A Tetration Function By Unconventional Means

James David Nixon
JmsNxn92@gmail.com

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Abstract

The author makes use of infinite compositions and a limiting function to sketch the construction of a holomorphic tetration function $\mathcal{F}(s) = e \uparrow\uparrow s$. As a tetration function, $\mathcal{F}$ satisfies $e^{\mathcal{F}(s)} = \mathcal{F}(s+1)$. Of it, $\mathcal{F}$ is holomorphic on the domain $\mathbb{C}/(\mathbb{R} + 2\pi i k)$ for $k \in \mathbb{Z}$; and $\mathcal{F}$ takes $(-2,\infty) \to \mathbb{R}$ bijectively with strictly monotone growth, and is continuously differentiable here.

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1 Introduction

The study of tetration can be traced back centuries. The first notable moment being, when Leonhard Euler proved that when $e^{-e} < \alpha < e^{1/e}$; the infinite tower converges [1]. Quote unquote, for such α,

$$\alpha^{\alpha^{\cdots(n\,\text{times})\cdots^{\alpha}}} \to A \text{ as } n \to \infty$$

Where $\alpha^A = A$ and $e^{-1} \leq A \leq e$. This was a monumentous moment in the study of iterated exponentials; and is probably the first truly publishable result in the history of tetration–hell, complex dynamics. Though the subject of tetration remains rather dormant; if we do not count numerous forays into the complex dynamics of the exponential function; it poses itself as a very interesting problem. The author would like to think, at the center of the study of iterated exponentials is the study of tetration functions; and the quest for a good and right solution should be a priority. But as the pulse of the subject is weak, he feels not many others feel the same way.

A tetration function is simple enough to describe, but proves a very difficult function to construct (if we want a good and right solution). We'll restrict our attention to tetration functions with the base e. For bases $b > e^{1/e}$ the result is similar. If $\mathcal{F}$ is a function such that $e^{\mathcal{F}(s)} = \mathcal{F}(s+1)$ and $\mathcal{F}(0) = 1$, then we call $\mathcal{F}$ a tetration function. It provides a similar continuation of the sequence $e, e^e, e^{e^e}, ...$ as e^s provides a continuation of the sequence $e, ee, eee, ...$ In contrast to the exponential function though, it is a much more volatile construction.

One facet of its volatility, tetration functions are highly non-unique. For any tetration function $\mathcal{F}$ the function $\mathcal{F}(s+\sin(2\pi s))$ is also a tetration function; as is $\mathcal{F}(s+\theta(s))$ for any 1-periodic function. Alors, a large problem with tetration, is qualifying a tetration function as unique, or as satisfying some property which characterizes it from other tetration functions.

There are quite a few trivial ways to construct a tetration function, for instance letting $\mathcal{F}(t) = 1 + t$ for $t \in [-1, 0]$ and extending $\mathcal{F}$ to $\mathbb{R}^+$ using the functional equation provides such a solution. But necessarily such a solution is not differentiable on the natural numbers. One can construct many continuous extensions in a parallel manner.

If we were to ask for a tetration function, we would at least require that it be analytic. Even better, that it be holomorphic on some domain in the complex plane including $\mathbb{R}^+$.

Of that end, Hellmuth Kneser was the first to undoubtedly provide a desirable solution to the tetration equation [2]. Insofar as it took the real positive line to itself and was holomorphic. He worked exclusively with the base $b = e$, and managed to construct a function $\mathcal{H}(s) = {}^s e$ holomorphic in $\mathbb{C}$ excluding the line $(-\infty, -2]$. His construction, although providing a stable solution, was highly esoteric and deeply expressed the difficulty of this problem. Constructing a holomorphic tetration function is no easy feat. To his talent, his sole goal was to construct h such that $h(h(z)) = e^z$ and h was real-valued; and he managed to construct the only true holomorphic tetration.

More recently, numerous attempts have been made to construct a simpler tetration function. There exists quite a few potential candidates for tetration; which exist scattered in the recesses of the internet. Since tetration has not gained widespread recognition as a notable problem–it is difficult to find published papers on the subject. Kneser's own paper [2] is in German, and there exists no English translations–but there are synopses and breakdowns.

The problem with most modern approaches to tetration seem to be that these candidates can be numerically verified, but never rigorously justified to exist or converge. And in contrast; Kneser's construction, which is correct, is a rather laborious numerical procedure. Even more troubling with these flurry of candidates to tetration; proofs of analycity become even more difficult–and in most instances do not exist. Alongside a lack of proof for mere convergence, this can be troubling. If that wasn't bad enough; we can't even prove if two candidate tetration functions equal or not. But sometimes our floating point accuracy can appear to suggest so (or dissuade so).

Nonetheless there is still great headway being made in the field. A treasured example is in the work of Dmitri Kouznetsov [3]. The author will not go into detail on Kouznetsov's method but will simply give a rough heuristic as motivation. The author will blatantly steal this heuristic–but he shall bypass a few of the obstacles he believes Kouznetsov built for himself. Though, in truth, much of the work parallels an extension of his trick.

Supposing we had a nice function $g(s)$, which has some desirable growth properties, and we took the limiting function,

$$G_n(s) = \log\log\cdots(n\,\text{times})\cdots\log g(s+n)$$

Then $e^{G_n(s)} = G_{n-1}(s+1)$. If the limit were to converge as $n \to \infty$, then we would have our tetration function $G_n \to G$. That is, upto a normalization constant ω where $\mathcal{F}(s) = G(s+\omega)$ to ensure $\mathcal{F}(0) = 1$.

Kouznetsov chose a very wonderful function g, such that numerically everything worked out and provides us with a calculator's version of a holomorphic tetration function. The function g was constructed through careful fixed-point analysis; and looks like an exponential sum of terms e^{nLs} for L a complex fixed point of e^s. Unfortunately, to rigorously justify convergence proves rather difficult. This becomes a sort of brick-wall. There is no in your hands proof that Kouznetsov's method actually works.

To that end, the goal of this paper is to construct our own g, and show the convergence of the above limit. Our choice of g will be very manufactured, and requires a familiarity with infinite compositions. To that end, we refer the reader to [5], where sufficient conditions are provided for an infinite composition to converge–and a familiarity with the subject is created. We shall not need anything from [5], but it exhibits the nuanced detail of the subject a bit more clearly. We will prove a modified form of the main result of [5]; to keep this paper self-contained; but we will give little to no motivating intuition in this paper.

The essential trick is to bypass the difficulty of constructing a tetration function using Kouznetsov's trick by constructing a function which satisfies a similar functional equation, but exhibits the same growth properties. As you may guess, tetration grows rapidly and rabidly in the complex plane, so we'll need something similarly chaotic.

To that end, our first goal is to construct an entire function $\phi(s)$ such that,

$$\phi(s+1) = e^{s+\phi(s)}$$

And take ϕ to be our g from above. The real novelty of the work is held in constructing ϕ. But it only really takes us writing out the equation for ϕ and justifying convergence. This is more of a taxing process than a difficult one. It is surprisingly simple to construct ϕ.

* * *

We introduce briefly the notation for nested compositions, which allows for our construction of ϕ. We will restrict from full generality, and only care about a subset of types of infinite compositions. Therein, if $h_j(s,z) : \mathbb{C}^2 \to \mathbb{C}$ is a sequence of entire functions in both variables, then,

$$\Omega_{j=1}^{n} h_j(s,z) \bullet z = h_1(s, h_2(s, ...h_n(s,z)))$$

Where we are interested in letting $n \to \infty$. The study of infinite compositions is very nuanced, and for that reason constructing ϕ will require care. The type of convergence we'll need is one which is a bit simpler than the general case. To wit, we will call,

$$\phi_n(s) = \Omega_{j=1}^{n} e^{s-j+z} \bullet z\Big|_{z=0}$$

Where if $h_j(s,z) = e^{s-j+z}$ then,

$$\phi_n(s) = h_1(s, h_2(s, ...h_n(s, 0)))$$

By design, $e^{s+\phi_n(s)} = \phi_{n+1}(s+1)$, so if this were to converge it would equal our desired function. The essential ingredient in our construction is for all compact disks $\mathcal{P}, \mathcal{K} \subset \mathbb{C}$,

$$\sum_{j=1}^{\infty} ||h_j(s,z)||_{s\in\mathcal{P}, z\in\mathcal{K}} < \infty$$

Where $||...||$ is taken to mean the supremum norm. So without further ado, we construct ϕ.

2 Constructing ϕ

The first thing we need to construct ϕ is a sort of normality condition. For all $\epsilon > 0$, there exists some N, such when $m \geq n > N$,

$$|\Omega_{j=n}^{m} e^{s-j+z} \bullet z| = |e^{s - n + e^{s-n-1+ ... e^{s-m+z}}}| < \epsilon$$

For $|z| < 1$, and s residing in some compact disk within $\mathbb{C}$. This then implies as we let $m \to \infty$, the tail of the infinite composition stays bounded. Forthwith, the infinite composition becomes a normal family, and proving convergence becomes simpler.

