Superfunction spaces

mphlee

2021 XX 16

Contents

1 Mar XX 2021, On superfunction-closed spaces

Exchange had at the Tetration forum. Thread n° XXX - ***"On superfunction-complete spaces"***

MphLee**, XX February 2021** This follows from the discussion held at

MphLee, **Generalized Kneser superfunction trick (the iterated limit definition)**, (January 21, 2021), Tetration Forum

1.0.1 Disclaimer

The present overview tries to identify a possible road to an holistic, high level, theory of hyperoperations. It is necessarily full of errors and inaccuracies. I highly encourage the readers to report any conceptual errors and plain wrong statements, publicly or privately. The intent of this document is to present an inspired selection, thus inevitably partial, of commented references and mathematical concepts that aspires to inform, motivate and trace an ideal route for a reasonable future approach to iteration theory in the field of hyperoperations. In the worst case this 'new route' will reveal itself as not new at all. Even if it's very likely that, given the very advanced state of the art in category theory and dynamics, these ideas will prove to be already well known or trivial cases of vastly more general and powerful results I hope this document will retain some value as an attempt to bridge hyperoperations to the mainstream of mathematics.

1.0.2 Brief comment on convergence notions

> If we can make your "convergence criteria" a little more absolute with an example of a space where this works? I.e: holding a space $f, g \in \mathcal{B}$ such we can always find functions $\phi \in \mathcal{B}$ such that $f\phi = \phi g$. (JmsNxn, 22 January 2021)

The convergence notions are a weakening of continuity: I'm sure it is not an original concept but I use is just a practical black-box. I'm using it because I find it a good way to black-box all the topology and analysis involved. I'll work more on this to make this a more solid bridge to the analysts-tribe. To be precise what I call *convergence notion* $\mathfrak{L}$ over a set X

is a triple

$$\mathfrak{L} = (X, X_{\mathfrak{L}}, L_{\mathfrak{L}})$$

made by a collection $X_{\mathfrak{L}} \subseteq X^{\mathbb{N}}$ of sequences that are *formally convergent* under the notion $\mathcal{L}$ and a function $L_{\mathfrak{L}} : X_{\mathfrak{L}} \to X$ that *formally assigns* limits to sequences x_n with notations

$$L_{\mathfrak{L}}[x_n] = l \quad \textit{or} \quad x_n \to_{\mathfrak{L}} l$$

and satisfies some reasonable axioms[1] that are still work in progress. Given another convergence structure $\mathfrak{J} = (Y, Y_{\mathfrak{J}}, L_{\mathfrak{J}})$ I call function $f : X \to Y$ $\mathfrak{L}/\mathfrak{J}$-*sequentially continuous* if, for every sequence $x_n \in X_{\mathcal{L}}$, f commutes the limits

$$f(L_{\mathfrak{L}}[x_n]) = L_{\mathfrak{J}}[f(x_n)]$$

Obviously some examples should be the usual converging notions $(X, \lim_{n\to\infty})$ over X Hausdorff spaces (where limits are unique), X metric spaces in particular and the interesting cases of normed vector spaces where we prove that lim is linear and its kernel is the preimages are strictly related to the asymptotic relation[2] and to the Landau o-notation.

1.1 Back to our real problem: the conjugacy problem

As long as someone smart enough can put an Hausdorff topology on the set of functions we are working with, turning them into a actual topological spaces, we can talk about convergence notions and work on the algebraic side of the superfunction business. For the superfunction tricks one will

[1] I'm still reasoning on this. The first two axioms should be that constant sequences have to be convergent and they converge to the constant: this means that the constants sequences belong to $X_{\mathfrak{L}}$ and the function $\kappa : X \to X^{\mathbb{N}}$, the assigns to every point $x \in X$ to the constant sequence $x, x, x, ...$, is a right inverse of $L_{\mathfrak{L}}$, i.e.

$$L_{\mathfrak{L}} \circ \kappa = \mathrm{id}_X$$

The third axiom may be something about downward closure to sub-sequences: is $x_n \in X_{\mathfrak{L}}$ then all its sub-sequences have to be in $X_{\mathfrak{L}}$ and

$$a_n \subseteq x_n \implies L_{\mathfrak{L}}[a_n] = L_{\mathfrak{L}}[x_n].$$

Conditions that somehow reminds me of ideals/ultrafilters properties.

