The Limits of a Family;
of Asymptotic Solutions to The Tetration Equation

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Abstract

In this paper we construct a family of holomorphic functions $\beta_\lambda(s)$ which are solutions to the asymptotic tetration equation. Each β_λ satisfies the functional relationship $\beta_\lambda(s+1) = \dfrac{e^{\beta_\lambda(s)}}{e^{-\lambda s}+1}$; which asymptotically converges as $\log \beta_\lambda(s+1) = \beta_\lambda(s) + \mathcal{O}(e^{-\lambda s})$ as $\Re(\lambda s) \to \infty$. This family of asymptotic solutions is used to construct a holomorphic function $\text{tet}_\beta(s) : \mathbb{C}/(-\infty,-2] \to \mathbb{C}$ such that $\text{tet}_\beta(s+1) = e^{\text{tet}_\beta(s)}$ and $\text{tet}_\beta : (-2,\infty) \to \mathbb{R}$ bijectively.

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1 Introduction

This paper will start with a general theorem the author has shown a multitude of times, but of which most recently appears in [4, 5]. In [4] it is shown for a specific case, and modified to a real analysis scenario a couple more times; but in [5] the general theorem is given. We'll use this introduction to introduce the theorem, and talk a little bit about the notation.

Theorem 1.1. [1] *Let $\{H_j(s,z)\}_{j=1}^\infty$ be a sequence of holomorphic functions such that $H_j(s,z) : \mathcal{S}\times\mathcal{G} \to \mathcal{G}$ where $\mathcal{S}$ and $\mathcal{G}$ are domains in $\mathbb{C}$. Suppose there exists some $A \in \mathcal{G}$, such for all compact sets $\mathcal{N} \subset \mathcal{G}$, the following sum converges,*

$$\sum_{j=1}^\infty ||H_j(s,z) - A||_{z\in\mathcal{N}, s\in\mathcal{S}} < \infty$$

Then the expression,

[1] We've added a proof of this theorem in the appendix 9.

$$H(s) = \lim_{n\to\infty} \Omega_{j=1}^{n} H_j(s,z) \bullet z = \lim_{n\to\infty} H_1(s, H_2(s, ...H_n(s,z)))$$

Converges uniformly for $s \in \mathcal{S}$ and $z \in \mathcal{N}$ as $n \to \infty$ to H, a holomorphic function in $s \in \mathcal{S}$, constant in z.

Upon which, this theorem provides a manner of proving holomorphy of an infinite composition of holomorphic functions; and it only requires that a certain sum converges. Where here, an infinite composition is denoted,

$$\lim_{n\to\infty} H_1(s, H_2(s, ...H_n(s,z))) = \Omega_{j=1}^{\infty} H_j(s,z) \bullet z$$

Much of the theory of infinite compositions, as the author has written about in [3, 4, 5, 6] depends on the behaviour of a sum which we compare with the infinite composition. In the case of this paper (as in [4, 5]), the infinite composition falls into a degenerate category. This is the case that the value in z of the infinite composition will be constant. And in contrast, in the non-degenerate category (as in [3, 6]), we'd have that our infinite composition is holomorphic in z and non-constant.

For this reason, we won't speak of z at all, except to denote the manner of composition (like binding a variable to an integral, and then tossing it away afterwords). Instead, we'll be talking about two variables, $s, \lambda \in \mathbb{C}$. And discussing infinite compositions with these two variables.

This will birth us a two variable holomorphic function $\beta_\lambda(s)$; which we'll call a family of solutions to the asymptotic tetration equation. Where, for us, we'll call a function $l(s)$ a solution to the asymptotic tetration equation if,

$$\log l(s+1) - l(s) \to 0 \text{ as } |s| \to \infty$$

Where we'll mostly be concerned with $|s| \to \infty$ while s is in a half-plane; and these things are holomorphic (or with countable singularities) unless stated otherwise. These asymptotic solutions, essentially look like tetration at infinity, but everywhere else they may not look like tetration. This allows us to talk about logarithms of these things at infinity in a nice manner. And if we are able to solve the equation $\log F(s+1) = F(s)$ for large s, repeatedly taking logarithms allows us to extend this definition almost everywhere in $\mathbb{C}$.

So the idea is to take our family of asymptotic solutions β_λ and construct an error term τ_λ which solves the tetration equation for large s. Since β_λ will be a well behaved solution to the asymptotic tetration equation; this is doable.

2 The family of functions β_λ

We will start our foray by pulling out of a hat the sequence of functions we want to infinitely compose to get β_λ. We'll denote this sequence of functions $q_j(s, \lambda, z)$; where the z value will disappear in the end. Write,

$$q_j(s, \lambda, z) = \frac{e^z}{e^{\lambda(j-s)} + 1}$$

Where the index $j \in \mathbb{N}$ and $j \geq 1$. Now, we're going to force $\Re(\lambda) > 0$ and that $\lambda(j - s) \neq (2k + 1)\pi i$ for all $k \in \mathbb{Z}$. The first restriction will be needed for the summation, and the second restriction ensures we have no poles. We'll call this domain of holomorphy $\mathbb{L}$, in which each $q_j(s, \lambda, z) : \mathbb{L} \times \mathbb{C} \to \mathbb{C}$, where $(s, \lambda) \in \mathbb{L}$ and $z \in \mathbb{C}$.