The direct analogy we can make to this is with sums and products. Suppose we have a sequence of numbers $a_j \in \mathbb{C}$ and we wish to show the their sum converges. Well the direct method is to prove,

$$|\sum_{j=n}^{m} a_j| < \epsilon$$

This is similar with products; for a sequence of numbers $b_j \to 1$; the direct method is to prove,

$$|\prod_{j=n}^{m} b_j - 1| < \epsilon$$

So, in the compositional case, we're doing something very similar. We're trying to bound the tail of the infinite composition. The sort of bounds we get are, not exactly but close to, of the form,

$$|\mathop{\Omega}_{j=n}^{m} e^{s-j+z} \bullet z| \leq \sum_{j=n}^{m} |e^{s-j+z}|$$

And in this, we are comparing the tail of the composition to the tail of a sum. It is a bit more subtle how we actually do this; supremum norms across different domains arise. The lemma may be a tad cryptic; but this is a good way to initialize the idea. We provide a quick proof of this.

Lemma 2.1. *For a compact disk $\mathcal{P} \subset \mathbb{C}$ and $|z| \leq 1$: for all $\epsilon > 0$, there exists some N, such when $m \geq n > N$*

$$||\mathop{\Omega}_{j=n}^{m} e^{s-j+z} \bullet z||_{\mathcal{P},|z|\leq 1} < \epsilon$$

Proof. Let $|z| \leq 1$ and $s \in \mathcal{P}$ be a compact disk in $\mathbb{C}$. Set $h_j(s,z) = e^{s-j+z}$ and set $||h_j(s,z)||_{s\in\mathcal{P},|z|\leq 1} = \rho_j$. Pick $1 > \epsilon > 0$, and choose N large enough so when $n > N$,

$$\rho_n < \epsilon$$

Denote: $\phi_{nm}(s,z) = \Omega_{j=n}^{m} h_j(s,z) \bullet z = h_n(s, h_{n+1}(s, ...h_m(s,z)))$. We go by induction on the difference $m - n = k$; which counts how many exponentials appear. When $k = 0$ then,

$$||\phi_{nn}(s,z)||_{|z|<1,s\in\mathcal{P}} = ||h_n(s,z)||_{|z|<1,s\in\mathcal{P}} = \rho_n < \epsilon$$

Assume the result holds for $m - n < k$, we show it holds for $m - n = k$. Observe,

$$\begin{aligned}
||\phi_{nm}(s,z)||_{|z|<1,s\in\mathcal{P}} &= ||h_n(s, \phi_{(n+1)m}(s,z))||_{|z|<1,s\in\mathcal{P}} \\
&\leq ||h_n(s,z)||_{|z|<1,s\in\mathcal{P}} \\
&= \rho_n < \epsilon
\end{aligned}$$

Which follows by the induction hypothesis because $|\phi_{(n+1)m}(s,z)| < \epsilon < 1$; i.e: $m - n - 1 < k$. ☐

The next step is to observe that $\Omega_{j=1}^{m} h_j(s,z) \bullet z$ is a normal family as $m \to \infty$, for $|z| < 1$ and $s \in \mathcal{P}$, an arbitrary compact disk. This follows because the tail of this composition is bounded. For $m > N$,

$$\mathop{\Omega}_{j=1}^{m} h_j(s,z) = \mathop{\Omega}_{j=1}^{N} h_j(s,z) \bullet \mathop{\Omega}_{j=N+1}^{m} h_j(s,z) \bullet z$$

But $|| \Omega_{j=N+1}^m h_j(s,z) \bullet z|| < \epsilon$. So as m grows the function stays within the neighborhood. Therefore we can say $|| \Omega_{j=1}^m h_j(s,z)||_{|z|<1, s\in\mathcal{P}} < M$ for all m.

The second way to ideate this, is to, again, compare it to a sum. What we have done is made a comparison,

$$| \underset{j=1}{\overset{\infty}{\Omega}} e^{s-j+z} \bullet z| \leq \sum_{j=1}^\infty |e^{s-j+z}|$$

Where on the right hand side |...| is a supremum norm across some compact set, and on the left hand side |...| is across a compact set. This is a nice way to think about it; where because the sum converges, so does the infinite composition. It may seem strange; but the idea is,

$$| \underset{j=1}{\overset{m}{\Omega}} e^{s-j+z} \bullet z - \underset{j=1}{\overset{n}{\Omega}} e^{s-j+z} \bullet z| \leq \sum_{j=n}^m |e^{s-j+z}| < \epsilon$$

Where on the right hand side |...| is a supremum norm for some compact set, and on the left hand side |...| is across a compact set. Which is very similar to the sum and product case, in which:

$$\begin{aligned} |\sum_{j=1}^m a_j - \sum_{j=1}^n a_j| &\leq \sum_{j=n}^m |a_j| < \epsilon \\ |\prod_{j=1}^m b_j - \prod_{j=1}^n b_j| &\leq A\sum_{j=n}^m |b_j - 1| < \epsilon \text{ for some } A \in \mathbb{R}^+ \end{aligned}$$

From this we can prove our infinite composition converges, and construct our entire function $\phi(s)$.

Theorem 2.2. *The expression*

$$\underset{j=1}{\overset{\infty}{\Omega}} e^{s-j+z} \bullet z\Big|_{z=0} = \phi(s)$$

is an entire function satisfying the identity $e^{s+\phi(s)} = \phi(s+1)$.

Proof. Since $\phi_m(s,z) = \Omega_{j=1}^m e^{s-j+z} \bullet z$ are a normal family; there is some constant $M \in \mathbb{R}^+$ such,

$$||\frac{d^k}{dz^k}\phi_m(s,z)||_{|z|<\frac{1}{2}, s\in\mathcal{P}} \leq M2^{k+1}\cdot k!$$

We can achieve this through Cauchy's Integral Theorem; and a couple uses of the supremum norm; because,

$$\begin{aligned}
\frac{d^k}{dz^k}\phi_m(s,z) &= \frac{k!}{2\pi i}\int_{|\xi|=1}\frac{\phi_m(s,\xi)}{(\xi-z)^{k+1}}\,d\xi \\
||\frac{d^k}{dz^k}\phi_m(s,z)||_{|z|<\frac{1}{2},s\in\mathcal{P}} &\leq \frac{k!}{2\pi}\int_{|\xi|=1}2^{k+1}M\,d\xi
\end{aligned}$$

When $|z| \leq 1/2$ and $|\xi| = 1$, $|\xi - z| \geq \frac{1}{2}$. Secondly, using Taylor's theorem; and expanding $\phi_{m+1}(s,z) = \phi_m(s, e^{s-m-1+z})$ about $\phi_m(s,z)$'s Taylor's series about z:

$$\begin{aligned}
\phi_{m+1}(s,z) - \phi_m(s,z) &= \phi_m(s, e^{s-m-1+z}) - \phi_m(s,z) \\
&= \sum_{k=1}^{\infty}\frac{d^k}{dz^k}\phi_m(s,z)\frac{(e^{s-m-1+z}-z)^k}{k!} \\
&= (e^{s-m-1+z}-z)\sum_{k=1}^{\infty}\frac{d^k}{dz^k}\phi_m(s,z)\frac{(e^{s-m-1+z}-z)^{k-1}}{k!}
\end{aligned}$$

This series converges for at least $|z| < \delta$ and $m > N$ large enough. We don't care about δ because we set $z = 0$. Then,

$$||\phi_{m+1}(s,0) - \phi_m(s,0)||_{s\in\mathcal{P}} \leq ||e^{s-m+1}||_{s\in\mathcal{P}}\sum_{k=1}^{\infty}M2^{k+1}||e^{s-m+1}||_{s\in\mathcal{P}}^{k-1}$$

The series on the right can be bounded by some $C \in \mathbb{R}^+$; because for large enough $m > N$ the term $||e^{s-m+1}||_{s\in\mathcal{P}} < \frac{1}{2}$. Applying the bounds,

$$||\phi_{m+1}(s,0) - \phi_m(s,0)||_{s\in\mathcal{P}} \leq C||e^{s-m-1}||_{s\in\mathcal{P}} = Ae^{-m}$$

For some $A \in \mathbb{R}^+$. Assume,

$$\sum_{j=n}^{m-1} e^{-j} < \frac{\epsilon}{A}$$

Then,

$$\begin{aligned}
||\phi_m(s,0) - \phi_n(s,0)||_{s\in\mathcal{P}} &\leq \sum_{j=n}^{m-1}||\phi_{j+1}(s,0) - \phi_j(s,0)||_{s\in\mathcal{P}} \\
&\leq \sum_{j=n}^{m-1}Ae^{-j} \\
&< \epsilon
\end{aligned}$$

And we can see the telescoping series converges and $\phi_m(s)$ must be uniformly convergent for $s \in \mathcal{P}$, and therefore defines a holomorphic function $\phi(s)$ as $m \to \infty$. Naturally $e^{s+\phi_m(s)} = \phi_{m+1}(s+1)$, and so therefore the functional equation is satisfied. Since $\mathcal{P}$ was an arbitrary compact set in $\mathbb{C}$, we know ϕ is entire. $\square$