[2] We could define formal asymptotic relation that extends to all the sequences: let X be also an abelian group with a compatible convergence notion and s, t be two arbitrary sequences. We define $s \sim_{\mathfrak{L}} t$ iff $t - s \in X_{\mathfrak{L}}$, i.e. if converges. This is an equivalence relation over $X^{\mathbb{N}}$.

eventually be able to talk about completeness and eventually completion (compactification?) in a second moment. But I let this work to analysts and use instead this *converge notions* black-box formalism in order to focus on the algebraic structure of the functions. You can use this loosely related analogy: instead of going directly from $\mathbb{N}$ to $\mathbb{R}$ we should better build the intermediate number systems $\mathbb{Z}, \mathbb{Q}$ before.

Using *convergence notions* is the point of view that tries to get at the *potential* solutions: we instead look for *actual* objects. Since we would like that if a solution to the equation $\chi f = g\chi$ is found, via limit or another technique, it have to sit inside $\chi \in \mathcal{B}$ we can instead firstly take the p.o.v. that a solution is already there or is not there at all: i. e. *algebraically*, reasoning in term of extensions (injection of algebraic structures) instead of *analytically*, approximating to a limit (potential) object.

Sadly, it tuns out that generally this is a really hard and fundamental problem [MSE13]. Integer iteration and finding functions $\phi \in \mathcal{B}$ such that $f\phi = \phi g$ are all special instances of a fundamental decision problems in group theory and monoids theory called **conjugacy problem** or Dehn's second decision problem. Back in 1911 Dehn posed this problems and solved some of them for some special groups presenting an algorithm. Since then many groups were discovered to have an undecidable or computationally intractable conjugacy problem. To make this clear, to make sense of the Abel's and superfunction's equations, your space $\mathcal{B}$ needs to have, at least, a monoid structure. Then, in some cases like contiuous iteration, we use the extra structure of $\mathcal{B}$ to solve the conjugacy problem partially or locally.

Examples A starting example is to consider the permutation group S_n. Two permutations $\sigma, \sigma' \in S_n$ that are said to be conjugated if exists a $x \in S_n$ s.t. $\gamma_x(\sigma) := x\sigma x^{-1} = \sigma'$ have the same structure, the same number of fixed points and same numbers of n-periodic points. Two permutation that are in the same conjugacy class are structurally the same.

A more common example comes from linear algebra: given a field k, we associate to k-linear operators $F : V \to V$ ($F \in \mathrm{End}(V)$) the characteristic polynomials $p_F \in k[X]$ as invariants that keep track of its conjugacy class: their zeroes are the eigenvalues of F, called spectrum of F. Two square matrices A, B are similar if they are conjugated by a third matrix Ψ that operates the change of basis $\Psi A = B\Psi$. To compute if a matrix $A \in \mathsf{GL}_n$

is diagonal is to compute his conjugacy problem respect to an unknown diagonal matrix D: to do that we have to compute the eigenspaces $V_\lambda(A)$ and if their dimensions sum up to the dimension of V we define an change of base matrix Ψ called diagonalization matrix that solves

$$\Psi D = A\Psi$$

Two matrices that are conjugate represent the same linear transformation in different bases!

It is not a surprise then finding that in the volume on *Iterative functional equations* of the Encyclopedia of Mathematics and its applications [IFE90] the general functional equation $\chi f = g\chi$ is called *"change of variable"* and, when g is linear, *equation of linearization* (EOL): Abel, Schroeder and Bottcher equations are "*the three most important*" EOLs.

The conjugacy problem can be extended to monoids and semigroups. If the monoid is $\mathbb{N}^\mathbb{N}$ then the recursion theorem tell us that for any g the conjugacy problem $\chi s = g\chi$ is solvable and the computation amounts to the choice of a base value $\chi(0) = b$: in fact $\chi(n) = g^n(b)$.

In the monoid $\mathbb{R}^\mathbb{R}$, let s be the successor on the reals, the conjugacy problem $\chi s = g\chi$ is always solvable and the computation amounts to the choice of a base function $\beta : [0,1) \to \mathbb{R}$: in fact $\chi(x) = g^{\lfloor x \rfloor}(\beta(\{x\}))$ and $\{\chi : \chi s = g\chi\} \simeq \mathbb{R}^{[0,1)}$. Here a summary with examples.