Observe that,

$$\sum_{j=1}^{\infty} |q_j(s, \lambda, z)| = \sum_{j=1}^{\infty} |\frac{e^z}{e^{\lambda(j-s)} + 1}| < \infty$$

But, even better than this, we have a normally converging sum. Let $\mathcal{K} \subset \mathbb{C}$ and let $\mathcal{U} \subset \mathbb{L}$ both be compact sets. Then,

$$\sum_{j=1}^{\infty} ||q_j(s, \lambda, z)||_{\mathcal{U},\mathcal{K}} < \infty$$

This should tell us that Theorem 1.1 is going to be useful, as q_j satisfies all the properties of H_j in the theorem's statement. Now, a small reminder is that Theorem 1.1 has no restriction on how many variables are involved, although it's only stated for one variable. For clarification of this, the reader is pointed to [5]. Therein, if we take,

$$\beta_\lambda(s) = \Omega_{j=1}^{\infty} q_j(s, \lambda, z) \bullet z$$

Then $\beta_\lambda(s)$ is holomorphic for $(s, \lambda) \in \mathbb{L}$. It's important to remember what β_λ looks like though, and why we'd even want this function. We write,

$$\beta_\lambda(s) = \Omega_{j=1}^{\infty} \frac{e^z}{e^{\lambda(j-s)} + 1} \bullet z$$

Then, if we shift the argument in s forward by 1, we get something magical.

$$\begin{aligned}
\beta_\lambda(s + 1) &= \Omega_{j=1}^{\infty} \frac{e^z}{e^{\lambda(j-s-1)} + 1} \bullet z \\
&= \Omega_{j=0}^{\infty} \frac{e^z}{e^{\lambda(j-s)} + 1} \bullet z \text{ We've shifted the index here} \\
&= \frac{e^{\Omega_{j=1}^{\infty} \frac{e^z}{e^{\lambda(j-s)}+1} \bullet z}}{e^{-\lambda s} + 1} \\
&= \frac{e^{\beta_\lambda(s)}}{e^{-\lambda s} + 1}
\end{aligned}$$

And from this, we're in a position to state that this is a family of solutions to the asymptotic tetration equation. That is,

$$\log \beta_\lambda(s+1) - \beta_\lambda(s) = -\log(1+e^{-\lambda s})$$

Which tends to 0 exponentially as $|s| \to \infty$ while $\Re(\lambda s) > 0$. Now this form of the β_λ family is difficult to compute, we need to compute a bunch of nested exponentials–at infinity no less, and so therefore it can be easier to make a change of variables $s = -\log(w)/\lambda$. The author would like to thank Sheldon Levenstein for doing this first; as the author rarely numerically evaluates, it was Sheldon's observation that this form is much less exhausting computationally.

Write,

$$g_\lambda(w) = \beta_\lambda(s)$$

Then this is holomorphic when $w \neq -e^{\lambda j}$,

$$g_\lambda(w) = \mathop{\Omega}_{j=1}^{\infty} \frac{we^z}{e^{\lambda j} + w} \bullet z$$

Upon which, calculating Taylor coefficients for $g_\lambda(w)$ at 0 are surprisingly simple; especially by the functional equation,

$$g_\lambda(e^\lambda w) = \frac{w}{w+1} e^{g_\lambda(w)}$$

Where now computing the Taylor coefficients at 0 is a relatively simple procedure; it's inductive. This will construct a Taylor-series valid for $|w| < e^{\Re\lambda}$. We leave the process to the curious reader who wants to numerically evaluate these functions. And to extend g_λ to its maximal domain we just iterate the functional equation.

Sheldon Levenstein, is again, to thank for this observation. This form of many of the equations the author has solved, make the solutions look like a kind of mock Schröder equation. Where in Schröder's case one would solve,

$$\Psi(Ls) = e^{\Psi(s)}$$

We are solving something similar, but adding a multiplicative factor to the construction. This helps tremendously at manipulating the complex dynamics of these objects. And the convergents can make a very complicated thing less so.

It's important to also note that $\beta_\lambda(s)$ has an exponential series in this form. Which is,

$$\beta_\lambda(s) = \sum_{k=1}^{\infty} a_k e^{k\lambda s}$$

Which is valid for $\Re(s) < 1$. Which implies that $\beta_\lambda(s + \frac{2\pi i}{\lambda}) = \beta_\lambda(s)$, so that our function is periodic in the s argument. You can also see this by inspection, plugging in the value in the infinite composition.

We compress all this knowledge into the existence of a family of functions which solve the asymptotic tetration equation.

Theorem 2.1. *There exists a family of functions β_λ which are holomorphic on $\mathbb{L} = \{(s,\lambda) \in \mathbb{C}^2 \,|\, \Re(\lambda) > 0,\ \lambda(j-s) \neq (2k+1)\pi i,\ j,k \in \mathbb{Z},\ j \geq 1\}$. These functions are expressible as,*

$$\beta_\lambda(s) = \underset{j=1}{\overset{\infty}{\Omega}} \frac{e^z}{e^{\lambda(j-s)}+1} \bullet z$$

Satisfy the functional equation,

$$\beta_\lambda(s+1) = \frac{e^{\beta_\lambda(s)}}{e^{-\lambda s}+1}$$

And the asymptotic relationship,

$$\log(\beta_\lambda(s+1)) - \beta_\lambda(s) = \mathcal{O}(e^{-\lambda s})$$

As $|s| \to \infty$, wherever $\Re(\lambda s) > 0$.

3 The exponential convergents

In this section we'll focus on better approximating tetration using $\beta_\lambda(s)$ at infinity. This is a difficult idea to intuit, but we're going to better understand its behaviour at infinity. The first thing we'd like to do is construct a sequence of convergents. Let's call,

$$\begin{aligned}
\log(\beta_\lambda(s+1)) - \beta_\lambda(s) &= \tau_\lambda^1(s) = -\log(1+e^{-\lambda s})\\
\log\log(\beta_\lambda(s+2)) - \beta_\lambda(s) &= \tau_\lambda^2(s)\\
\log\log\log(\beta_\lambda(s+3)) - \beta_\lambda(s) &= \tau_\lambda^3(s)\\
&\vdots\\
\log^{\circ n}\beta_\lambda(s+n) - \beta_\lambda(s) &= \tau_\lambda^n(s)
\end{aligned}$$

Upon which, the asymptotic relationship,

$$\log(1+\frac{\tau_\lambda^n(s+1)}{\beta_\lambda(s+1)}) - \tau_\lambda^{n+1}(s) = \log(1+e^{-\lambda s}) = \mathcal{O}(e^{-\lambda s})$$

Which is another side effect of being an asymptotic solution to tetration; but it's required we have exponential convergence of τ as $\Re(s) \to \infty$. To make sure everything stays well behaved in the iterated logarithm. It is here we provide a quick theorem.