3 The correction term τ

The main philosophy of our approach to constructing tetration is to add a corrective term to ϕ such that it becomes a tetration function. The function ϕ already looks very close to tetration, satisfying a similar functional equation,

$$\phi(s+1) = e^{s+\phi(s)}$$

We will introduce a sequence of correction terms as follows:

$$\log\log\cdots(n\,\text{times})\cdots\log\phi(s+n) = \phi(s) + \tau_n(s)$$

Where inductively, starting with $\tau_1(s) = s$ and $\tau_0(s) = 0$; τ_n can be defined,

$$\begin{aligned}
\tau_{n+1}(s) &= \log\big(\phi(s+1) + \tau_n(s+1)\big) - \phi(s)\\
&= \log\phi(s+1) + \log\big(1 + \frac{\tau_n(s+1)}{\phi(s+1)}\big) - \phi(s)\\
&= s + \phi(s) + \log\big(1 + \frac{\tau_n(s+1)}{\phi(s+1)}\big) - \phi(s)\\
&= s + \log(1 + \frac{\tau_n(s+1)}{\phi(s+1)})
\end{aligned}$$

Our choice of log is defined implicitly by the relation,

$$e^{\phi(s)+\tau_{n+1}(s)} = \phi(s+1) + \tau_n(s+1)$$

And the restriction that $\tau_n(s)$ is real on the real-line. The first thing to note, is that for $s = t \in \mathbb{R}^+$, the sequence of functions τ_n converge uniformly on bounded intervals greater than T. This is because $0 < \tau_{n+1}(t+1)/\phi(t+1) < 1$ and using the relation $|\log(1+A) - \log(1+B)| \leq |A - B|$,

$$
\begin{aligned}
|\tau_{n+1}(t) - \tau_n(t)| &\leq |\log(1 + \frac{\tau_n(t+1)}{\phi(t+1)}) - \log(1 + \frac{\tau_{n-1}(t+1)}{\phi(t+1)})| \\
&\leq \frac{1}{\phi(t+1)}|\tau_n(t+1) - \tau_{n-1}(t+1)| \\
&\leq \frac{1}{\phi(t+1)\phi(t+2)}|\tau_{n-1}(t+2) - \tau_{n-2}(t+2)| \\
&\vdots \\
&\leq \frac{1}{\prod_{k=1}^{n}\phi(t+k)}|\tau_1(t+n) - \tau_0(t+n)|
\end{aligned}
$$

Recalling that $\tau_1(t) = t$ and $\tau_0(t) = 0$ then,

$$
|\tau_{n+1}(t) - \tau_n(t)| \leq \frac{t+n}{\prod_{k=1}^{n}\phi(t+k)}
$$

And here $\phi(t)$ is monotone increasing and unbounded, so for some T with $t > T$ it is $\phi(t) > \lambda > 1$. Now to show the telescoping sum converges uniformly on bounded intervals; pick an interval $\mathcal{I}$; and pick $n, m > N$ such,

$$
\sum_{j=n}^{m-1} \frac{||t+j||_{\mathcal{I}}}{\lambda^j} < \epsilon
$$

Then,

$$
\begin{aligned}
||\tau_m(t) - \tau_n(t)||_{\mathcal{I}} &\leq \sum_{j=n}^{m-1} ||\tau_{j+1}(t) - \tau_j(t)||_{\mathcal{I}} \\
&\leq \sum_{j=n}^{m-1} \frac{||t+j||_{\mathcal{I}}}{\lambda^j} \\
&< \epsilon
\end{aligned}
$$

And we are given a function $\tau : \mathbb{R}^+_{t>T} \to \mathbb{R}^+$ such that,

$$
\widetilde{\mathcal{F}}(t) = \phi(t) + \tau(t)
$$

And $\widetilde{\mathcal{F}}$ is a tetration function, albeit not yet normalized to $\widetilde{\mathcal{F}}(0) = 1$, but there is an appropriate ω such that $\mathcal{F}(t) = \phi(t+\omega) + \tau(t+\omega)$ is a true tetration function. Also by taking logarithms, the domain can be extended to its maximal $(-2, \infty)$. Call this function $\mathcal{F}$.

To assure that this construction hits no singularities along the way, we need that the derivative is monotone. Going through the same motions as above one can derive that $\mathcal{F}'$ is a continuous function and that $\tau_n' \to \tau'$ uniformly on bounded intervals. This is really no different then what we've already written. The function,

$$\tau_{n+1}'(t) = 1 + \Big(\frac{1}{1+\frac{\tau_n(t+1)}{\phi(t+1)}}\Big)\Big(\frac{\tau_n'(t+1)}{\phi(t+1)} - \frac{\tau_n(t+1)}{\phi(t+1)^2}\phi'(t+1)\Big)$$

Grinding the gears of this expression we get $|\tau_{n+1}' - \tau_n'|$ is a convergent series of the same form as above. This is more of a task than a problem. It is left to the reader.

We need that $\mathcal{F}'(t) > 0$ for all $t \in (-2,\infty)$. From the expression above, $\tau'(t) - 1 \to 0$ as $t \to \infty$. So, eventually $\tau'(t) > 0$ for some $t \geq T$. It is no hard fact to notice $\phi'(t) > 0$ for all $t \in \mathbb{R}$. Ergo, for $t \geq T$,

$$F'(t) = \phi'(t+\omega) + \tau'(t+\omega) > 0$$

Therefore, since,

$$\mathcal{F}'(t-1) = \frac{\mathcal{F}'(t)}{\mathcal{F}(t)} > 0$$

Thereby $\mathcal{F}'(t) > 0$ for $t \geq T-1$. By infinite descent we must have $\mathcal{F}'(t) > 0$ everywhere $\mathcal{F}(t+1) > 0$ which is $t > -2$. Therefore of this nature we have a differentiable inverse $\mathcal{A} = \mathcal{F}^{-1}(t) : \mathbb{R} \to (-2,\infty)$. In laymen's terms, amongst the jargon of people who study tetration; one calls this the super-logarithm. It is a continuously differentiable Abel function of e^t. En drame,

$$\mathcal{A}(e^t) = \mathcal{A}(t) + 1$$

These facts will be reinforced throughout this paper. Nonetheless it helps to introduce them when they can be conveniently introduced. We state this less than drastic theorem below.

Theorem 3.1. *For some $\omega \in \mathbb{R}$ there exists a continuously differentiable tetration function $\mathcal{F}(t) : (-2,\infty) \to \mathbb{R}$ such that $\mathcal{F}'(t) > 0$ and $\mathcal{F}$ is a bijection. This tetration function $\mathcal{F}(t)$ can be expressed as,*

$$\mathcal{F}(t) = \lim_{n\to\infty} \log\log\cdots(n\ times)\cdots\log\phi(t+\omega+n)$$

Where,

$$\phi(t) = \underset{j=1}{\overset{\infty}{\Omega}} e^{t-j+z} \bullet z\Big|_{z=0}$$

This provides us with a continuously differentiable tetration function $\mathcal{F}$ defined for $(-2,\infty)$, but it sadly says nothing of the case for complex numbers. This proves to be a much more exhausting challenge. But the challenge is perfectly manageable.

To set the stage we'll work on an easier case. The function ϕ is periodic with period $2\pi i$, and therefore a very similar argument as that of above allows us to construct $\mathcal{F}(s)$ for $s \in \mathbb{R} + 2\pi ik$ for $k \in \mathbb{Z}$ when $k \neq 0$. $\mathcal{F}$ will not be real valued on these lines, but ϕ will be, which allows for the same bounds. The same initial conditions are given as $\tau_1(s) = s$ and $\tau_0(s) = 0$, so the convergence follows for $t > T$.

As that,

$$\begin{aligned} |\tau_{n+1}(t+2\pi ik) - \tau_n(t+2\pi ik)| &\leq \frac{|\tau_n(t+1+2\pi ik) - \tau_{n-1}(t+1+2\pi ik)|}{|\phi(t+1)|} \\ &\leq \frac{|t+2\pi ik+n|}{\prod_{j=1}^n |\phi(t+j)|} \end{aligned}$$

Therefore,

$$\mathcal{F}(t+2\pi ik) = \phi(t+\omega) + \tau(t+\omega+2\pi ik)$$

Is a continuously differentiable function. Going further, if $\mathcal{F}(s) = 0$ then $s = -1$ necessarily; orbits of log on complex numbers with non zero imaginary part stay away from 0. (This statement will be made much clearer in the coming sections.) Therefore, we can extend $\mathcal{F}$ to $\mathbb{R} + 2\pi ik$ for all $k \in \mathbb{Z}$ with $k \neq 0$ by taking repeated logarithms.

An important note, as $s \to -\infty$, the functions $\tau(s+2\pi ik) \to L_k$ where L_k is a fixed point of the exponential map. The derivation of this comes from the limit $\lim_{s\to-\infty} \phi(s) = 0$ so,

$$\begin{aligned} \lim_{s\to-\infty} e^{\phi(s)+\tau(s+2\pi ik)} &= \lim_{s\to-\infty} \phi(s+1) + \tau(s+1+2\pi ik) \\ \lim_{s\to-\infty} e^{\tau(s+2\pi ik)} &= \lim_{s\to-\infty} \tau(s+1+2\pi ik) \\ e^{\tau(-\infty+2\pi ik)} &= \tau(-\infty+2\pi ik) \\ e^{L_k} &= L_k \end{aligned}$$

The fixed points satisfy the conjugate identity $\overline{L^k} = L^{-k}$. This can be derived because $\overline{\mathcal{F}(s)} = \mathcal{F}(\overline{s})$. The idea then, depending on how we limit to negative infinity, we get different fixed points. This relationship will be mirrored for $\lim_{t\to-\infty} \mathcal{F}(t+iy) = L$ for arbitrary $y \in \mathbb{R}$. In that, as $\Re(s) \to -\infty$ the function $\mathcal{F}(s) \to L$ (or we hit a branch-cut along the way) where L is a fixed point of the exponential map. This can be better codified if one thinks: iterated logarithms converge to a repelling fixed point of e^s or we get a branching problem (I.e: F hits zero somewhere).