structure $\mathcal{B}$	**problem**	**solution set**	**relation**
group G	*conjugacy problem*	$G(g,h) = \{\chi : \gamma_\chi(g) = h\}$	*conjugation* $\sim_G$
$\mathsf{GL}_n(k)$	*diagonalization*	*eigenbases matrices*	*similarity* $\sim$
abelian grp. A	*equivalence problem*	$A(a,b) \neq \varnothing$ *iff* $a = b$	*equality* $\sim_A \equiv =$
monoid M	*conjugacy problem*	$M(g,h) = \{\chi : \chi g = h\chi\}$	*conjugation*
$\mathbb{R}^\mathbb{R}$	*change of variable*		
	Abel eq.	*indefinite slog*	
	Schroeder eq.	$\mathbb{R}^\mathbb{R}(g,c) = \{\chi : \chi g = c \cdot \chi\}$	
$\mathbb{N}^\mathbb{N}$	*iteration*	$\mathbb{N}^\mathbb{N}(s,h)$	

1.1.1 A first observation on general recursion

The factorial and Fibonacci-like recursions, thus the iterated compositions, and the full primitive recursion schme seem can not fit in this scheme. There is a good reason for this: the congugacy problem is in some sense *iterational*, and with that I mean that congugacy is a problem of finding a morphism χ that respect two endofunction g, f. We can picture this like an arrow between functions.

$$f \xrightarrow{\chi} g$$

In the case of general primitive recursion, e.g. factorial, Fibonacci and iterated compositions, the conjugation heuristic seems to break but a close inspection shows that it does not fail as badly as it may seem. In fact we can still conceive the recursion as a problem of finding an appropriate morphism between two *something*, but this time these two something are not just simple endofunctions but more complex structures. This is not the sign that we are on the wrong way but instead it is a door that brings us toward a more coherent and broad conceptual understanding of all those problems.

In what follows I will focus solely on the conjugation problem and on the goal of describing such a species of ideal spaces where conjugation is possible.

> With that out of the way, I'm going to keep thinking about this as operations on $\mathcal{F}$ and functors; but to me they make sense as functors on $\mathcal{F}$; or subgroups, or different versions or whatever. What I mean is, can we think of $\mathcal{F}$ as an almost IDEAL space. Like the best space possible; where all the algebra is simple. Rather than monsters like e^x we look at simple amoebas like $x^2 + x$. And build from the bottom up. (JmsNxn, 28 January 2021)

Simplifying algebra seems a great starting point. As you notice, we can do it in at least two ways. We can simplify the algebra on $\mathcal{F}$ by building a family toy-models, some kind of test-theory where we study simple phenomenons and constructions that we would like to perform on monsters like e^x and general holomorphic functions, call this *the synthetic approach*); or we can instead build up from the bottom up considering some building blocks and performing closures on our space in an analogous fashion of studying first polynomials or elementary functions, call this *the analytic approach*.

1.1.2 Synthetic approach

A synthetic theory of these spaces would, for example, strip $\mathcal{F}$ of all his topology and additional structure, focus only on the composition structure, forget that every element is a function and ask that every function in $\mathcal{F}$ is invertible: this theory would be the study of conjugacy in groups and their representations and the theory of "*complete spaces*" would reduce to the group theory of groups G in which there is only one conjugacy class[3], i.e. conjugation action on G has a single orbit or, in other words, acts transitively. The number of conjugacy classes of G depends crucially on the theory of representation of G. A very rich theory, indeed, but too narrow to study exponentiation and hyperoperations directly. The function exp, for example, is not invertible, hence if we want it in our space we need to not restrict our theory to monoids to groups.

Needles to say that this the gateway to category theory.

1.1.3 Analytic approach

An analytic theory of superfunction spaces would resemble very much the theory of recursive functions or that of polynomial over a fixed ring. We could start with a basic stock of real valued functions like, for example, $\mathcal{F}_0 = \{x; x+1\}$ or $\mathcal{F}_0 = \{x+1\}$ and define the successive closures $\mathcal{F}_k$ under sum, composition and k-fold solution of conjugacy equation. This reduces to classical real dynamics but on top of that we can develop the interaction between the grading of the family $\mathcal{F}_k$ and the dynamics of the functions themselves. On those classes one would like to study the dynamics of the various conjugation maps and define many numerical invariants and discrete metrics. A preliminary exploration of this approach yield some interesting examples: let f be an invertible real valued function, define its *height* relative to s to be a possible infinite natural number $\mathfrak{h}_s(f) \in \mathbb{N} \cup \{+\infty\}$ defined as the cardinality of the set $\langle f \rangle_s$ minus one.