Theorem 3.1 (The Unbounded Theorem)**.** *The function $\beta_\lambda(s) \to \infty$ as $\Re(s) \to \infty$.*

Proof. The function $\beta_\lambda(s)$ looks like the orbits of e^z as $\Re(s) \to \infty$. The accumulation points of e^z for z almost everywhere are the orbit $1, e, e^e, e^{e^e}, ...$; the orbit of the exponential at 0; which diverge to infinity. This fact is cited from Milnor [1]; but was actually shown by Lyubich [2] and Rees [7].

For our case, since $\beta(s+1) = e^{\beta(s)}(1+e^{-\lambda s})^{-1} = e^{\beta(s)-\log(1+e^{-\lambda s})} = e^{\beta(s)+\epsilon}$; for very large $\Re(s)$ we eventually converge towards the orbit $1, e, e^e, e^{e^e}, ...,$. As these orbits are in ϵ-neighborhoods of each other at infinity–with $\epsilon \to 0$. □

We are going to call upon the sequence of functions $\tau_\lambda^n(s)$, which are holomorphic on some subset $\mathbb{L}$ (we'll get to that later), and try to express them.

To begin, we know that each $\tau_\lambda^n(s)$ has exponential decay to 0 as $\Re(s) \to \infty$. To show this, we need only look at the functional equation. Assume it follows for n and go by induction to get $n+1$. Take,

$$\tau_\lambda^{n+1}(s) = -\log(1+e^{-\lambda s}) + \log(1 + \frac{\tau^n(s+1)}{\beta_\lambda(s+1)})$$

Since we know that, by The Unbounded Theorem 3.1,

$$\frac{1}{\beta_\lambda(s+1)} \to 0 \text{ as } \Re(s) \to \infty$$

We know that $\tau_\lambda^{n+1}(s)$ must look like $-\log(1+e^{-\lambda s}) + \mathcal{O}(e^{-\lambda s})$. Which certainly means each τ has exponential convergence to 0.

To begin,

$$\tau_\lambda^n(s) = \log^{\circ n} \beta_\lambda(s+n) - \beta_\lambda(s)$$

And we want to show by induction that this thing is an exponential series. These functions satisfy the identity,

$$\tau_\lambda^{n+1}(s) = \log(\beta_\lambda(s+1) + \tau_\lambda^n(s+1)) - \beta_\lambda(s)$$

We have begun this iteration with $\tau_\lambda^0(s) = 0$ and,

$$\tau_\lambda^1(s) = \sum_{k=1}^\infty \frac{(-1)^k}{k} e^{-k\lambda s}$$

Now the goal is, when talking about $\tau_\lambda^2(s)$, we can also express it as an exponential series. We start with,

$$\frac{\tau_\lambda^2(s)}{e^{-\lambda s}} \to c_{21} \text{ as } \Re(s) \to \infty$$

Then,

$$\tau_\lambda^2(s) - c_{21}e^{-\lambda s} = \mathcal{O}(e^{-2\lambda s})$$

Upon which we induct to get,

$$\frac{\tau^2_\lambda(s) - \sum_{j=1}^{k-1} c_{2j}e^{-j\lambda s}}{e^{-k\lambda s}} \to c_{2k} \text{ as } \Re(s) \to \infty$$

Which will give us an exponential series,

$$\tau^2_\lambda(s) = \sum_{k=1}^\infty c_{2k}e^{-k\lambda s}$$

And there's nothing special about τ^2_λ, we can do this for all $\tau^n_\lambda(s)$. This allows us to write,

$$\tau^n_\lambda(s) = \sum_{k=1}^\infty c_{nk}(\lambda)e^{-k\lambda s}$$

For a sequence of functions $c_{nk}(\lambda)$; and this thing will definitely converge in a neighborhood of infinity. But we don't know where else as of yet. To prove all of this, we have two choices. The first, we can make the change of variables $s = -\log(w)\lambda^{-1}$ where in a neighborhood of zero, this can be thought of as Taylor's theorem. We can think of using our function g_λ from above to express this; where we also have a new sequence of function u^n_λ.

This is to say,

$$u^{n+1}_\lambda(w) = \log(g_\lambda(\frac{e^\lambda}{w}) + \sum_{k=1}^\infty c_{nk}e^{-\lambda}w^k) - g_\lambda(\frac{1}{w})$$

In which $u^{n+1}_\lambda(0) = 0$ and for $|w| < \delta$ is holomorphic. This doesn't mean $g_\lambda(\frac{1}{w})$ will be holomorphic here. It means the expression above has a removable singularity at 0. Describing this construction will be further pursued in Section 5. For the moment, the limiting process takes care of everything, and so the result follows.

The second way to prove the existence of an exponential series; we can use $\beta_\lambda(s)$'s periodicity. Which, implies a Fourier series exists; and then it must be of this form by its convergence at infinity. Which is to say, $\tau^n_\lambda(s + 2\pi i/\lambda) = \tau^n_\lambda(s)$, so,

$$\tau^n_\lambda(s) = \sum_{k=-\infty}^\infty c_{nk}e^{-k\lambda s}$$

And we know that $c_{nk} = 0$ for $k < 1$ because, that's the only way we have convergence to 0 as $\Re(s) \to \infty$. I dislike this proof in comparison to the last, as it's not as enlightening in the method of construction. Especially as we pass deeper in the well.

This gives us a way of talking about approximate solutions to the tetration equation.