Of the following discussion we can see that $\mathcal{F}(s)$ is continuously differentiable on the lines $\mathbb{R} + 2\pi ik$ for $k \neq 0$ and continuously differentiable on $(-2, \infty)$. We were able to obtain this result because $\tau_{n+1} - \tau_n$ was bounded by a product of the form $\dfrac{|t+n+2\pi ik|}{\prod_{k=1}^{n}\phi(t+k)}$. Since $\phi(t)$ grows super-exponentially, this product is very well behaved. We had yet to mention this fact but $\dfrac{1}{\phi(t)}$ is smaller than any iterate of the exponential as t grows. Which is, for all n, $\frac{1}{\phi(t)} \leq \frac{1}{\exp^{\circ n}(t)}$ for large enough $t \geq T$ depending on n.

In order to get τ to be holomorphic it would be required that we can control this product well in the complex plane. Doing this is a bit of a hassle. It necessitates us understanding how ϕ grows. The main hurdle of this paper comes in what follows.

4 Obtaining lower bounds on ϕ

This section will focus on the function $\phi(s)$ in the strip $0 \leq \Im(s) = y \leq \pi$ for large positive values of $t = \Re(s)$. The function $\phi(s)$ is real-valued and periodic with period $2\pi i$. So, $\overline{\phi(s)} = \phi(\overline{s})$, and $\phi(s+2\pi i) = \phi(s)$. To show $|\phi(s)|$ grows as t grows requires only looking at when $0 \leq y \leq \pi$.

Now this can't be all that simple. The function ϕ is entire, and it looks super-exponential, and satisfies a super-exponential-like functional equation. So, to stop and think for a second, we'd like to ask the worst that can happen. The author would like to thank Sheldon Levenstein whole-heartedly for pointing the following out to me. Let's look for values $|\phi(s)| = 1$.

Firstly, since $\phi(s)$ is entire, it gets arbitrarily close to all complex numbers as $|s| \to \infty$, and $\lim_{|s|\to\infty} \frac{\phi(s)}{e^{s-1}} = 1$ while $\pi/2 < |\arg(s)| \leq \pi$. This coupled with $2\pi i$ periodicity, the function ϕ is very well behaved in the left half plane–it looks like e^{s-1} almost exactly. This means, like e^{s-1}, it has to get arbitrarily large as $\Re(s) \to \infty$–and has to be in a neighborhood of any arbitrarily large number. The trouble arises because ϕ has a super-exponential functional equation.

The function $\phi(s)$ must get arbitrarily close to $e^{2\pi i\ell}$ for some $\ell \in [0,1)$ as $\Re(s)$ increases. This is very troublesome because it assures us that $|\phi(s)| = 1$ and surely this means that $|\phi(s)|$ cannot grow uniformly as $\Re(s)$ grows.

As to this, a weird paradox arises. These points in which $|\phi(s)| = 1$ actually cluster and happen nearest $s \in \mathbb{R} + 2\pi ik$. It can be understood as, where we have the sharpest growths, we also have the sharpest drops.

To elucidate, consider the strip $0 < \Im(s) < \delta$ for a small number δ. Surely the real part of $\phi(s)$ is still super-exponentially large for small enough δ. But then,

$$\Im\phi(s+1) = e^{\Re(s)+\Re\phi(s)}\sin(\Im(s)+\Im\phi(s))$$

On a good day, this might be alright, but usually that means the imaginary part is going to shoot off super-exponentially. And in doing this, it can force the real-part to zero in the next iteration. Think, $\cos(\Im(s)+\Im\phi(s)) \approx 0$. And then the next iteration, because the real part was obliterated; maybe the real part has jumped from super exponential to looking like $-\Re(s)$, then $|e^{s+\phi(s)}| \approx 1$.

We are pretty much guaranteed numerically that $\phi(s)$ gets arbitrarily close to 1 as $\Re(s)$ grows. Which, equates to solving for $\phi(s-1) = -\Re(s-1) + i\ell$ So what we want to say, is that it happens for $\Im(s) < \delta$ for δ shrinking as $\Re(s)$ grows.

The idea is to show for all $\epsilon > 0$ there exists some large value T and a value $\delta > 0$, such the function $|\phi(t+iy)| > 1+\epsilon$ for all $t > T$ and $\delta < y \leq \pi$. Despite: the simplicity of this statement; the fact ϕ grows super-exponentially on the real line; this statement proves to be the most difficult part of this construction to digest.

* * *

As some preliminary remarks; this result is highly non-trivial. It meshes well with intuition, but requires a good amount of heavy lifting. We'll need to reduce the case for all $0 \leq y \leq \pi$ to the case $y = \pi$; which in essence is the worst our function can grow like–other than a clustering of $|\phi(s)| = 1$ near the real-line.

The author had a more ambitious idea for this section, but kept on failing; and upon failure gathered what he could to prove that eventually $\phi(t + i\pi)$ is greater than $1+\epsilon$ for all $\epsilon > 0$ and stays greater. The ambitious proof the author, at first, thought was self-apparent, was that $\phi(t+\pi i)$ grows in a certain manner. And that which he cannot prove for the life of him. But luckily, we can at least prove it grows, eventually. And that's really all that's needed–fortunately.

In short; ϕ does not grow super-exponentially in the complex plane as it does on the real positive line. In fact, if $|\phi(t + \pi i)| > t$ then $|\phi(t + 1 + \pi i)| = e^{t-|\phi(t+\pi i)|} \leq 1$. As such $\phi(t + \pi i)$ must stay less than t, and limit like $|\dfrac{\phi(t+\pi i)}{t}| < 1 - \rho$ for $0 < \rho < 1$. This will also force $1 > |\phi'(t + \pi i)| > \rho$ as $t \to \infty$.

The only fact the author can think of to explain why ϕ experiences drops in its growth relies on a heavy study of complex dynamics. But, if $\mathcal{N}$ is a neighborhood in $\mathbb{C}$ then the orbits $\exp^{\circ k}(\mathcal{N})$ are dense in $\mathbb{C}$ [4]. Therefore, the neighborhoods $\mathcal{F}(\mathcal{N} + k)$ must be dense in $\mathbb{C}$. And to do this, $\mathcal{F}$ must grow relaxed in some parts. In accordance, ϕ must grow slow in some parts; or otherwise the entire construction will collapse.

The author couldn't derive an exact asymptotic for ϕ, but he guesses the main term is something just less than t; he hopes something like $\log^{\circ\rho}(t)$; which can be expressed as $\mathcal{F}(\mathcal{A}(t) - \rho)$, which looks something like $(1 - \rho)t$.

* * *

To better accustom ourselves to the behaviour of ϕ in the complex plane we can rewrite our function. It'll look a bit stranger in this form but,

$$\psi_y(t) = e^{-iy}\phi(t+iy) = \Omega_{j=1}^{\infty} e^{t-j+e^{iy}z} \bullet z\Big|_{z=0}$$

Then,

$$\psi_y(t) = e^{t} - 1 + e^{iy} e^{t} - 2 + e^{iy} e^{t - e^{iy} \cdots}$$

We have bumped the problem term e^{iy} into the exponent just to simplify some of the problems typographically. Our recursion looks a tad different in this form, but it's simpler. The function ψ_y,

$$\psi_y(t+1) = e^{t} + e^{iy} \psi_y(t)$$

And we can obtain the crude estimate,

$$|\psi_y(t+1)| \geq e^{t} - |\psi_y(t)|$$

Now this estimate gets its worse when $y = \pi$ where,

$$\psi_\pi(t+1) = e^{t} - \psi_\pi(t)$$

So, the heuristic is, if we could construct a lower bound for this case we could bound from below the other cases. So the manner of proof will involve working firstly on the case where $y = \pi$. As a preliminary remark, if the reader hasn't noticed: When $y = \pi$ the function $\psi_\pi(t)$ is real valued and strictly positive.

To this end,

$$\psi(t) = \psi_\pi(t) = \mathop{\Omega}_{j=1}^{\infty} e^{t-j-z} \bullet z\Big|_{z=0}$$

Where we notice that all we've really done is swap the sign in the z term from our definition of $\phi(t)$. A subtle difference, which as Dorothy would put it–I don't think we're in Kansas anymore, Toto. It is hereupon we have to fiddle with our notation a bit.