$$\mathfrak{h}_s(f) := |\langle f \rangle_s| - 1$$

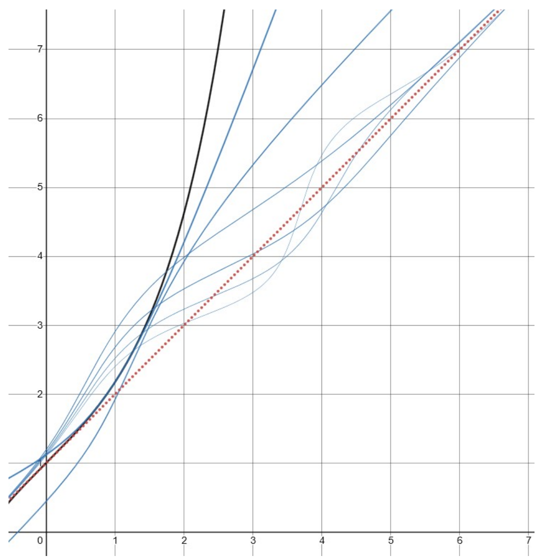
[3]Note that this is the opposite of the theory of abelian groups since if G is abelian every conjugacy class is the singleton and this kind of synthetic concept of superfunction is meaningless.

Where $\langle f \rangle_s = \{f(x); f(1+f^{-1}(x)); \ldots\}$ is the subset generated by f under the subfunction operator. It is easy to see that $\mathfrak{h}_s(b \cdot x) = 2$. Take the function $f(x) = b/x$, we have that $\mathfrak{h}_s(f) = +\infty$ [Mph13]. For every f invertible we have:

$$\mathfrak{h}_f(f) = 0 \qquad \textit{and}$$

$$\textit{if} \quad f(h(x)) = g(f(x)) \quad \textit{then} \quad \mathfrak{h}_h(g) \leq \mathfrak{h}_h(f)$$

It seems even possible to extend this concepts to non invertible real functions and develop the problem of the height in a more local manner. For example $f(x) = \sinh x + 1$ has, probably, an infinite height but, in some sense that can be made precise, it behaves locally as a function of height 3. If we define $f_0(x) := \sinh x + 1$ and $f_{k+1}(x) = f_k(1 + f_k^{-1}(x))$ we obtain the set $\langle \sinh \rangle_s = \{f_0; f_1; \ldots; f_k; \ldots\}$

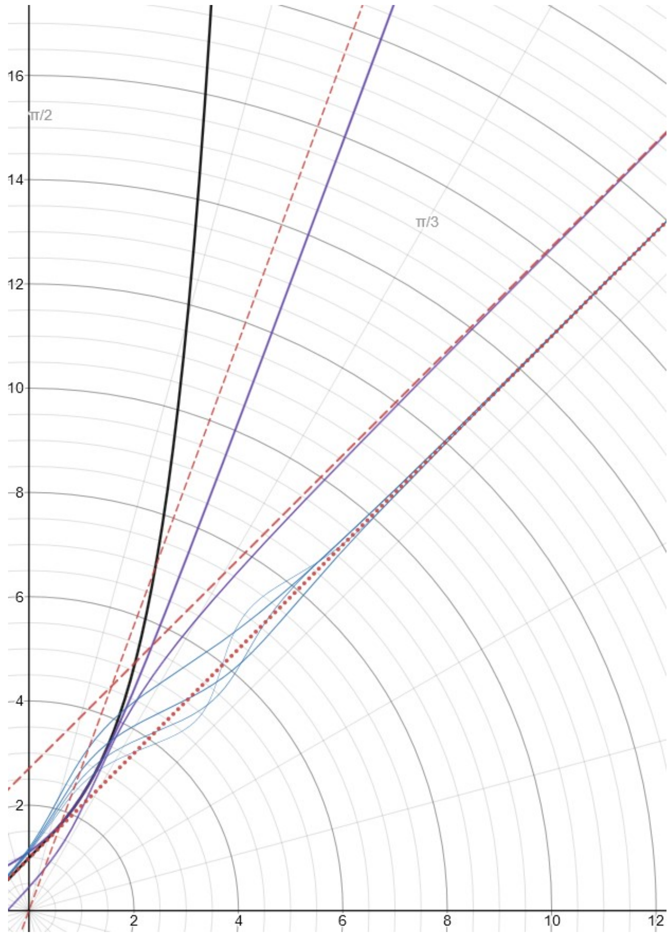


In **black** the function $\sinh x + 1$, in **dot-red** the successor, in **blue** the other elements of $\langle \sinh x + 1 \rangle_s$ with decreasing thickness for $0 < k \leq 7$.