Theorem 3.2. *There exists a sequence of holomorphic functions $c_{nk}(\lambda)$ for $n, k \geq 1$ such that,*

$$\sum_{k=1}^{\infty} c_{nk}(\lambda)e^{-k\lambda s} = \log^{\circ n} \beta_\lambda(s+n) - \beta_\lambda(s)$$

4 Choosing the proper Riemann mapping

The author would like to take the reader through a brief segue in the history of tetration. The year is 1950, and Kneser has published a treatise on iterating the exponential function. In constructing this exponential he had a god knows how revelation. Before the dawn of computers, before the dawn of an efficient way at calculating these things, Kneser saw a solution to tetration.

Now the idea isn't so far out there now. But this idea is still what us "tetrationers" think of when we think of a nice solution to tetration. Hell, as far as most of us are concerned, it's the nicest tetration. Hell, at this point in history, it's still *the* tetration.

And what was Kneser's *je ne sais quoi* that flipped everything on its head... A Riemann mapping. Nothing more, nothing less. It was the key to his tetration; a Riemann mapping.

The idea was simple enough; we take the inverse Schröder function Ψ of e^z about a fixed point $L \in \mathbb{C}$. This function Ψ is entire, and satisfies,

$$e^{\Psi(z)} = \Psi(Lz)$$

Where, the standard way to iterate this is to write,

$$F(z) = \Psi(e^{Lz})$$

Upon which,

$$F(z+1) = e^{F(z)}$$

Where, this may betray the simplicity, it looks right, it should be our solution. The problem is, by construction, this function $F : \mathbb{R} \not\to \mathbb{R}$. This tetration is not real-valued. We definitely want the iteration of exponentiation to be real-valued. So, this form is pretty useless.

Now, Kneser's idea was simple, but breathtaking. Instead of talking just about a fixed point L, we also talk about its conjugate pair $\overline{L}$. And we want to create a Riemann mapping which glues the two tetrations from above into a single entity. And this will be real-valued, so long as we remember to keep them conjugate similar.

It's difficult to find analyses of Kneser's method, as his paper is in German, and it's yet to be translated. However many people have re-explained his work. The most acute, I find, is in Sheldon Levenstein's interpretation. He simplifies looking for the Riemann mapping by framing it as a search for a periodic function. In many ways, he refers to Kneser's construction in a much more hands-on approach.

To explain, if we call $\mathbb{H} = \{z \in \mathbb{C} \,|\, \Im(z) > 0\}$, there is a 1-periodic function $\theta(z) : \mathbb{H} \to \mathbb{C}$ such that,

$$\theta(z) = \sum_{k=0}^{\infty} p_k e^{2\pi i k z}$$

$$\text{tet}_K(z) = \Psi(e^{Lz}\theta(z))$$

Where,

$$\overline{\text{tet}_K(z)} = \overline{\Psi(e^{Lz}\theta(z))}$$

Is precisely,

$$\text{tet}_K(\overline{z})$$

Where the conjugate of Ψ is the inverse Schröder function about the fixed point $\overline{L}$, and the conjugate θ function works in the same manner here.

Now, finding this function θ is very difficult, and no matter how you slice it, you need to compute a Riemann mapping. That Riemann mapping being a miraculous thing. Where firstly, Kneser finds a nice simply connected domain $\mathbb{E}$, and then produces the Riemann mapping to $\mathbb{H}$, all while respecting the Abel functional equation, and all while making sure this will be real-valued.

And so, at this point in our paper, we have to pay respect that in constructing a holomorphic tetration–Kneser had to pull out a very complicated mapping theorem. And so, any solution to tetration avoiding some kind of complexity similar to this, would either be miraculous or just wrong.

In our approximate solutions to tetration, in the asymptotic, they have singularities. They have singularities everywhere. When we talk about,

$$F_n(s) = \beta_\lambda(s) + \tau_\lambda^n(s)$$

We have to account for a plethora of singularities which occur when $\lambda(j-s) = (2k+1)\pi i$. These are very ugly singularities too, where if,

$$\begin{aligned} \beta_\lambda(s) &= \infty \\ \beta_\lambda(s+1) &= e^\infty \end{aligned}$$

And, if we go by inspection, we must have,

$$\beta_\lambda(1 + \frac{(2k+1)\pi i}{\lambda}) \text{ are simple poles}$$

But,

$$\beta_\lambda(j + \frac{(2k+1)\pi i}{\lambda}) \text{ are essential singularities for } j \geq 2$$

So, if we have any hope in constructing a tetration function $\text{tet}_\beta(s)$ which is holomorphic on $s \in \mathbb{H}$, which is real-valued; satisfies $\overline{\text{tet}_\beta(s)} = \text{tet}_\beta(\overline{s})$; we'll need to get rid of these singularities somehow. And, we'll need a Riemann mapping on $\mathbb{L}$ to do this. Now, given, we don't need as miraculous a Riemann mapping as Kneser. We just have to avoid the poles somehow. Using the $\sqrt{\cdot}$ function will suffice.

The way we're going to do this is actually pretty simple. We're going to use a function $\lambda(s) : \{s \in \mathbb{C} \,|\, |\arg(s)| < \theta, 0 < \theta < \pi/2\} \to \mathbb{L}_2$. Where $\mathbb{L}_2$ is a projection into the second component of $\mathbb{L}$. This can be better explained, such that $(s, \lambda(s)) \in \mathbb{L}$ is satisfied for $|\arg(s)| < \theta$. We'll require that $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$. And of course that λ is holomorphic.

From here, we'll define a sequence of functions,

$$F_n(s) = \log^{\circ n} \beta_{\lambda(s+n)}(s+n)$$

Where we use all of the above expressions from the previous sections, to prove that the exponential series $\tau^n_{\lambda(s+n)}(s)$ converges as $n \to \infty$. And then we use this to prove that $F_n \to F$. And here, F will be holomorphic for $|\arg(s)| < \theta$. But as soon as F is holomorphic for $\Im(s) = A$ it is necessarily holomorphic for $\Im(s - j) = A$ by the functional relationship and the non-zero nature of F. And so, this will construct a tetration function $F(s) : \mathbb{C}/(-\infty, X] \to \mathbb{C}$ which is real-valued.