The following will be a result in real-analysis, as such we don't need to consider complex z. Let $z = x \in [0,1]$ and write,

$$\psi_m(t,x) = \mathop{\Omega}_{j=1}^{2m+1} e^{t-j-x} \bullet x$$

The reader should note the $2m+1$ in the upper index of our composition. And they should note that $\psi_m(t,0)$ converges uniformly to a continuously differentiable function as $m \to \infty$. Where now our recursion is a tad different but,

$$\psi_{m+1}(t,x) = \psi_m(t, e^{t} - 2m - 1 - e^{t-2m-2-x})$$

We've doubled up on the exponentials here because we want to use that,

$$e^{t+1-e^{t-x}}$$

Starts to look like $e^{-\delta e^{t}}$ for some $\delta > 0$; and this has very fast convergence to zero $t \to \infty$. But additionally,

$$e^{t-2m-1-e^{t-2m-2-x}}$$

Tends to zero like $\mathcal{O}(e^{-m})$ as $m \to \infty$ as well. We've chosen the upper index $2m+1$ because each of these $\psi_m(t)$ grow exponentially. This is really the entire trick of the following theorem. Observe the sequence,

$$\begin{aligned}
\psi_0(t,x) &= e^{t-1-x} \to \infty \text{ as } t \to \infty \\
\psi_1(t,x) &= e^{t-1-e^{t-2-e^{t-3-x}}} \to \infty \text{ as } t \to \infty \\
&\vdots \\
\psi_m(t,x) &= e^{t-1-^{\dots(2\text{m}+1\,times)\dots^{e^{t-2m-1-x}}}} \to \infty \text{ as } t \to \infty
\end{aligned}$$

This leads us to a more regular idea of what our infinite compositions behave like; which is,

$$\psi_m(t,x) \to \psi(t) = \lim_{m\to\infty} \psi_m(t,0)$$

Which means, regardless of what x-value we choose the result still converges to the same limit as when we set $x = 0$. We can think of this as iterations converging to a fixed point locally in x. This is perhaps the best way to think of it. We will frame this in the more general tone of complex analysis.

Theorem 4.1. *The function $\phi(s)$ can be represented as,*

$$\phi(s) = \mathop{\Omega}_{j=1}^{\infty} e^{s-j+z} \bullet z$$

for all $z \in \mathbb{C}$.

Proof. The limit function,

$$\begin{aligned}
\phi(s,z) &= \mathop{\Omega}_{j=1}^{\infty} e^{s-j+z} \bullet z \\
&= \mathop{\Omega}_{j=1}^{m} e^{s-j+z} \bullet \mathop{\Omega}_{j=m+1}^{\infty} e^{s-j+z} \bullet z \\
&= \mathop{\Omega}_{j=1}^{m} e^{s-j+z} \bullet \epsilon \bullet z \text{ per Lemma 2.1} \\
&\to \mathop{\Omega}_{j=1}^{\infty} e^{s-j+z} \bullet z\Big|_{z=0} \text{ as } m \to \infty
\end{aligned}$$

Because as $m \to \infty$ we get $\epsilon \to 0$. ∎

With that we can obtain lower bounds on ψ.

Theorem 4.2. *Let,*

$$\psi_\pi(t) = \underset{j=1}{\overset{\infty}{\Omega}} e^{t-j-z} \bullet z$$

Then for all $\epsilon > 0$ there exists some $T > 0$ such that, for all $t > T$:

$$\psi_\pi(t) \geq 1 + \epsilon$$

Proof. Let $0 < x < 1$ and $t \in \mathbb{R}$ and,

$$\psi_m(t,x) = \underset{j=1}{\overset{2m+1}{\Omega}} e^{s-j-x} \bullet x$$

Now, for some $m > M$ we can take $0 < x < 1$ and $2M + 3 > t > T$ such that $\inf_{0<x<1} |\psi_m(t,x)| > 1 + \epsilon$ for arbitrary ϵ. From this,

$$|\psi_{m+1}(t,x)| = |\psi_m(t, e^{t - 2m - 1 - e^{t-2m-2-x}})| > 1 + \epsilon$$

Because, $0 < e^{t - 2m - 1 - e^{t-2m-2-x}} < 1$ for large enough $m > M$ and $2M + 3 > t > T$; where the use of the upper bound of t is only temporary, to assure the term $e^{t - 2m - 1 - e^{t-2m-2-x}}$ doesn't grow past 1. Since we're growing $M \to \infty$ it disappears when we limit $m \to \infty$. Therefore, by induction; for all $m > M$,

$$\psi_m(t,x) > 1 + \epsilon$$

And necessarily, $\lim_{m\to\infty} \psi_m(t,x) \geq 1 + \epsilon$.

□

The second result we get from this, is more of a corollary than a lemma, but as it's dire to the following proofs–we write it as a Lemma.

Lemma 4.3. *There exists large enough T such for all $t \geq T$,*

$$\psi'_\pi(t) > 0$$

Proof. The function $\psi_\pi(t)$ has to grow. If eventually $\psi(t_0) \geq t_0$, then $\psi(t_0+1) \leq e^{t_0-t_0} \leq 1$, which is a contradiction. Thereby $\psi_\pi(t) < t$, for large enough $t \geq T$. We want to show that $\psi_\pi(t)/t < 1 - \rho$ for large enough $t \geq T$. This can only not happen if $\limsup \psi_\pi(t)/t \to 1$. But as t grows so does ψ_π and the moment it clusters near t it tends to 1. Which is a contradiction. Then necessarily $\psi_\pi(t)/t < 1 - \rho$.

But then, by a L'Hôpital kind of argument,

$$\psi'_\pi(t) < 1 - \rho$$

And necessarily,

$$\psi'_\pi(t+1) = \psi_\pi(t+1)(1-\psi'_\pi(t)) > \rho\psi_\pi(t+1) > 0$$

□

To derive the result for the general case, we need to run a cranking mechanism. We want to show that this function ψ_π acts as a lower bound to ψ_y. But, this can't happen everywhere; we get arbitrarily close to 1; and ψ_π stays past 1. What we want to say, is as t gets arbitrarily large, the domain in which it is a minimum increases. And here is where our brains need to turn on. We write the following theorem:

Theorem 4.4. *The function $\psi_y(t) = e^{-iy}\phi(t+iy)$ satisfies,*

$$\frac{d}{dy}|\psi_y(t)| = 0 \text{ if } y = k\pi \text{ for } k \in \mathbb{Z}$$

Proof. Again, we can restrict ourselves to $0 \le y \le \pi$. All we need to show is the result for $y = 0, \pi$. First of all,

$$\lim_{t\to-\infty} \frac{|\psi_y(t)|}{e^{t-1}} = 1$$

and therefore,

$$\frac{d}{dy}|\psi_y(t)| = 0 \text{ as } t \to -\infty$$

Now as we continue this iteration,

$$\begin{aligned} \frac{d}{dy}|\psi_y(t+1)| = \frac{d}{dy}|e^{t+e^{iy}\psi_y(t)}| &= 0 \\ \frac{d}{dy}\Re(e^{iy}\psi_y(t)) = \frac{d}{dy}\big(\cos(y)\Re\psi_y(t) - \sin(y)\Im\psi_y(t)\big) &= 0 \\ \sin(y)\Re\psi_y(t) + \cos(y)\frac{d}{dy}\Re\psi_y(t) - \cos(y)\Im\psi_y(t) - \sin(y)\frac{d}{dy}\Im\psi_y(t) &= 0 \end{aligned}$$

Now when $y = 0, \pi$, we know $\sin(y) = 0$ and $\Im\psi_y(t) = 0$ (ψ_y is real valued when $y = 0, \pi$). So we only care about,

$$\begin{aligned} \frac{d}{dy}\Re\psi_y(t) = \frac{d}{dy}\Big(|\psi_y(t)|\cos(\arg\psi_y(t))\Big) &= 0 \\ \Big(\frac{d}{dy}|\psi_y(t)|\Big)\cos(\arg\psi_y(t)) + |\psi_y(t)|\sin(\arg\psi_y(t))\frac{d}{dy}\arg\psi_y(t) &= 0 \end{aligned}$$

Again, $\arg(\psi_y) = 0$ when $y = 0, \pi$. So we've reduced it to, if we know $\frac{d}{dy}|\psi_y(t)| = 0$ it implies $\frac{d}{dy}|\psi_y(t+1)| = 0$ when $y = 0, \pi$. But we know $\frac{d}{dy}|\psi_y|$ approaches zero as $t \to -\infty$. □

Now the essential idea is that when,

$$\frac{d}{dy}|\psi_y| = 0$$

For $y = k\pi$, the even values $y = 0, \pm 2\pi, \pm 4\pi, ...,$ correspond to maxima. This is essentially where ϕ is super-exponential. It grows faster than any tetration. And in line, when $y = \pm\pi, \pm 3\pi, \pm 5\pi, ...$ is an odd multiple of π it is a local minima. This means we can bound from below in a neighborhood of these points.

This paper would be much simpler if ψ_π was a global minimum, but it is not. We'd like to thank Sheldon Levenstein, again, for pointing this out. In the first iteration we had made this mistake. We get other maxima and other minima. This is because $|\phi(s)| = 1$ happens infinitely often as we increase $\Re(s)$.