The set $\langle \sinh \rangle_s$ is approximately that of exponentiation

$$\langle \sinh x + 1 \rangle_s = \{f_0; f_1 \sim ex; f_2 \sim e + x; f_3 \sim x + 1; f_4 \sim x + 1; \ldots\}$$

Locally, i.e. on some $U \subseteq \mathbb{R}$, it looks like the superfunction of $e \cdot x$.



In **red** the functions $e + x = y$, $ex = y$ and $x + 1$, now in **purple** f_1 and f_2. Polar coordinates to highlights the slopes.

Another possibility, that is in-between, is to consider as a space the groups of diffeomorphisms $(\mathrm{Diff}(M), \circ)$.

Those points of view however are not new or innovative at all: especially when it comes to consider an initial stock of primitive elements and performing closure on them! It is helpful for someone, like me, who is still trying to organize all this information, to briefly explore the history of the emergence of the analytic and the synthetic approach to iteration and superfunction closed spaces.

1.2 The emergence of the concept of superfunction-complete spaces

First of all I make clear that when I talk of "emergence" of this circle of ideas I'm referring explicitly to the pubblic discussion surrounding Hyperoperations and the Tetration forum. The basic concepts were in fact intrinsic in many constructions belonging to what I'll call here pre-history of Hyperoperations.

1.2.1 A partial pre-history

While the use of recursive definitions has not a clear origin, it seems that the first time the process called primitive recursion was abstracted can be traced back to Dedekind (1888). Now, the human ability to build new objects (*read functions*) from known one (*read computable*) is quite interesting. As much to push mathematicians to wonder what is computable in general, what are the limits for the brains and machines to compute: a business that goes from recursion theory, the Church theses to the beginning of the information age. But what does it mean to be computable?

Back then, it was probably not clear enough what was computable and what wasn't. In fact, the procedure of recursion (primitive recursion) was not enough to generate all the computable functions: already in 1926 Hilbert hypothesized that a certain function defined by double recursion, an extension of primitive recursion, was total computable but not primitive recursive and in 1928 Ackermann, one of his former student, proved it. We know the name of that function.

Kurt Gödel (1933), with Jacques Herbrand, defined a class of functions, call it R, generated by some basic functions (*read rank* 0+*constants*) and that have to be closed under composition (*it's a monoid!*), closed under (primitive) recursion (*!!*) and under minimization (μ-recursion/operator). They proposed this class as a class containing everything that can be "effectively computed". Later, Alonzo Church created λ-calculus (1936), a formal system centered around function evaluation. Imagine it roughly as a foundation of mathematics where instead of $\in$ we have function application and where everything we can define we can compute (λ-computability). Call R_λ the class of such functions. The same year Turing defined what was later called a Turing Machine: if a function f can be evaluated by a Turing-machine then it is Turing-computable. Lets say $f \in \mathsf{R}_T$. Church and Turing proved these classes of functions coincide

$$\mathsf{R} = \mathsf{R}_\lambda = \mathsf{R}_T$$

The Church-Turing thesis is instead the claim that R coincides with the, unknown, class of what can actually be computed in our universe by using all possible means.

Have you noticed something interesting here? Call PR the class of primitive recursive functions. First of all PR is strictly smaller than R because the latter contains the Ackermann-like functions, the Goodstein function (1947) [Goo47], and, in general, R is closed under the μ-recursion. Secondly, PR was organized by Grzegorczyk (1953) into a hierarchy of classes $\mathcal{E}^n$ that "covers"[4] it [Grz53]. The classes are closed to a limited number of applications of the recursion operator. Thirdly, and crucially, while the μ-operator may produce partial functions from common functions, thus the class R of general recursive functions is not a common monoid of function, $(\mathsf{PR}, \circ)$ is! It is a monoid that is closed under the recursion procedure.

The key point here is that, abstractly, we need some set of generators, an abstract monoid operation (*read category, or group for a toy model*) and on that we define something that has what seems a primitive recursion operator. Where does the complexity go out of control? It happens when instead of considering recursion, that is unique and computable on natural numbers we ask solutions to arbitrary functional equations (*read continuous iteration, continuous sums, 'compositorials'*). In these cases we do not have a well defined operator of recursion but some improper construction procedures of iteration/superfunction (think of indefinite integrals aka antiderivatives) that gives possibly empty solution sets.
We can summarize the analogies adding a column for the conjugacy problem in diffeomorphism groups, linking us back to the previous section.