Hereupon we take the value $x_0 \in (X, \infty)$ in which $F(x_0) = 1$ as a transfer value so that $F(s + x_0) = \text{tet}_\beta(s)$; which will be our desired tetration. Which is quite the mouthful; but summarizes pretty clearly our line of attack.

So we write the desired theorem,

Theorem 4.1 (The Desired Mapping Theorem)**.** *The function* $\lambda(s) = \dfrac{1}{\sqrt{1+s}}$, *is holomorphic for* $|\arg(s)| < \theta$ *for some* $0 < \theta < \pi/2$, *and* $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$, *where the pair* $(s, \lambda(s)) \in \mathbb{L}$ *for* $|\arg(s)| < \theta$.

We'll prove this in a bit. First we need to better understand the family of functions $\beta_\lambda(s)$.

5 Normality theorem at infinity

When discussing the convergence of $\beta_\lambda(s)$ so far; we've only referred to convergence on compact sets. When discussing infinite compositions at infinity, the author points the reader to [3], where asymptotics were used on infinite compositions. Where the general idea was, we had uniform convergence wherever the sum converges; even at infinity.

Ipso, if we call a $\mathcal{U} \subset \mathbb{L} \cup \infty$ in which $(s, \lambda) \in \mathcal{U}$ satisfy,

$$\sum_{j=1}^{\infty} \frac{1}{||e^{\lambda(j-s)} + 1||_{\mathcal{U}}} < \infty$$

Then, $\beta_\lambda(s)$ is continuous on $\mathcal{U}$, including the point at infinity (it becomes a removable singularity). In this case, conveniently, we can think of the point at infinity as $\Re(s) \to -\infty$ or $\Re(s) \to \infty$. And so, in a half plane of $\mathbb{L}$ we have the identification that,

$$\beta(-\infty) = 0$$

Because when $\Re(s) = -\infty$ we have that,

$$\sum_{j=1}^{\infty} |\frac{e^z}{e^{\infty + \lambda j} + 1}| = 0$$

And this can be done compactly, where if we include this point at infinity, we know that the sum converges uniformly. And additionally that $\beta_\lambda(s)e^{-\lambda s} = 1$ as $\Re(s) \to -\infty$. This was explored deeply in [3]; where we derived asymptotic relationships for infinite compositions.

This transformation can be thought similarly to how we've constructed $g_\lambda(w)$. The value of $g_\lambda(0) = 0$ is the value of $\beta_\lambda(s)$ at $(s, \lambda) = -\infty$; and the value of $g_\lambda(\infty) = \infty$ is the values of $\beta_\lambda(s)$ at $(s, \lambda) = \infty$. This means, exactly like $e^{\lambda s}$ our function $\beta_\lambda(s) \not\to 0$ as $(s, \lambda) \to \infty$, but tends to zero as $(s, \lambda) \to -\infty$. Since $\mathbb{L}$ is in two complex dimensions, we can assign two points at infinity. In such a sense, we are adding considering $\widehat{\mathbb{L}} = \mathbb{L} \cup \{\infty, -\infty\}$

Now, if we take the $g_\lambda(w)$ approach at understanding $\widehat{\mathbb{L}}$, the real trouble is when we look at the real ∞ in question. This is when $g_\lambda(w)$ is in a neighborhood of infinity; which is when our asymptotic kicks in.

In this space, we again consider $g_\lambda(1/w)$, which we give by, yet again, an infinite composition. We'll begin now to denote this,

$$f_\lambda(w) = g_\lambda(1/w) = \Omega_{j=1}^{\infty} \frac{e^z}{e^{\lambda j} w + 1} \bullet z$$

This function is holomorphic on $\mathbb{C}^\times = \{w \in \mathbb{C} \,|\, w \neq 0,\, w \neq -e^{-\lambda j}\}$; but we don't need such an expansive domain. Instead we'll focus on the unit disk $\mathbb{D}$ subtracting our bad points. The set in question is,

$$\mathbb{D}^\times = \{w \in \mathbb{D} \,:\, 0 < |w| < 1,\, w \neq -e^{-\lambda j}\}$$

And on this punctured disk (sort of, forgive the abuse),

$$f_\lambda(e^{-\lambda} w) = \frac{e^{f_\lambda(w)}}{w + 1}$$

Where in this space,

$$\log f_\lambda(e^{-\lambda}w) = f_\lambda(w) - \log(1+w)$$

Where here, all our discussions of convergence are linearized. We have the contraction $\mathcal{T} : p(w) \mapsto p(e^{-\lambda}w)$ and a sum $z \mapsto z + \log(1+w)$; and our exponential $e^{-\lambda s} \mapsto w$. What we want to say, which is incredibly simple in this space, is that this asymptotic relationship is satisfied for $\mathbb{D}^\times \cup \{0\}$. It isn't much, but the holomorphy of tetration follows from this. This identity is satisfied for all $\mathbb{D}^\times$ and since $\log(1+w)$ is holomorphic on $\mathbb{D}$ we know that,

$$\log f_\lambda(e^{-\lambda}w) - f_\lambda(w) = -\log(1+w) \sim -w \ |w| \to 0$$

And when we pull back into the space $\mathbb{L}$, it means that,

$$\log \beta_\lambda(s+1) - \beta_\lambda(s) = 0 \text{ when } (s,\lambda) = +\infty$$

And it means this uniformly. In such a sense we can assign a point at $\pm\infty$ in $\mathbb{L}$ in which $\beta(-\infty) = 0$ and $\beta(\infty) = \infty$; if we keep this with the correlation that $\pm\infty = \Re s$; and at infinity the statement $\log(\beta(\infty+1)) - \beta(\infty) = 0$ is actually pretty meaningful. Whereupon, in $\widehat{\mathbb{L}}$ this is fully rigorous $\log(\beta(\infty)) = \beta(\infty)$.

