi)$

$$\begin{aligned} \alpha_{n+1} &= g\alpha_n r && (\textit{by definition}) \\ \alpha_{n+1} f &= g\alpha_n r f \\ &= g\alpha_n && (\textit{r is a retract}) \\ \alpha &\in [f,g]_\Delta && (\textit{by definition}) \end{aligned}$$

(1b) assume f has a section and $\alpha, \alpha' \in [f,g]_\Delta$. If $\alpha_0 = \alpha'_0 = \phi$ then the sequences are equal $\alpha = \alpha'$. Let $j : X \to X$ be a section of f: by definition $\alpha_{n+1} f = g\alpha_n$ and $\alpha'_{n+1} f = g\alpha'_n$ thus we have

$$\begin{aligned} (\alpha_{n+1} f j = g\alpha_n j) &\wedge (\alpha'_{n+1} f j = g\alpha'_n j) \\ (\alpha_{n+1} = g\alpha_n j) &\wedge (\alpha'_{n+1} = g\alpha'_n j) && (\textit{j is a section}) \\ \alpha = \mathrm{rec}_{\gamma_{(g;j)}}(\phi) &\wedge \alpha' = \mathrm{rec}_{\gamma_{(g;j)}}(\phi) && (\textit{by hypothesis}) \\ \alpha &= \alpha' && (\textit{by rec. theorem}) \end{aligned}$$

By the recursion theorem, recursion is a function so we conclude injectivity; (1c) follows trivially.

(2a) this is the dual of (1a) and the proof is the same. For clarity I'll prove this differently: let $\phi : X \to Y$ and $j : Y \to Y$ a section of g. Define recursively $\beta_n := j^n \phi f^n$, this time we proceed by induction. For $n = 0$ we have $\phi f = gj\phi f = g\beta_1$; assume $\beta_n f = g\beta_{n+1}$

$$\begin{aligned}
\beta_{n+1} f &= (gj)\beta_{n+1} f && (j\textit{is a section})\\
\beta_{n+1} f &= gj(j^{n+1}\phi f^{n+1}) f && (\textit{by definition})\\
\beta_{n+1} f &= gj^{n+2}\phi f^{n+2} && (j\textit{is a section})\\
&= g\beta_{n+2}
\end{aligned}$$

we conclude that β is indeed in $[f,g]^{op}_\Delta$; (2b) and (2c) can be easily done by the reader.

(3) if $\phi \in [f,g]$ then it satisfy the equation $\phi f = g\phi$. Define the constant sequence $\phi! : \mathbb{N} \to Y^X$ as usual $\forall n : \phi_n := \phi$. Injectivity is trivial.

(4) Let's pick three sequences $\gamma \in [f,f]_\Delta$, $\delta \in [g,g]_\Delta$ and $\chi \in [f,g]_\Delta$. By definition, for every natural number we have

$$\gamma_{n+1} f = f\gamma_n, \quad \chi_{n+1} f = g\chi_n \quad \text{and} \quad \delta_{n+1} g = g\delta_n$$

obviously $g(\gamma_n\chi_n\delta_n) = \gamma_{n+1} g\chi_n\delta_n = \gamma_{n+1}\chi_{n+1} f\delta_n = (\gamma_{n+1}\chi_{n+1}\delta_{n+1)} f$ □

This is just the shadow of a much more general mechanism: the sets $[f,g]_\Delta$ are just hom-sets in the category of constant covariant functors from the ordered set $\mathbb{N}$, seen as a category, to the category of set; the sets $[f,g]^{op}_\Delta$ are hom-sets in the category of the contravariant ones. Therefore we are just composing natural transformations and as in all the categories we have a composition map from the hom-sets

$$[g,h]_\Delta \times [f,g]_\Delta \to [f,h]_\Delta$$

[todo]

Appendix B: Convergence Notions

Definition 1. *Let X be a set, a convergence notion over X is a pair (X_L, L) where $X_L \subseteq X^{\mathbb{N}}$ and $L : X_L \to X$ such that*

1. *the constant sequences X_K on X are in X_L;*

2. *L is a left-inverse of the constant sequence map, i.e. $L \circ k_X = \mathrm{id}_X$, i.e.*

$$L(k_X(X)) = L(x, x, x, x, ...) = x$$

3. *if $\varsigma \mathbb{N} \to \mathbb{N}$ is increasing $X_L \varsigma \subseteq X_L$.*

Where $k_X : X \to X_L$ is the composition $X \to X^{\mathbb{N}} \subseteq X_L$

Definition 2. *Let $\mathcal{C}$ be a category, a convergence notion over C consists of this data*

- *a subcategory $i_L : \mathcal{C}_L \hookrightarrow \mathcal{C}^{\mathbb{N}}$;*
- *a functor $L : \mathcal{C}_L \to \mathcal{C}$*

such that

1. *exists an unique $j : \mathcal{C}_K \hookrightarrow \mathcal{C}_L$ such that $i_L j = i_k$;*
2. *$Lj = K$ that is $L \circ (jK^{-1}) = 1_{\mathcal{C}}$;*

Where K is the restriction of the canonical functor $\mathcal{C} \to \mathcal{C}^{\mathbb{N}}$ induced contro-variantly by the terminal functor $1 \leftarrow \mathbb{N}$.

Appendix C: category-theoretic rephrasing

Conjecture 1. *Let $(X_\bullet, f_\bullet)$, $(Y_\bullet, g_\bullet)$ be two constant sequences over the category $\mathcal{C}$ and let ϕ be a natural transformation from between them, i.e. a chain morphism,*

$$\begin{array}{ccccccccccccc}
X & \xrightarrow{f} & X & \xrightarrow{f} & X & \xrightarrow{f} & \dots & \xrightarrow{f} & X & \xrightarrow{f} & X & \xrightarrow{f} & \dots \\
{\scriptstyle \phi_0}\downarrow & & {\scriptstyle \phi_1}\downarrow & & {\scriptstyle \phi_2}\downarrow & & & & {\scriptstyle \phi_n}\downarrow & & {\scriptstyle \phi_{n+1}}\downarrow & & \\
Y & \xrightarrow{g} & Y & \xrightarrow{g} & Y & \xrightarrow{g} & \dots & \xrightarrow{g} & Y & \xrightarrow{g} & Y & \xrightarrow{g} & \dots
\end{array}$$

let $\mathcal{L}$ be a convergence notion over the hom-set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ then: if ϕ is $\mathcal{L}$-convergent then

$$\mathcal{L}(\phi) \circ f = g \circ \mathcal{L}(\phi)$$