[4]In symbols $\mathsf{PR} = \bigcup_{n<\omega} \mathcal{E}^n$.

dictionary	**Recursion theory**		**Real iteration**
		on 1-ary functions	*on $\mathcal{C}^1$ – functions*
base object	$\{\mathbb{N}^k\}_{k\in\mathbb{N}}$	$\mathbb{N}$	$\mathbb{R}$
elements	$F:\mathbb{N}^k\to\mathbb{N}$	$f:\mathbb{N}\to\mathbb{N}$	$f:\mathbb{R}\tilde{\to}\mathbb{R}$
	$\bigcup_{k\in\mathbb{N}}\mathbb{N}^{\mathbb{N}^k}$	$\mathbb{N}^{\mathbb{N}}$	$\mathcal{C}^1(\mathbb{R},\mathbb{R})^*=\mathrm{Diff}(\mathbb{R})$
name	*~ clone over* $\mathbb{N}$	*Baire space* $\mathcal{N}$	*diffeomorphisms group*
generators	$\{\kappa^k_n;\pi^k_i,s\}$	$\{\kappa_n;\mathrm{id}_{\mathbb{N}},s\}$	$\{\mathrm{id}_{\mathbb{R}};s\}$
constants	$\kappa^k_n:\mathbb{N}^k\to\mathbb{N}$	$\kappa_n:\mathbb{N}\to\mathbb{N}$	
	$\kappa^k_n(\mathbf{m})\overset{\text{def}}{=}n$		
projections	$\pi^k_i:\mathbb{N}^k\to\mathbb{N}$	$\mathrm{id}_{\mathbb{N}}:\mathbb{N}\to\mathbb{N}$	$\mathrm{id}_{\mathbb{R}}:\mathbb{R}\to\mathbb{R}$
	$\pi^k_i(\mathbf{m})\overset{\text{def}}{=}m_i$		
successor	$s:\mathbb{N}\to\mathbb{N}$	$s:\mathbb{N}\to\mathbb{N}$	$s:\mathbb{R}\to\mathbb{R}$
operator	$\circ$ – *substitution*	$\circ$ – *composition*	$\circ$ – *composition*
closure	*clone*	*monoid*	*group*
operator	*recursion*	*iteration*	*superfunction??*
	$\mathrm{rec}:\mathbb{N}^{\mathbb{N}^k}\times\mathbb{N}^{\mathbb{N}^{k+2}}\to\mathbb{N}^{\mathbb{N}^{k+1}}$	$\mathrm{ite}:\mathbb{N}\times\mathbb{N}^{\mathbb{N}}\to\mathbb{N}^{\mathbb{N}}$	??
closure	PR	PR_1	*superf. space??*
hierarchies	*Grzegorczyk's ranks* $\mathcal{E}^n$	*1-ary ranks*	*real functions ranks??*

Anyways, as we have seen, everything needed was already in the air, i.e. the shift to considering higher oderd construction over spaces of functions is very old but only since recursion theory it has begun to take such a shape and a form that is now intuitively and conceptually close enough to the theory of Hyperoperations to be recognized as relevant.

1.2.2 A partial history

Now we can finally turn to the emergence of this point of view applied to continuous iteration in the context of tetration and hyperoperations. The very concept of identifying a process that links the three arithmetical operations and the task of building a closure under that construction procedure is the very reason there is a concept of hyperoperation family to begin with. The claim that this task is useful or even meaningful is debated: there are reasons to suspect that looking for such a construction procedure the first place is conceptually misleading [Dev08][5], [QYu11a] and [QYu11b]. Anyways we can't fight with the aesthetic drive of a mathematician, so let's

[5] I thank Gottifred Helms for this reference

ignore these critiques. Let's find this link.

$$+ \longrightarrow \times$$

In the year 1915, in his famous paper "*Note on an operation of the third grade*"[Ben15], Bennett asks himself if the the right procedure linking addition and multiplication should be the same linking multiplication to exponentiation or, instead, the operation of inducing an homomorphic image via exponentiation. He proceeds considering the closure of his initial stock of operations $\{+;\times\}$ obtaining what we know today as Bennet's (Commutative) Hyperoperations. Much later Rubtsov in the monograph [KAR96], in the period that goes from 1989 to 1996, vastly generalized Bennet's construction extending it on what he called mathematical spaces ω_j: a continuous spectrum of classes that is the closure of the universe of common practice mathematical objects ω_0 under the Bennet's mapping procedure. He also compares this family with another family of well known operations, i.e. the common Goodstein's one. It is not precisely what we are looking for but it is a first conceptual upgrade in our direction. The first explicit step in this direction that I'm aware of is maybe Trappmann's dissertation [Tra07] on arborescent numbers (2007): given a binary operation $*$ on X he defines it's "*successor operation*" $*'$ as a kind of *left action* of the set $\mathbb{B}$ of binary trees on X encoding the bracketing/iteration of $*$. From there he achieves closure just by setting $X = \mathbb{B}$ and goes on on the rest of the paper studying the arithmetic on these new "*arborescent numbers*"[6]. Chronologically we are at the beginning of the Tetration Forum era.