What we want to do is look at the maxima nearest the minima $y = \pi$ and show, as t gets larger, it gets closer towards 0. Which is to say, all the ups and downs, and sharp drops towards $|\phi(s)| = 1$ happen closer and closer towards the real-line. To write this mathematically, for the numbers y_k,

$$\frac{d}{dy}\Big|_{y=y_k}|\psi_y(t)| = 0$$

Then, as we increase t, the value $y_k \to 0$. This corresponds to the other, unwanted, maxima/minima appearing arbitrarily near the real line and clustering there as $t \to \infty$. The following proof is quite exhausting; I apologize in advance to the reader.

Theorem 4.5. *For all solutions $0 < y_k < \pi$ to the equation,*

$$\frac{d}{dy}\Big|_{y=y_k}|\psi_y(t)| = 0$$

For all $\delta > 0$, there exists T such that, for all $t > T$, the solutions y_k satisfy $0 < y_k < \delta$.

Proof. Start by taking the solution $0 < y(t) < \pi$ which satisfies,

$$\frac{d}{dy}\Big|_{y=y(t)}|\psi_y(t)| = 0$$

Where $y(t)$ is the nearest local maximum to the minimum at $y = \pi$. All other minima and maxima y_k must satisfy $y_k < y(t)$. The function y is at least piece-wise continuously differentiable in t by the implicit function theorem. We are trying to show that $y \to 0$ as $t \to \infty$. Begin by observing,

$$\frac{d}{dy}\big(\cos(y)\Re\psi_y(t-1) - \sin(y)\Im\psi_y(t-1)\big) = 0$$

Which is derived from the functional equation,

$$|\psi_y(t)| = e^{t + \cos(y)\Re\psi_y(t-1) - \sin(y)\Im\psi_y(t-1)}$$

The general idea is that solutions to this equation attract towards 0 as $t \to \infty$. This is because there is an attracting solution at $y = 0$; where as at $y = \pi$ this solution is repelling. This equation can be rewritten:

$$\begin{aligned}
0 &= \frac{d}{dy}\Big(|\psi_y(t-1)|\big(\cos(y)\cos(\arg\psi_y(t-1)) - \sin(y)\sin(\arg\psi_y(t-1))\big)\Big) \\
0 &= \frac{d}{dy}\big(|\psi_y(t-1)|\cos(y + \arg\psi_y(t-1))\big) \\
0 &= \big(\frac{d}{dy}|\psi_y(t-1)|\big)\cos(y+\arg\psi_y(t-1)) + \\
&\quad |\psi_y(t-1)|\sin(y+\arg\psi_y(t-1))\big(1 + \frac{d}{dy}\arg\psi_y(t-1)\big)
\end{aligned}$$

We will split into cases here. Now, as $t \to \infty$, the function $\frac{d}{dy}|\psi_y(t-1)| \to 0$ as $y(t) - y(t-1) \to 0$. Assuming otherwise, that $y(t) - y(t-1) \not\to 0$; this case will be solved in the second half of the proof. But momentarily assume that $y(t) - y(t-1) \to 0$. Using the following.

$$|\psi_y(t-1)|\sin(y+\arg\psi_y(t-1))\big(1+\frac{d}{dy}\arg\psi_y(t-1)\big) \to 0$$

This can only occur if $y \to 0$ or $y \to \pi$; which are solutions to this equation for all t. The only other way is if $\frac{d}{dy}\arg\psi_y(t-1) \to -1$, which means $\arg\psi_y(t) \to -t$; I hope the reader can see how preposterous this is. All that's left is to show that $y \not\to \pi$, and that $\lim_{t\to\infty} y$ converges. We must show that no local maxima appear near $y = \pi$ as $t \to \infty$.

By running a cranking mechanism,

$$\begin{aligned}
\frac{d}{dy}\Big|_{y=y(t)}\psi_y(t) &= \frac{d}{dy}\Big(|\psi_y(t)|e^{i\arg\psi_y(t)}\Big) \\
&= \big(\frac{d}{dy}|\psi_y(t)|\big)e^{i\arg\psi_y(t)} + |\psi_y(t)|e^{i\arg\psi_y(t)}\frac{d}{dy}i\arg\psi_y(t) \\
&= \psi_y(t)\frac{d}{dy}i\arg\psi_y(t) \\
\frac{\frac{d}{dy}\psi_y(t)}{\psi_y(t)} &= \frac{d}{dy}i\arg\psi_y(t) \\
\frac{d}{dy}\log\psi_y(t) &= \frac{d}{dy}i\arg\psi_y(t) \\
\log\psi_y(t) &= i\arg\psi_y(t) + C \\
\psi_y(t) &= Ae^{i\arg\psi_y(t)} \\
|\psi_y(t)| &= A
\end{aligned}$$

This implies our minima/maxima arise when the absolute value of $|\psi_y(t)|$ is constant–it occurs on a level set. But, $\psi_\pi(t)$ grows arbitrarily large as $t \to \infty$, and $\psi'_\pi(t) > 0$ for large enough $t > T$, and $\psi_\pi(t)$ a local minima; so in a neighborhood $|\psi_y(t)| \geq \psi_\pi(t)$. But if this is so $|\psi_y|$ can't remain constant while $y \to \pi$ as $t \to \infty$–because $\psi_\pi(t) \to \infty$. Therefore, as $t \to \infty$, the function $y(t) \not\to \pi$–it must tend to 0.

Now we've played a little fast and loose here, because these $y_k(t)$ may intersect; and necessarily, only locally $|\psi_y(t)|$ has a constant absolute value; but it implies each individual solution must tend to 0 if it tends to anything. Showing they necessarily tend to something will complete the proof.

Now, we've done the above assuming that $y(t) - y(t-1) \to 0$. Assume $y(t) - y(t-1) \not\to 0$, and we'll go by cases. In the first case $|\psi_{y(t)}(t)| = A$ is globally constant, and is the nearest maximum to $\psi_\pi(t)$, A is always bounded below by ψ_π–forcing y to 0.

The second case, and more difficult case, is that $|\psi_{y(t)}(t)|$ is only locally constant; and is made of piecewise functions, where if we were to continue each piece it tends to 0. Now,

$$\begin{aligned} \frac{d}{dt}|\psi_{y(t)}(t)| &= \frac{d}{dy}|\psi_{y(t)}(t)|\frac{dy}{dt} + \frac{\partial}{\partial t}|\psi_{y(t)}(t)| \\ &= \frac{\partial}{\partial t}|\psi_{y(t)}(t)| \end{aligned}$$

Therefore the total change in t equals the partial change in t. But $|\psi_{y(t)}(t)|$ is locally constant. So,

$$\frac{\partial}{\partial t}|\psi_{y(t)}(t)| = 0$$

But if this happens it implies,

$$|\psi_{y(t)}(t+\delta)| = |\psi_{y(t)}(t)| + \mathcal{O}(\delta^2)$$

Which can be better written,

$$|\psi_{y(t+\delta)}(t)| = |\psi_{y(t)}(t)| + \mathcal{O}(\delta^2)$$

As such, as $t \to \infty$ and $\delta \to 0$,

$$\frac{|\psi_{y(t+\delta)}(t)| - |\psi_{y(t)}(t)|}{\delta} = \mathcal{O}(\delta) \to 0$$

This implies $\frac{dy}{dt} \to 0$. Which implies $y(t+\delta) - y(t) \to 0$, and with that, the result. □

After this proof, it gives us a way of saying that $\psi_\pi(t)$ is a global minima on the strip $\delta < y < 2\pi - \delta$, $t > T$. Which is to mean $\frac{d}{dy}|\psi_y(t)| \neq 0$ in this strip except at $y = \pi$. So we know instantaneously that $|\psi_y(t)| \geq \psi_\pi(t)$ in this strip. Where $\delta \to 0$ as $T \to \infty$. We can compress this into the following theorem:

Theorem 4.6. *For all $\delta > 0$ there exists T such for all $\delta < y < 2\pi - \delta$ and $t > T$:*

$$|\psi_y(t)| \geq \psi_\pi(t)$$

From henceforth, we'll write $\lambda = \frac{1}{1+\epsilon} < 1$. And we can see that; in this strip,

$$\frac{1}{|\phi(t+iy)|} \leq \lambda < 1$$

5 Showing τ is holomorphic

To show τ is holomorphic requires a lot of careful observations. Firstly, for all $\delta > 0$ there exists some $T \in \mathbb{R}^+$ such that for all $t = \Re(s) \geq T$ and $\delta \leq \Im(s) \leq 2\pi - \delta$ the value $|\phi(s)| \geq 1 + \epsilon$. Therefore, of this form, if $\lambda = \frac{1}{1+\epsilon}$,

$$\frac{1}{\prod_{k=1}^{n} |\phi(s+k)|} \leq \lambda^n$$

The nature of which is that this product geometrically converges to 0. It is not quite super-exponential–like what happens on the real-line, but nonetheless we've done fairly well for ourselves. This is a very convenient bound. Therefore when we look at our τ functions; we can readily apply Banach's fixed point theorem. Looking at the sequence of τ's,

$$\begin{aligned}
\tau_0(s) &= 0\\
\tau_1(s) &= s\\
\tau_2(s) &= s + \log(1 + \frac{s+1}{\phi(s+1)})\\
\tau_3(s) &= s + \log\big(1 + \frac{s+1+\log(1+\frac{s+2}{\phi(s+2)})}{\phi(s+1)}\big)\\
&\vdots\\
\tau_n(s) &= s + \log(1 + \frac{\tau_{n-1}(s+1)}{\phi(s+1)})
\end{aligned}$$