It's important to remember we are absolutely not talking about the points $\Im s = \pm\infty$. We have absolutely nothing to say about these points when we make the change of variables back to s. We are only focused on shifting the real argument to the left or the right. In fact, in the complex plane this will be much much uglier.

To translate this into a more standard way of thinking $\beta_\lambda(s) \to \infty$ as $\Re(s) \to \infty$, but not necessarily on other paths towards infinity. Since we are keeping λ finite, these are *sort of* equivalent statements. We may get ourselves in a bind if we push this isomorphism to the extreme, but it works fine for what we need it for.

Now when we talk about our functions τ_λ^n in this different correspondence; everything is straight forward. Once everything is linearized we get that,

$$u_\lambda^{n+1}(w) = \log(f_\lambda(e^{-\lambda}w) + u_\lambda^n(e^{-\lambda}w)) - f_\lambda(w)$$

Each $u_\lambda^n(w)$ is holomorphic on $\delta\mathbb{D}$ for an appropriately small δ and has a removable singularity at 0; which is a fixed point $u_\lambda^n(0) = 0$ with multiplier $u_\lambda^{n\prime}(0) = -1$. The function $u_\lambda(e^{-\lambda}w)$ is also holomorphic on $\delta\mathbb{D}$, and has a fixed point at 0. The difference $u_\lambda^{n+1} - u_\lambda^n$ has the equation,

$$|u_\lambda^{n+1}(w) - u_\lambda^n(w)| \leq A|u_\lambda^n(e^{-\lambda}w) - u_\lambda^{n-1}(e^{-\lambda}w)|$$

With a fixed point at $w = 0$. The constant $1 < A < 1+\epsilon$ is eventual for $|w| < \delta$; where $\epsilon \to 0$ as $\delta \to 0$; because $\frac{u_\lambda^n(w)}{w} \to -1$ and $\frac{1}{f(e^{-\lambda k}w)} \to 0$ as $k \to \infty$. This implies the Lipschitz constant satisfies at least $A \to 1$. Now if we take a compact set $\mathcal{B} = \overline{\delta\mathbb{D}}$, then $||u_\lambda^n(e^{-\lambda}w) - u_\lambda^{n-1}(e^{-\lambda}w)||_{w\in\mathcal{B}} \leq q||u_\lambda^n(w) - u_\lambda^{n-1}(w)||_{w\in\mathcal{B}}$

for $e^{-\Re\lambda} < q < 1$. And therefore u_λ^n converges uniformly as $n \to \infty$ on the compact set $\mathcal{B} = \{w \in \mathbb{C} \,|\, |w| \leq \delta\}$ because $0 < Aq < 1$. And this is precisely our tetration existence theorem. Where upon $u_\lambda(w) : \mathcal{B} \to \mathbb{C}$, in which the functional equation $f_\lambda(e^\lambda w) + u_\lambda(e^\lambda w) = \log(f_\lambda + u_\lambda)$ extends this function almost everywhere.

Now, the compact set $\mathcal{B}$ depends on δ, and we can shrink δ. The value δ controls our Lipschitz constant A and our contracting constant q. Our contracting constant q can get as close as we want to $e^{-\Re\lambda}$; similarly with $A \to 1$. But additionally, so long as $e^{-\Re\lambda} \leq e^{-\rho}$, we can alter δ uniformly in λ in which $qA < 1$. And as such, the convergence is uniform in λ as well as w, because δ can be chosen uniformly in λ.

Theorem 5.1 (Tetration Existence Theorem)**.** *For $(s, \lambda) \in \mathcal{L} \subset \mathbb{L}$ there exists a tetration function $F_\lambda(s)$ such that,*

$$F_\lambda(s+1) = e^{F_\lambda(s)}$$

Where $\mathbb{L}/\mathcal{L}$ is a measure zero set in $\mathbb{C}^2$.

Proof. The correct way to observe this is a bit more tacit. This is the quick run through of everything we've done above. We start with the result,

$$\Big|\log\big(\frac{f_\lambda(w) + w}{f_\lambda(w) + w'}\big)\Big| \leq A|w - w'|$$

Where the smaller $|w - w'| < \delta$ is, the closer $1 < A < 1 + \epsilon$ shrinks to $\epsilon = 0$. And so the recursive process we've derived,

$$u_\lambda^n(w) = -w + \mathcal{O}(w^2)$$

Because the initial derivative is $u_\lambda^1(w) = -\log(1+w)$, and since the further iterates will converge to $u_\lambda^1(w)$ as $w \to 0$ we know $u_\lambda^n(w)/w \to -1$. Hence, $A \to 1$. For the second half of the proof; parce que,

$$u_\lambda^n(e^{-\lambda}w) = -e^{-\lambda}w + \mathcal{O}(e^{-\lambda}w)^2$$

We know that the operator,

$$\mathcal{T}u_\lambda(w) = u_\lambda(e^{-\lambda}w)$$

Is a contraction, subject to Banach's theorem. This theorem can be applied uniformly. So choose a compact set $\mathcal{B} = \{w \in \mathbb{C} \,|\, |w| \leq \delta\}$, mais soi,

$$||u_\lambda^n(e^{-\lambda}w) - u_\lambda^{n-1}(e^{-\lambda}w)||_{w\in\mathcal{B}} \leq q||u_\lambda^n - u_\lambda^{n-1}||_{w\in\mathcal{B}}$$

And from this, we can choose $e^{-\Re\lambda} < q < 1$ for $\delta > 0$; where as $\delta \to 0$ we have $q \to e^{-\Re\lambda}$. Choose δ such that $0 < qA < q(1+\epsilon) < 1$, and then,

$$
\begin{aligned}
||u_\lambda^{n+1} - u_\lambda^n||_{\mathcal{B}} &\leq A||u_\lambda^n(e^{-\lambda}w) - u_\lambda^{n-1}(e^{-\lambda}w)||_{\mathcal{B}} \\
&< qA||u_\lambda^n - u_\lambda^{n-1}||_{\mathcal{B}}
\end{aligned}
$$