This is not an history of hyperoperations so the following observation seems a little bit tangential to our goals here but we can already notice that a general definition of hyperoperations family was already in the air.

Definition 1. *Let X be a number system, e.g. $\mathbb{N}$ or $\mathbb{C}$, let I an ordered set and $i_1 \leq i_2 \leq i_3$ three distinct indexes in I. A function $h : I \to X^{X\times X}$ is said to be an* ***hyperoperations family*** *(HOF) if and only if*

1. *$h(i_1) = +_X$ and $h(i_2) = \times_X$ and*

[6] Trappmann's hyperoperations family is a vast generalization that can express classical and lower hyperoperations.

2. $h(i_3)$ *is exponentiation in* X*.*

In this case I will call I *the set of ranks. If* h *satisfies only condition 2. I would call it a* ***weak hyperoperations family*** *(wHOF).*[7]

This formal definition was sketched by Robbins in the Hyperoperations' Wikipedia page, later removed, and even if it is not present in any literature is hardly original. As Robbins observes on the Tetration forum [TF14]

> I do not believe that the definition of the "general sense" is original research because it is a natural translation of other people's terminology from 50 years ago into today's terminology. [...] think of it as: "this is the definition that everyone has been talking about with different words".

What brings us back to the emergence of superfunction complete spaces is the observation that this function h is routinely defined recursively in most of the model proposed by setting $I = \mathbb{N}$ and that the recursive definition, often called **mother law (ML)** of the HOF, is some kind of iteration.

$$\dots \longrightarrow + \xrightarrow{\textbf{ML}} \times \longrightarrow \dots$$

The conceptual shift from thinking of rank as a mere index to rank as iteration argument of an operator on functions, i.e. iteration of the **mother law**, traces back to the forum thread [Tra08]. The poster makes clear and explicit the problem of defining such an operator, denoted as E, on a domain of formal power series such that

$$E(f)(z) = f^{\circ z}(1),$$

the problems of computing the operator in an unique way and, I add, to chose a domain that is closed under the iteration of this operator. It is clear that E depends on the iteration method used but, crucially for our discussion, E^{-1} is well defined

> [...] is unique (independent of the method of E and independent on the initial condition) [Tra08]

and, I highlight, its **function conjugation**!

$$E^{-1}(f)(x) = f(1 + f^{-1}(x)),$$

[7]Goodstein and lower hyperoperations are HOFs. Bennet's family is a wHOF.

The same stance is clear in the July 2008 version[8] of Trappmann and Robbins's *Tetration Reference* [TR08] where they make explicit that they think of Abel, Schroeder, Böttcher and Julia functions as result of an higher order procedure.

A slightly different route starts by assuming first that our space H contains everything needed, i.e. there is a "T-parametrized curve" $\mathfrak{o}$ in the space H, where $T = \mathbb{N}$, $\mathbb{R}$ or $\mathbb{C}$, and then asking the points $\mathfrak{o}_t \in H$ of our curve (sequence) to satisfy some set of mutual relationships. The approach should remind of the formal definition hyperoperations families in that does not relies on defining an hypothetical *increase-rank-by-one*-operator on H. This paradigm is exemplified in [Nxn13]. Here the space is generated by a countable "basis" of "points" that we have parametrized by $\mathbb{N}$ and happen to be the reciprocals of our dear binary functions we know and love, i.e. $1/[n]$. This step, at least conceptually, is fundamental, in my opinion, in generalizing away from the need of a concrete operator E acting on formal series or functions.

I tried to make a summary of this story and its relevance on non-integer ranks hyperoperations in [Mph14] and [Mph15]. There I sketch the the importance of having a space closed under the iteration process in order to approach the ranks problem from an iteration point of view.

Paulsen in [Pau16] considers a metric on the set of solution of the Abel equation in order to find natural solutions. The metric is constructed using the fundamental "affine distance" that is intrinsic to every set of solutions of a conjugacy equation[9].

