1 Showing infinite differentiability

The infinite differentiability of τ is a fun exercise in compositional analysis. We really have to pull out all the stops; the idea is inductive; and the proofs are mostly just lemmas. It's a good exercise in bounding compositions with sums. It's a good exercise in exotic iterative procedures, and the quickness of Ω-notation. For that, the work to follow may seem terse; or abnormal. The author will be diligent in explaining every conclusion to the best of his ability. But he'd like it if the reader could appreciate the malleability of the work.

Since ϕ is so well behaved, and $\frac{1}{\phi}$ is smaller than any iterate of the exponential, we can make cake work of this. If we take each τ_m then,

$$|\tau_m(t) - t| \leq \sum_{j=1}^{m} \frac{t+j}{\prod_{c=1}^{j} \phi(t+c)}$$

For all $t \geq T$. This relationship is also satisfied for all derivatives. We'll write this a tad simpler,

$$|\frac{d^l}{dt^l}\tau_m(t) - \frac{d^l}{dt^l}t| \leq A_{lm}e^{-e^t}$$

This can be shown by induction on m for each l. To visualize the argument, use a sequence of L'Hôpital/Bernoulli comparisons, and prove them inductively on m using the functional equation of $\tau_m = t + \log(1 + \frac{\tau_{m-1}(t+1)}{\phi(t+1)})$,

$$\begin{aligned}
\frac{\tau_m(t)}{t} &= 1 + \mathcal{O}(e^{-e^t}) \\
\tau_m'(t) &= 1 + \mathcal{O}(e^{-e^t}) \\
\tau_m''(t) &= \mathcal{O}(e^{-e^t}) \\
&\vdots \\
\tau_m^{(l)}(t) &= \mathcal{O}(e^{-e^t})
\end{aligned}$$

This isn't far fetched considering that $\frac{1}{\phi(t)} \leq \frac{1}{\exp^{\circ n}(t)}$, and the decay is uniform as we take derivatives (in fact it's more than Schwartz). This sequence of bounds is pretty weak; but stronger would be over-kill. As they stand, how strong these bounds are, isn't so much needed either; but it reduces some of the complexity of the problem.

Continuing:

$$\sum_{c=1}^{\infty} ||\tau_m^{(l)}(t+c) - \frac{d^l}{dt^l}(t+c)||_{t\geq T} \leq \sum_{c=1}^{\infty} A_{lm}e^{-e^c} < \infty$$

For T large enough. Since we know τ is differentiable, we only care about $l > 1$. Making this simpler. For $l > 1$,

$$\sum_{c=1}^\infty ||\tau_m^{(l)}(t+c)||_{t \ge T} < \infty$$

Now we remember that,

$$\tau_{m+1}(t) = t + \log\big(1 + \frac{\tau_m(t+1)}{\phi(t+1)}\big)$$

If we differentiate this l times; there is a function F_l in $l+1$ variables, such that,

$$\tau_m^{(l)}(t) = F_l(t+1, \tau_{m-1}(t+1), \tau_{m-1}'(t+1), ..., \tau_{m-1}^{(l)}(t+1))$$

And the sequence F_l is generated by,

$$F_{l+1}(t, x_0, ..., x_{l+1}) = \frac{\partial F_l}{\partial t} + \sum_{d=0}^{l} x_{d+1}\frac{\partial F_l}{\partial x_d}(t, x_0, ..., x_l)$$

Starting with,

$$F_0(t, x_0) = t + \log\big(1 + \frac{x_0}{\phi(t+1)}\big)$$

It's up to the reader to check that $F_0(t) - t = \mathcal{O}(e^{-e^t})$, $F_1(t) - 1 = \mathcal{O}(e^{-e^t})$, and $F_l(t) = \mathcal{O}(e^{-e^t})$ for $l > 1$. The philosophy of our approach is to boil all our questions into the well performed behaviour of F_l. Using infinite compositions, we arrive at the iterative formula:

$$\tau_m^{(l)}(t) = \Omega_{c=1}^{m} F_l(t+c, \tau_{m-c}(t+c), ..., \tau_{m-c}^{(l-1)}(t+c), x) \bullet x\Big|_{x=0}$$

Notice that this expression only uses the derivatives up to $l-1$, so if this infinite composition converges when we take $m \to \infty$, we're all done. Insofar as, a proof by strong induction on l would suffice. This infinite composition looks a tad different than what we are used to so far–but it's not much of a different beast.

We can first note that we still have a summability criterion. For all $X \in \mathbb{R}^+$ and for large enough $t \ge T$; with $l > 1$,

$$\lim_{m\to\infty} \sum_{c=1}^{m} ||F_l(t+c, \tau_{m-c}(t+c), ..., \tau_{m-c}^{(l-1)}(t+c), x)||_{t \ge T, X \ge x \ge 0} < \infty$$

And from this we can derive a normality theorem–as we've done before. And then the rest differs very little from before. We show this iteration converges in the next proof.