Where we can think of this similarly to the iteration,

$$
\begin{aligned}
\mu_0(s) &= 0\\
\mu_1(s) &= s\\
\mu_2(s) &= s+\log(1+\lambda(s+1))\\
\mu_3(s) &= s+\log\big(1+\lambda(s+1+\log(1+\lambda(s+2))\big)\\
&\vdots\\
\mu_n(s) &= s+\log(1+\lambda\mu_{n-1}(s+1))
\end{aligned}
$$

Now μ_n certainly converges for compact sets way off in the right-half plane; and τ_n is no different. If anything, our convergence is better. In fact, we can *almost* sandwich τ_n between two of these μ_n. Write:

$$
\begin{aligned}
\mu'_0(s) &= 0\\
\mu'_1(s) &= s\\
\mu'_2(s) &= s+\log(1+\frac{s+1}{\phi(t+1)})\\
\mu'_3(s) &= s+\log\big(1+\frac{s+1+\log(1+\frac{s+2}{\phi(t+2)})}{\phi(t+1)}\big)\\
&\vdots\\
\mu'_n(s) &= s+\log(1+\frac{\mu'_{n-1}(s+1)}{\phi(t+1)})
\end{aligned}
$$

Then, it almost looks like we could get that,

$$|\mu'_n(s)| \leq |\tau_n(s)| \leq |\mu_n(s)|$$

Because,

$$\frac{1}{\phi(t+1)} \leq \frac{1}{|\phi(s+1)|} \leq \lambda$$

But our use of iterated complex log's turned the author off this approach. But it does make for good reference as an intuition. We get something of the same species of this. Instead we'll grind the gears of τ_n more.

The only problem we could have is if $\tau_n(s) = -\phi(s)$. For the moment, forgo this possibility; even better assume we have the inequality $|\frac{\tau_n(s)}{\phi(s)}+1| > \kappa$. It is always possible to choose $\Re(s) > T$ and $\delta < \Im(s) < 2\pi - \delta$ where this is true. This is because if $\phi(s) = -s$, then $\phi(s+1) = 1$; but this is impossible in our compact strip, per the last section. Continuing the iteration of τ, this property continues. Observationally, remember that $\tau_n(s) = s + \mathcal{O}(\log(s))$ and this continues in the limit.

Now, we haven't said much about λ so far. But we can choose T as large as we want so that λ is as small as possible. Take λ small enough such that $\frac{\lambda}{\kappa} = \mu < 1$. Using a multiplier argument,

$$\begin{aligned}
|\tau_{n+1}(s) - \tau_n(s)| &\leq |\log(1 + \frac{\tau_n(s+1)}{\phi(s+1)}) - \log(1 + \frac{\tau_{n-1}(s+1)}{\phi(s+1)})| \\
&\leq \frac{1}{\kappa|\phi(s+1)|}|\tau_n(s+1) - \tau_{n-1}(s+1)| \\
&\leq \mu|\tau_n(s+1) - \tau_{n-1}(s+1)|
\end{aligned}$$

Because $|\log(1+z_1) - \log(1+z_2)| \leq \frac{1}{\kappa}|z_1 - z_2|$ so long as $|1+z_1|, |1+z_2| \geq \kappa$. Remembering that $\tau_0(s) = 0$ and $\tau_1(s) = s$,

$$|\tau_{n+1}(s) - \tau_n(s)| \leq \mu^n |s+n|$$

The telescoping argument applies again. For all $\epsilon > 0$, pick $m, n > N$ such that, for a compact set $\mathcal{K} = \{s \in \mathbb{C} : \delta \leq \Im(s) \leq 2\pi - \delta, T \leq \Re(s) \leq T'\}$ the inequality,

$$\sum_{j=n}^{m-1} \mu^j ||s+j||_{\mathcal{K}} < \epsilon$$

Then,

$$\begin{aligned}
||\tau_m(s) - \tau_n(s)||_{\mathcal{K}} &\leq \sum_{j=n}^{m-1} ||\tau_{j+1}(s) - \tau_j(s)||_{\mathcal{K}} \\
&\leq \sum_{j=n}^{m-1} \mu^j ||s+j||_{\mathcal{K}} \\
&< \epsilon
\end{aligned}$$

This implies $\tau(s)$ is holomorphic on this set. The next step is to extend τ's domain of holomorphy, from these boxes within a strip to a maximal domain. Using the functional equation, since,

$$\phi(s+1) + \tau(s+1) = e^{\phi(s) + \tau(s)}$$

By taking logarithms τ can be extended. The only difficulty we could have is if $\tau(s+1) = -\phi(s+1)$; wherein no logarithm can be taken and τ must have a singularity. We need to prove this is only possible when $\Im(s) = 2\pi ik$. But for the moment, we know that this still defines τ on $0 < \Im(s) < 2\pi$ upto a nowhere dense set; i.e: it is locally holomorphic almost everywhere. Further; the singularities s_n in which $\tau(s_n + 1) = -\phi(s_n + 1)$ can only accumulate at $-\infty$. Each compact box in $0 < |\Im(s)| < 2\pi$ can only have a finite number of singularities. This allows us to show that,

$$\mathcal{F}(s) \text{ is holomorphic on } \mathbb{C}/\mathcal{L}$$

With this we state our titular theorem. Where what is left to prove is that τ is actually holomorphic on $0 < \Im(s) < 2\pi$; proving this one note about singularities is quite exhaustive of complex analysis.

Theorem 5.1 (The Tetration Existence Theorem). *The tetration function $\mathcal{F}$ from Theorem 3.1 can be analytically continued to a function which satisfies the following properties:*

1. *$\mathcal{F}$ is holomorphic for $\Im(s) \neq 2\pi i k$ for $k \in \mathbb{Z}$*
2. *$\mathcal{F}(s+1) = e^{\mathcal{F}(s)}$*
3. *$\mathcal{F}(0) = 1$*
4. *$\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}^+$, and $\mathcal{F}' : \mathbb{R}^+ \to \mathbb{R}^+$–$\mathcal{F}$ is continuously differentiable.*

$\mathcal{F}$ is best described; for some $\omega \in \mathbb{R}$; as,

$$\lim_{n\to\infty} \log\log ...(n\, times)... \log \phi(s + \omega + n) = \mathcal{F}(s) = e \uparrow\uparrow s$$

Proof. We have to prove that $\phi(s) \neq -\tau(s + 2\pi i k)$ for all $0 < \Im(s) < 2\pi$ from the result that $\phi(s) \neq -s + i\ell$ for $\delta < \Im(s) < 2\pi - \delta$ for large enough $\Re(s) > T$ and $\ell \in \mathbb{R}$. To first show we know this,

$$\phi(s + 1) = e^{s+\phi(s)} = e^{i\ell}$$

But then $|\phi(s + 1)| = 1$, which is a contradiction to our bounds–so $\phi(s) \neq -s + i\ell$; in fact it must stay in a neighborhood away from these points, as $\Re(s) \to \infty$. Each $\tau_n \to s$ as $\Re(s) \to \infty$; alors, for large enough $\Re(s)$ our compact sum argument works, and we have a function:

$$e \uparrow\uparrow s : \{0 < \Im(s) < 2\pi\}/\mathcal{L} \to \mathbb{C}$$

For some nowhere dense set $\mathcal{L}$. We have to show that $\mathcal{L} = \emptyset$. This will require a reframing of our problem. We'll have to better reaccustom ourselves with the iteration.

* * *

Start with the recursion of τ,

$$\tau(s) = s + \log(1 + \frac{\tau(s+1)}{\phi(s+1)})$$

So the problem reduces to showing that,

$$\Re\log(1 + \frac{\tau(s+1)}{\phi(s+1)}) \neq 0$$

Which means we have to show,

$$1 + \frac{\tau(s+1)}{\phi(s+1)} \neq e^{i\ell}$$

We know this is true for large enough $\Re(s) > T$ and $\delta < \Im(s) < 2\pi - \delta$. And so, we must show it continues to hold as $\Re(s)$ shrinks. This result can be more aptly written,

$$|\phi(s+1) + \tau(s+1)| \neq |\phi(s+1)|$$

Now, as $\Re(s) \to -\infty$ we can assure this doesn't happen because τ will tend to L a non-zero fixed point of e^s and $\phi \to 0$. It is here-upon where we have to take an inverted form of our recursion. This is a tad tricky, but perfectly doable.

We've been solving for τ based on its behaviour as $\Re(s) \to \infty$–it is now time to solve for τ based on its behaviour as $\Re(s) \to -\infty$. Start by writing,

$$e \uparrow\uparrow s = \exp\exp\cdots(n\, times)\cdots\exp\phi(s+\omega-n) + \tau(s+\omega-n)$$

For large $n > N$ we know that $\tau(s+\omega-n) \sim L + Aq^n$. To refresh,

$$\tau(s+\omega-n) = \log\log\cdots(n\,\text{times})\cdots\log e\uparrow\uparrow s - \phi(s+\omega-n)$$

Of which $\log\log\cdots(n\,\text{times})\cdots\log\xi \to L$ geometrically. Iterates of functions about an attractive fixed point tend to that fixed point geometrically [4]. The iterated logarithm of complex numbers tend to an attractive fixed point (almost everywhere). And $\phi(s+\omega-n) \to 0$ as $n \to \infty$.