Which concludes the convergence of $u_\lambda^n(w) = \tau_\lambda^n(s)$ by Banach's Fixed Point Theorem. The function $\tau_\lambda(s) = \log(\beta_\lambda(s+1) + \tau_\lambda(s+1)) - \beta_\lambda(s)$; upon which we can iterate this process to $s \in \mathbb{C}$ for all $\tau_\lambda(s)$, except for branching points. Additionally, this proof was done in a uniform manner so that we have local holomorphy in λ; giving us the theorem. □

To better explain the situation with λ, we suggest the reader look at John Milnor's [1] treatment of the holomorphy of the Schröder function in its multiplier. Which is a limit process described by a multiplier, where the result is holomorphic in the multiplier; which is what we have here. Except the multiplier is written $e^{-\lambda}$.

Now this theorem isn't exactly what we want. We want to take the limit $\lim_{n\to\infty} u_\lambda^n(w)$ as λ depends on w. This is perfectly doable by the above analysis; since convergence is uniform in λ and w. But we need $u_\lambda^n(w)$ to converge in a manner where $\lambda \to 0$ as $n \to \infty$; this is not covered by this theorem. We want a mapping; a nice enough function $\lambda \mapsto \lambda(s)$ to wash away all the problems.

6 Finding an appropriate mapping

It's helpful to look at this problem from two different angles. The first being a mapping $\lambda(w) : \mathbb{D}^\times \to \Re(\lambda) > 0$ and the second being a mapping $\lambda : \mathbb{L}_1 \to \mathbb{L}_2$. Recall that $\mathbb{D}^\times = \{w \in \mathbb{C} \,|\, 0 < |w| < 1,\, w \neq -e^{\lambda j},\, j \geq 1\}$ and $\mathbb{L} = \mathbb{L}_1 \times \mathbb{L}_2 = \{(s,\lambda) \in \mathbb{L} \,|\, \Re(\lambda) > 0,\, \lambda(j-s) \neq (2k+1)\pi i,\, j \geq 1,\, k \in \mathbb{Z}\}$.

In the second form, we want a mapping $\lambda : \{s \in \mathbb{C} \,|\, |\arg(s)| < \theta < \pi/2\} \to \mathbb{L}_2$ in which $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$. We want to convert this into a restriction on $\lambda(w)$. The answer to this riddle isn't too difficult.

To find an appropriate mapping, we just want a mapping that expands the lines where,

$$\lambda(j-s) = (2k+1)\pi i$$

This can be done with the function $\sqrt{s}$, and many functions like this will work, but $\sqrt{s}$ is simple enough. To visualize this, when λ is constant we have a lattice of points in the right half plane. And if we were to multiply that lattice by $\frac{1}{\sqrt{1+s}}$ we'd be able to place a sector $|\arg(s)| < \theta$ within it. Or rather,

$$\frac{j-s}{\sqrt{1+s}} = (2k+1)\pi i$$

Then,

$$s = j + \sqrt{1+s}2(k+1)\pi i$$

And there's a sector $|\arg(s)| < \theta < \pi/2$ in which this is true for $j \geq 1$. So with this we're going to consider the function,

$$\beta(s) = \Omega_{j=1}^{\infty} \frac{e^z}{e^{\frac{j-s}{\sqrt{1+s}}} + 1} \bullet z$$

And it's alternate form on $\mathbb{D}^\times$,

$$f(w) = \Omega_{j=1}^{\infty} \frac{e^z}{e^{\frac{j}{\sqrt{1+s}}} w + 1} \bullet z$$

But,

$$\begin{aligned} w &= e^{-\lambda s} \\ s &= -\log(w)/\lambda \\ s &= -\frac{\log(w)}{\sqrt{1+s}} \end{aligned}$$

So that $h(\log(w)) = \frac{1}{\sqrt{1+s}}$ for some algebraic function h,

$$f(w) = \Omega_{j=1}^{\infty} \frac{e^z}{e^{h(\log(w))j} w + 1} \bullet z$$

Now, this expression looks very cryptic. But we know that $u_\lambda^n(w)$ as $n \to \infty$ converges uniformly for $w \in \delta\mathbb{D}$ and $\lambda \in \mathcal{K} \subset \mathbb{C}_{\Re(\lambda)>0}$–where these are compact sets. When we translate this back into $\beta_\lambda(s)$; where $\lambda = \frac{1}{\sqrt{1+s}}$ we get that, first of all $\beta(s) : \{|\arg(s)| < \theta < \pi/2\} \to \mathbb{C}$; and secondly that,

$$\tau^n(s) = \log^{\circ n} \beta(s+n) - \beta(s) \to 0$$

Should converge uniformly for $|\arg(s)| < \theta < \pi/2$ as $|s| \to \infty$ like $\mathcal{O}(e^{-\lambda s}) = \mathcal{O}(-|s|^{1/2})$. As such we still have our identity $\log\beta(\infty) - \beta(\infty) = 0$. Where this again equates to a compact set in $\widehat{\mathbb{L}} = \mathbb{L} \cup \{-\infty, \infty\}$. This will imply we have a tetration function F which satisfies,

$$F(s+1) = e^{F(s)}$$

For all $|\arg(s)| < \theta$. But then, for every $s \in \mathbb{C}$ there exists some n such that $s+n$ is in this sector, therefore we can undo this by taking n logarithms. This will define our tetration function almost everywhere–excluding logarithmic branch-cuts/singularities. Since it will be real valued, we know that there is a real value $x_0 \in \mathbb{R}$ such that,

$$
\begin{aligned}
\text{tet}_\beta(s) &= F(s+x_0) : (-2,\infty) \to \mathbb{R} \text{ bijectively} \\
\text{tet}_\beta(0) &= 1
\end{aligned}
$$

This gives us a tetration function which definitely has singularities at the negative integers and isn't holomorphic on the line $(-\infty,-2]$. But there may be other singularities which appear elsewhere, it is our job to show this doesn't happen; which we'll do in the next section.

