Composition, bullet notation and the general role of categories: *the softest introduction ever made*

mphlee

2021 02 02

Exchange had at the Tetration forum. Thread n° 1293 - ***"Composition, bullet notation and the general role of categories"***

This follows from the discussion held at

MphLee, **Generalized Kneser superfunction trick (the iterated limit definition)**, (January 21, 2021), Tetration Forum.

> But if I did all this bullet stuff with ∘-that's not really how ∘ is usually used, so I'd be overriding the meaning of an existent symbol within this context. Better to use a new symbol and be fresh. This is especially beneficial when we talk about $ds \bullet z$ which is almost like a differential form. Writing $ds \circ z$ would be going a step too far I think.

It is clear to me where you are coming from. Your solution is pleasant, pretty, comfortable and the notation, as usually happens with good notation, hints at new developments, e.g. the differential forms. Even if I don't get n-forms yet and exterior algebra feels alien to me I feel it's a similarity worth considering: for this and another reason, I like your choice. I feel like you are aiming, not secretly at all, to a general *infinitesimal compositional calculus*.

But said that you shouldn't be overly confident about the variable vs function distinction:

$$f \circ g \circ z$$

> Wtf is that nonsense? lol

What nonsense is this? It's abstract nonsense!

In category theory, a land where only composition and arrows make the whole ontology, writing $f \circ g \circ z$ makes perfect sense, not only that, it means exactly what you expect it should.

More than that: **I claim that the natural home for general iterated compositions are categories!** In the last part I'll offer moral reasons for that but first let's inspect your "nonsensical" composition.

About evaluation. In general categories morphisms are just abstract arrows, not functions, and evaluation of them doesn't generally make sense because not every object can be conceived as a bag of something, e.g. points. The philosophy of category theory is exactly this: ignore what's inside, the inner structure of things, and solely observe how your things interact with each other.

There are some very special categories where objects are indeed made of points, e.g. the category of topological spaces, of vector spaces, of abelian groups or the category of bare sets: with this I mean that some categories have among all the objects a special "point object" $*$.

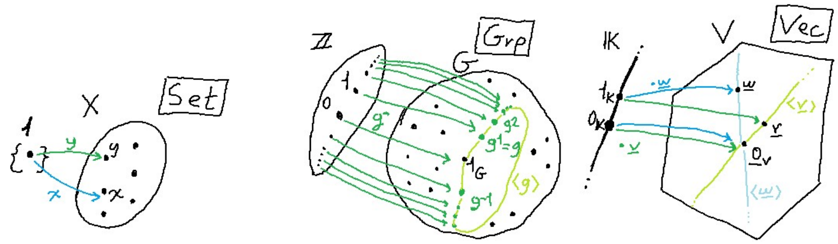
In these particular categories a morphism from this ***objectified abstract point*** to an ***arbitrary object*** X can be thought as (a choice of) ***a point*** x in X

$$* \xrightarrow{x} X$$

therefore defining the set points of X to be the (hom-)set of arrows

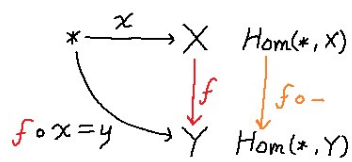
$$\mathrm{Points}(X) := \mathrm{Hom}(*, X) = \{x \mid * \xrightarrow{x} X\}$$

For example, for a bare set X we have $X^1 \simeq X$ where 1 is a singleton; the set of group homomorphisms from the group of integers to G is in bijection with the set of group elements of G, i.e. $\mathrm{Hom}_{\mathrm{Grp}}(\mathbb{Z}, G) \simeq G$; the linear applications from a field $\mathbb{K}$, seen as a vector space, to a $\mathbb{K}$-vector space V are in bijection with vectors of V, i.e. $\mathrm{Hom}_{\mathrm{Vec}}(\mathbb{K}, V) \simeq V$



and all of this without actually being able to look inside our objects ***nor knowing what set membership is***!

In the case you make, given an abstract arrow $X \overset{f}{\to} Y$ and a point $* \overset{x}{\to} X$ we can evaluate f at x composing the two and producing a new point $* \overset{y}{\to} Y$

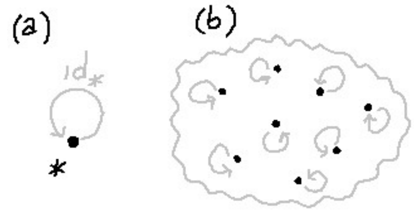


On categories: the softest introduction. Habitually, the prototypical example of category made is "sets and set functions". I believe this is not a wise example at all!

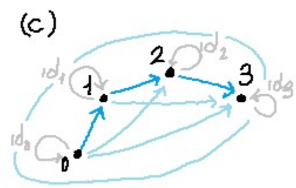
In a category we have a bunch of points (called objects) and a bunch of arrows. Arrows are "things":

1. of which we can say from where to where they are going;
2. that are closed under concatenation, i.e. if there are two consecutive arrows $x \to y \to z$ then there must exist a third arrow $x \to z$ and we can point to it;
3. that include an unique special loop arrow $\underset{\circlearrowleft}{x}$ for every point (object) x such that when concatenated with other consecutive arrows behaves like an identity.