So we can almost write,

$$A_n(s) = \exp\exp\cdots(n\,\text{times})\cdots\exp\phi(s+\omega-n) + L$$

Which, in the limit, we're trying to get that,

$$\lim_{n\to\infty} A_n(s) = e\uparrow\uparrow s$$

Or at least, something close to this. This converges somewhere in the strip $0 < \Im(s) < 2\pi$, by the relation,

$$e\uparrow\uparrow s = \exp\exp\cdots(n\,\text{times})\cdots\exp\phi(s-n) + \tau(s-n)$$

And recalling $\tau(s-n) \to L$ somewhere; i.e: upto a nowhere dense set.

* * *

The goal of this proof is to show this limit converges everywhere; and not that it converges to τ, but that it is normal to τ. Which is to say, since this iteration converges well, and is close to τ; τ too, must be holomorphic. And consequently we bypass all of these inequality problems. If τ is holomorphic for $\Re(s) < -T$ for large T and $0 < \Im(s) < 2\pi$; by the functional equation it's holomorphic as $\Re(s)$ grows:

$$\tau(s+1) = e^{\phi(s)+\tau(s)} - \phi(s+1)$$

So to begin, and paint a picture of what we're going to do: the function A_n satisfies the recursion,

$$\begin{aligned}
A_n(s+1) &= \exp\exp\cdots(n\,\text{times})\cdots\exp\Big(\exp(s-n+\phi(s-n))+L\Big)\\
&= \exp\exp\cdots(n\,\text{times})\cdots\exp\Big[\big(L\exp\phi(s-n)\big)\big(1+\frac{e^{s-n}}{L}\big)\Big]\\
&= \exp\exp\cdots(n\,\text{times})\cdots\exp\Big[\exp\big(\phi(s-n)+L\big)\big(1+\frac{e^{s-n}}{L}\big)\Big]\\
&= A_{n+1}(s+1)+\mathcal{O}(\lambda^n)
\end{aligned}$$

For some $0 < \lambda < 1$. But this is a tad loose. Instead, consider the functions:

$$\begin{aligned}
\upsilon_0(s) &= L\\
\upsilon_1(s) &= \exp(\phi(s-1)+L)-\phi(s)\\
\upsilon_2(s) &= \exp\big[\phi(s-1)+\upsilon_1(s-1)\big]-\phi(s)\\
&\vdots\\
\upsilon_n(s) &= \exp\big[\phi(s-1)+\upsilon_{n-1}(s-1)\big]-\phi(s)
\end{aligned}$$

Now we pull out more infinite compositions. If we write,

$$\lim_{n\to\infty}\upsilon_n = \upsilon(s) = \Omega_{j=1}^{\infty}\exp[\phi(s-j)+z]-\phi(s+1-j)\bullet z\Big|_{z=L}$$

Then,

$$\phi(s+1)+\upsilon(s+1) = e^{\phi(s)+\upsilon(s)}$$

Of course, this is tetration's functional equation–υ looks a lot like τ. This expression needs to be studied carefully though. To ensure this infinite composition converges, and where it converges, we have to analyze a certain sum better. For arbitrary compact sets $\mathcal{P} \subset \mathbb{C}$; and an arbitrary sequence $z_j \to L$ and $\sum_j |z_j - L| < \infty$,

$$\sum_{j=1}^{\infty} \left|\left| e^{\phi(s-j)+z_j} - \phi(s+1-j) - L \right|\right|_{s\in\mathcal{P}} < \infty$$

This implies our infinite composition defining υ converges everywhere for $s \in \mathbb{C}$; where $\sum_j |\upsilon(s-j) - L| < \infty$ (see Appendix A). Aussi, $z_j = L$ is the trivial summable sequence–$z_j = \tau(s-j)$ is a non-trivial summable sequence. This hints to us one thing, since $\sum_{j=1}^{\infty} |\tau(s-j) - L| < \infty$ almost everywhere in the strip $0 < \Im(s) < 2\pi$. And in doing so, $|\upsilon(s) - \tau(s)| < \delta$ for large enough $\Re(s) < -T$ and $0 < \Im(s) < 2\pi$. This can't happen if τ has singularities, because υ is holomorphic; implying τ is holomorphic here, and consequently is holomorphic for all $0 < \Im(s) < 2\pi$. □

In Conclusion

We have sketched the construction of a tetration function, but there lies a more daunting problem. It is necessary to describe this tetration function more acutely. The author is unsure of the behaviour $e \uparrow\uparrow t + iy$ as $y \to \infty$, nor of which fixed points of e^s the function $e \uparrow\uparrow t + iy$ tends to as $t \to -\infty$ depending on y; though each strip $2\pi k < \Im(s) < 2\pi(k+1)$ corresponds to a different fixed point.

Just as well, he knows no method of constructing a uniqueness criterion for this function–certainly not one depending entirely on its behaviour on the real positive line. Though, he suspects the *eventual* monotonicity of each derivative may suffice–perhaps hinging on some extraneous growth condition. We haven't proved this here; the author considers what he has an insufficient theorem. As Sheldon Levenstein pointed out, it's very likely that $\mathcal{F}$ is not analytic on $\mathbb{R} + 2\pi ik$–and we've yet to prove its more than once differentiable.

It would be nice to have monotonicity on all of $\mathbb{R}^+$ for all $\mathcal{F}^{(n)}$–not just eventually for large enough $t > T$, if indeed $\mathcal{F}$ is $\mathcal{C}^\infty$. *Can the reader prove this: For all n there exists T such for $t > T$ the function $\mathcal{F}^{(n)}(t) > 0$? If they can, can they prove the stronger result that $\mathcal{F}^{(n)}(t) > 0$ for all $t \in \mathbb{R}^+$? The author would be eternally grateful.*

We do not know much about this solution qualitatively. Other than this solution shouldn't agree with Kneser's solution which is holomorphic everywhere except $(-\infty, -2]$. Does this solution agree with any other solution? There are no zeroes of $e \uparrow\uparrow s$ in its domain of holomorphy, and consequently there are no zeroes of $\frac{d}{ds} e \uparrow\uparrow s$. This should allow us to construct a sequence of inverse function $\text{slog}_k(s)$, holomorphic almost everywhere on $\mathbb{C}$ sending to $2\pi k < \Im(s) < 2\pi(k+1)$. Can we describe what this looks like? Where are the branch cuts?

The author doesn't know a great deal about these questions; but he hopes to find out; and hopes others do as well.

A Better Normality Conditions

We'd like to briefly point out a subtlety of the construction of infinite compositions. Taking the construction of $\phi(s)$ we sacrificed much generality. If we wanted to add twenty or so pages we could've added a more general construction. But nonetheless, the summation condition, which allowed for the construction of $\phi(s)$; namely,

$$\sum_{j=1}^{\infty} ||e^{s-j+z}||_{\mathcal{S},\mathcal{K}} < \infty$$

For arbitrary compact sets $\mathcal{K},\mathcal{S} \subset \mathbb{C}$; can be drastically weakened. If we were to only use that, for any sequence $z_j \to 0$ in which $\sum_j |z_j| < \infty$,

$$\sum_{j=1}^{\infty} ||e^{s-j+z_j}||_{\mathcal{S}} < \infty$$

We can still construct ϕ using only this. So when we write,

$$\upsilon(s) = \mathop{\Omega}_{j=1}^{\infty} e^{\phi(s-j)+z} - \phi(s+1-j) \bullet z\Big|_{z=L}$$

We know this is holomorphic because, for all summable sequences $L - z_j$,

$$\begin{aligned}
& \sum_{j=1}^{\infty} \Big|\Big|e^{\phi(s-j)+z_j} - \phi(s+1-j) - L\Big|\Big|_{s\in\mathcal{P}} \\
= & \sum_{j=1}^{\infty} \Big|\Big|e^{\phi(s-j)+L+z_j-L} - \phi(s+1-j) - L\Big|\Big|_{s\in\mathcal{P}} \\
= & \sum_{j=1}^{\infty} \Big|\Big|Le^{\phi(s-j)+z_j-L} - \phi(s+1-j) - L\Big|\Big|_{s\in\mathcal{P}} \\
< & \sum_{j=1}^{\infty} \Big|L(1+|L-z_j|+Cq^j) - Cq^j - L\Big| \\
\leq & \sum_{j=1}^{\infty} C|L-z_j| + Cq^j \\
< & \infty
\end{aligned}$$

Which amounts to nothing more than a more general theorem than the construction of ϕ–doing the exact same motions. This implies,

$$\mathop{\Omega}_{j=1}^{n} e^{\phi(s-j)+z} - \phi(s+1-j) \bullet z\Big|_{z=z_n}$$

Converges uniformly as $n \to \infty$ if $\sum_j |z_j - L| < \infty$. Setting $\tau(s-j) = z_j$, this sequence qualifies. It's summable like this almost-everywhere, and since the limit will converge everywhere; the limit is holomorphic here.

The rest is left to you.

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