[8]For a the thread with versions (link).

[9]Let G be a group of function or just an abstract group. Define the sets $G(f,g) = \{\chi : \chi f = g\chi\}$. Notice that $G(f,f) = C(f)$ is always a subroup of G and is called the centralizer of f. Since $G(g,f) = G(f,g)^{-1}$ and since the group operation of G defines a map $G(g,f) \times G(f,g) \to C(f)$ the difference $\chi^{-1}\psi$ of two solutions $\chi, \psi \in G(f,g)$ is always in $C(f)$: we define a map

$$\delta : G(f,g) \times G(f,g) \to C(f) \quad ; \quad \delta(\chi,\psi) := \chi^{-1}\psi$$

that satisfies

$$\delta(\chi,\chi) = 1 \quad ; \quad \delta(\chi,\psi) \cdot_G \delta(\psi,\xi) = \delta(\chi,\xi).$$

1.2.3 A chapter in the long march towards non-integer ranks: problems

1.3 Conjugation and dynamics: from monoids to categories

1.3.1 From $\mathrm{Diff}(\mathbb{R})$ to Diff: conjugacy of dynamical systems

1.3.2 From $\mathsf{GL}_n(k)$ to Vec_k: linear algebra

1.3.3 From $\mathrm{Aut}(X)$ to Set: $\mathbb{N}$-systems

1.4 Categories of endomorphisms: from $\mathcal{C}$ to $\mathcal{C}^{\circlearrowleft}$

1.4.1 Superfunction clousure=connectedness

1.4.2 Conceptual approach

1.5 Canonical extension of iteration $f \setminus \mathcal{C}^{\circlearrowleft} \to \mathcal{C}^{\mathcal{E}^1_f}$

1.6 Road to hyperoperations, categorically

1.6.1 Stellar systems $(C^{\circlearrowleft})^\star$

1.6.2 Hyperstellar systems and hyeroperations

References

[Ben15] A. A. Bennett, *Note on an operation of the third grade*, Annals of Mathematics, vol. 17, no. 2, 1915, pp. 74–75. JSTOR, www.jstor.org/stable/2007124. Accessed 19 Feb. 2021.

[Goo47] R.L. Goodstein, *Transfinite Ordinals in Recursive Number Theory*, The journal of Symbolic Logic Vol 12, No 4 (Dec. 1947), pp. 123-129

[Grz53] A. Grzegorczyk, *Some classes of recursive functions*, Rozprawy matematyczne, Vol 4, pp. 1–45. (1953)

[IFE90] Kuczma, M., Choczewski, B., Ger, R. (1990). *Iterative Functional Equations (Encyclopedia of Mathematics and its Applications)*, Cambridge: Cambridge University Press

[KAR96] Rubtsov K.A., *New mathematical objects*, BelGTASM, 1996, NPP Informavtosim: 251 p

[Tra07] Trappmann H., *Arborescent Numbers: Higher Arithmetic Operations and Division Trees*, 2007

[Tra08] bo198214, *non-natural operation ranks*, May 02, 2008

[TR08] Trappmann H., Robbins A., *Tetration reference*

[Dev08] K. Devlin, *It Ain't No Repeated Addition*, June 2008, at Devlin's Angle, MAA

[QYu11a] Qiaochu Yuan, *Why are addition and multiplication commutative, but not exponentiation?*, 28 April 2011, MathSE

[QYu11b] Qiaochu Yuan, *Is There a Natural Way to Extend Repeated Exponentiation Beyond Integers?*, 10 August 2011, MathSE

[Nxn13] JmsNxn, *Hyper operator space*, August 12, 2013

[MSE13] user1729 *Theorems with the greatest impact on group theory as a whole*, MathSE, August 29, 2013

[Mph13] MphLee, *Sets* $f_n \in A_f$ *where* $f_{n+1} = f_n \circ S \circ f_n^{\circ(-1)}$ *and operator* $\alpha(f_n) = f_{n+1}$", April 13, 2013

[Mph14] MphLee, *Comment on Negative, Fractional, and Complex Hyperoperations*, 30 may 2014

[TF14] Thread: *References about the formalization of the Hyperoperations*, July 17, 2014

[Mph15] MphLee, *On non-integer rank Hyperoperations*, 19 April 2015, MathSE

[Pau16] Paulsen, *FINDING THE NATURAL SOLUTION TO f(f(x))=exp(x)*, Korean J. Math.24(2016), No. 1, pp. 81–106