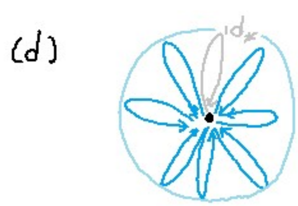
A category is just a bunch of arrows between a bunch of points. So when you think of a category here are some good examples to build an intuition on:



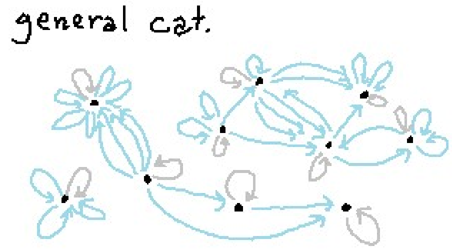
A single point (a) is the simplest category: you can think of it as **pure absolute being**. If you have only points but no arrows except for the identities (b) you just have **pure discreteness**: nothing moves and nothing becomes anything. That is just a bare set without any shape or structure, where every point is just itself.



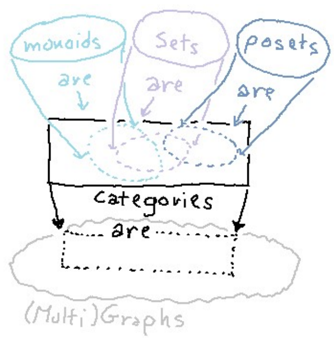
In (c) we have just a finite ordinal number, i.e. a bunch of points and a relation of linear order between them: the existence of composition expresses transitivity of the order; the existence of identity expresses reflexivity.



Here things become interesting! When you have only one point and many arrows (d) you have **pure motion**! We have many ways in which the arrows go from the point to itself, think of them as modes of transforming the object in itself, like symmetries. Closure under composition means we obtain a magma, associativity of composition expresses being a semigroup and existence of identities implies we have a monoid;



As we have seen, sets are merely a static kind of categories; monoids (thus groups) are just categories with a single point; ordered sets are just "slim" categories where there exists at most a single arrow between two points. To sum it up, a general a category is much like a special (multi)graph, special because edges are closed under concatenation and every vertex has a personal loop-edge.



Graphs and multigraphs come with a jungle of definitions.

What it's meant here precisely is a **di**rected **multi**graph, or **multi di**graph: **directed** because edges are oriented and **multi** because there can be multiple edges between any two nodes/vertices: these kind of graphs are called **QUIVERS**[1], for the friends.

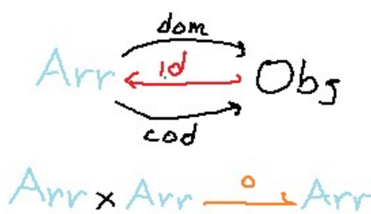
We are ready to produce a first informal definition of what is a category! A category is something that looks a lot like a quiver where you can concatenate edges!

[1] The n-**lab** defines quivers to be the most general kind of graphs: directed pseudographs, or directed loop multigraphs or "digraphs".

Definition of quiver A **quiver** is just a set E of **edges**, a set V of **vertexes** and two function $s, t : E \to V$ called **source** and **target**.

Definition of category A category is just a quiver where we call the edges Arr(-ows), the vertices Ob(-jects) and we have further structure:

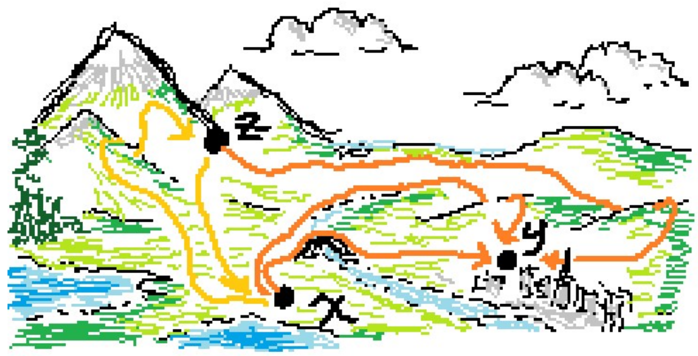
- an **injection map** of objects into arrows $\mathrm{id}_{-} : \mathrm{Ob} \to \mathrm{Arr}$ called identity;
- a **partial binary associative map** on arrows $\mathrm{Arr} \times \mathrm{Arr} \rightharpoonup \mathrm{Arr}$ called composition.



All that structure must satisfy some natural properties that I'm not going to write down formally but comprise the fact that identities are loops, that identities behave like identities and that composition is possible only for consecutive arrows.

Conclusive notes. It's now evident that the standard example of a category is the most involved one: in fact we usually take as objects the class of models of a mathematical theory $\mathbb{T}$, e.g. theory of sets, theory of groups or theory of topological spaces, and as arrows the model-theoretic homomorphisms between them, e.g. functions, group homomorphisms or continuous functions.

Maybe the most natural example is our world (space)! Just immagine every point (position) to be an object and an arrow is just a route, i.e. path, from a point to another:



Another insightful possibility is to imagine categories as partially defined monoids with multiple identity elements. Doing so we have that monoids are categories with a single object and groups are groupoids with one object:

$$
\begin{array}{ccc}
\text{Monoids} & \xrightarrow{\textit{every arrow is invertible}} & \text{Groups} \\
{\scriptstyle \textit{add more objects}}\Big\downarrow & & \Big\downarrow{\scriptstyle \textit{add more objects}} \\
\text{Categories} & \xrightarrow{\textit{every arrow is invertible}} & \text{Groupoids}
\end{array}
$$

I hope I've provided enough motivation for the claim made at the beginning. One day I could continue with functors and natural transformations.