Theorem 1.1. *The function τ is infinitely differentiable.*

Proof. Let's go by induction on the order of the derivative. Start by assuming that, for all $j < l$ that $\frac{d^j}{dt^j}\tau(t+\omega)$ is continuous on $(-2,\infty)$. And additionally, that there is some A_j such that,

$$||\frac{d^j \tau_m}{dt^j} - \frac{d^j t}{dt^j}||_{t\geq T} \leq A_j e^{-e^t}$$

And,

$$\sum_{m=1}^{\infty} ||\tau_{m+1}^{(j)} - \tau_m^{(j)}||_{t\geq T} < \infty$$

For all $T > \omega - 2$. Choose T large enough so that, $0 \leq |\tau_m^{(j)}(t) - t^{(j)}| \leq 1$. Now, as before, we compare the composition to a sum,

$$||\underset{p=1}{\overset{m}{\Omega}} q_p(t,x) \bullet x|| \leq \sum_{p=1}^{m} ||q_p(t,x)||$$

Where the supremum norms are across different sets on either side of the inequality. Insofar,

$$\begin{aligned} &\Big|\Big| \underset{c=1}{\overset{m}{\Omega}} F_l(t+c, \tau_{m-c}(t+c), ..., \tau_{m-c}^{(l-1)}(t+c), x) \bullet x \Big|\Big|_{t\geq T, 1\geq x\geq 0} \leq \\ \leq \quad &\sum_{c=1}^{m} ||F_l(t+c, \tau_{m-c}(t+c), ..., \tau_{m-c}^{(l-1)}(t+c), x)||_{t\geq T, X\geq x\geq 0} < \infty \end{aligned}$$

Since the right hand side is bounded for all m; we get the bound,

$$||\tau_m^{(l)}(t)||_{t\geq T} \leq \Big|\Big| \underset{c=1}{\overset{m}{\Omega}} F_l(t+c, \tau_{m-c}(t+c), ..., \tau_{m-c}^{(l-1)}(t+c), x) \bullet x \Big|\Big|_{t\geq T, 1\geq x\geq 0} \quad \leq \quad M$$

This tells us that our sequence $\tau_m^{(l)}$ is bounded and normal for $t \geq T$. We can strengthen this for large enough $t \geq T$ so that $M = 1$, $||\tau_m^{(l)}(t)||_{t\geq T} \leq 1$ for all m. We are going to use the mean-value theorem; but in l variables; and in-order to do so we have to derive a bound on the partial derivatives of F_l. Recalling that $l > 1$, we have uniform decay to 0 of the derivatives. So we are given the bounds, for large enough T,

$$\Big|\Big| \frac{\partial F_l}{\partial x_j}(t, x_0, ..., x_l) \Big|\Big|_{t\geq T, 0\leq |x_0 - t|, |x_1 - 1|, x_2, ..., x_l \leq 1} \quad \leq \quad \lambda_j < 1$$

For some sequence of numbers $0 < \lambda_0, \lambda_1, ..., \lambda_l < 1$. Provided x_j and x_j' are in the compact set $0 \leq |x_j - t^{(j)}| \leq 1$; we arrive at the bound,

$$\Big|\Big|F_l(t, x_0, ..., x_l) - F_l(t, x_0', ..., x_l')\Big|\Big|_{t\geq T} \leq \sum_{j=0}^{l} \lambda_j |x_j - x_j'|$$

Since $0 \leq ||\tau_m^{(l)}|| \leq 1$ sits happily as a normal family now; and we already know $0 \leq ||\tau_m^{(j)} - t^{(j)}|| \leq 1$, we can use the bound:

$$\begin{aligned}
& ||\tau_{m+1}^{(l)}(t) - \tau_m^{(l)}(t)||_{t\geq T} \leq \\
\leq \quad & ||F_l(t+1, \tau_m(t+1), ..., \tau_m^{(l)}(t+1)) - F_l(t+1, \tau_{m-1}(t+1), ..., \tau_{m-1}^{(l)}(t+1))|| \\
\leq \quad & \sum_{j=0}^{l} \lambda_j ||\tau_m^{(j)}(t+1) - \tau_{m-1}^{(j)}(t+1)||_{t\geq T}
\end{aligned}$$

Which can be re-written, by continuing the iteration, remembering $\tau_1^{(l)} = \tau_0^{(l)} = 0$ for $l > 1$,

$$||\tau_{m+1}^{(l)}(t) - \tau_m^{(l)}(t)||_{t\geq T} \quad \leq \quad \sum_{c=0}^{m-1} \sum_{j=0}^{l-1} \lambda_j \lambda_l^c ||\tau_{m-c}^{(j)}(t+c+1) - \tau_{m-c-1}^{(j)}(t+c+1)||_{t\geq T}$$

Par quoi, this is summable by the induction hypothesis. To visualize, compare $||\tau_m^{(j)}(t) - \tau_{m-1}^{(j)}(t)||_{t\geq T}$ to $\mathcal{O}(q^m)$ for $0 < q < 1$–and break out a proof showing the Cauchy product of summable geometric sequences $a_m = \lambda_l^m$, $b_m = \mathcal{O}(q^m)$ is itself summable. Concluding the proof.

$\square$

This is a very brief proof, and the language was chosen rather abstractly. This language will help us immensely in what follows. We want to be able to recycle this proof for higher order hyper-operations. It's not exceptionally hard to construct a proof of $\mathcal{C}^\infty$ for $e \uparrow\uparrow t$; but choosing a proof which generalizes well; allows us to iterate this procedure into higher order hyper-operators.