Before the dramatic conclusion, we have to take a closer look at τ_λ^n. As you'll note, for a varying λ, τ_λ is holomorphic; but in the case of interest, we have a sequence of $\lambda_n \to 0$ as $n \to \infty$ and 0 is on the boundary of our domain $\mathbb{L}$. So we have to use that $\lambda_n = \mathcal{O}(n^{-1/2})$ somewhere in our construction, and show that the limit is still holomorphic in s.

This is a tricky job. And of it we'll show in the following theorem.

Theorem 6.1. *The function $F_n \to F$ is holomorphic on $|\arg(s)| < \theta$; and convergence is uniform on compact sets of this.*

Proof. The goal of this proof is to control the Lipschitz constant A and the contraction constant q from The Tetration Existence Theorem 5.1. Now, when we shrink δ from our compact set we know that $q \to e^{-\Re\lambda}$ and that $A \to 1$, which ensures convergence. In the current situation, we know that $\Re(\lambda) \to 0$ like $\mathcal{O}(n^{-1/2})$. This implies that $\prod_{j=1}^n e^{-\Re\lambda} \to 0$ because $\sum_{j=1}^n \mathcal{O}(j^{-1/2}) \to \infty$ as $n \to \infty$, and this will be $-\infty$ because $\Re\lambda > 0$. This implies the product of our contractions looks like $e^{-Bn^{1/2}}$ for some $B > 0$ which is enough to ensure convergence of the final result if we can keep $q \to e^{-\Re\lambda}$ well enough, which we can because δ can be chosen uniformly for λ.

But in order for this work, we need $A \to 1$ in a fast enough manner to not affect this convergence. We need that $A \to 1$ like $e^{\mu n^{-1/2}}$ for $0 < \mu < 1$. Now we know we can choose A such that $qA < 1$, and this choice can be done uniformly. This implies that $A \leq e^{\mu\Re\lambda}$ for $0 \leq \mu < 1$.

So, when we perform the iteration,

$$||u_\lambda^{n+1} - u_\lambda^n||_\mathcal{B} \leq A_n q_n ||u_\lambda^n - u_\lambda^{n-1}||_\mathcal{B}$$

And here,

$$A_n q_n = e^{-(1-\mu)\mathcal{O}(n^{-1/2})}$$

And therefore,

$$\prod_{j=1}^n A_j q_j = e^{-(1-\mu)\sum_{j=1}^n \mathcal{O}(j^{-1/2})} = e^{-\mathcal{O}(n^{1/2})} = e^{-Dn^{1/2}}$$

For a constant $D = (1-\mu)B > 0$. Therefore by induction,

$$||u_\lambda^{n+1} - u_\lambda^n||_{\mathcal{B}} \leq e^{-Dn^{1/2}}||\log(1+w)||_{\mathcal{B}} = Me^{-Dn^{1/2}}$$

Wherefore, $\log(1+w) = u_\lambda^1 - u_\lambda^0$. For some constant $M = ||\log(1+w)||_{\mathcal{B}}$. Now, for all $\epsilon > 0$ there exists N such for $n, m > N$ we can show,

$$\sum_{j=n}^{m} e^{-Dj^{1/2}} < \epsilon/M$$

And therefore,

$$||u_\lambda^m - u_\lambda^n||_{\mathcal{B}} < \epsilon$$

Which concludes the proof. □

7 tet_β is non-zero in the upper half-plane

This section is devoted to showing that $\text{tet}_\beta(s)$ is non-zero for $s \in \mathbb{H} = \{s \in \mathbb{C} \,|\, \Im(s) > 0\}$. This can be equivalently said, that the only zero of $\text{tet}_\beta(s)$ is at $s = -1$. This equates to there being no singularities for $\Im(s) > 0$.

The only way a singularity arises is if $\log \text{tet}_\beta(s_0 + 1)$ is singular; which implies $\text{tet}_\beta(s_0 + 1) = \infty, 0$. Where we know for large enough N for all $n > N$ the function $\text{tet}_\beta(s_0 + n)$ is non-singular. Therein, the only thing that can start this chain of singularities is if $\text{tet}_\beta(s_0 + k) = 0$ for some k. Upon which $\text{tet}_\beta(s_0 + k - j) = \infty$ for all $j \geq 1$.

So we need a theorem that tet_β is non-zero in the upper-half plane, and we've simultaneously showed that tet_β is holomorphic in the upper half-plane. By conjugation, we'll know that tet_β is holomorphic in the lower half-plane. And then $\text{tet}_\beta : \mathbb{C}/(-\infty, -2] \to \mathbb{C}$ will be our maximal domain of holomorphy for tet_β.

To do this, we need to understand what happens when $\log(\text{tet}_\beta(s_0)) = 0$. It can only happen if $\text{tet}_\beta(s_0) = 1$, but it doesn't necessarily happen if $\text{tet}_\beta(s_0) = 1$. There exists a curve C in a neighborhood of s_0 in which $\text{tet}_\beta(C) \in \mathbb{R}$. Now, supposing that $C = s_0 + t$ for $t \in (-\delta, \delta)$ then when we continue to iterate this procedure, necessarily $\text{tet}_\beta(s_0 + t)$ will be real-valued as t grows. This will force $\text{tet}_\beta(s_0 - 1) = 0$. Assuming that C is not a line, then the line $\text{tet}_\beta(s_0 + t)$ is not real-valued for $t \in (-\delta, \delta)$ excepting at $t = 0$. This means the logarithm $\log(\text{tet}_\beta(s_0 + t))$ must be complex valued, and this means that $\text{tet}_\beta(s_0 - 1)$ must be in a neighborhood of $2\pi i k$ for $k \neq 0$, and it is non-zero.

So all we have to do is focus our attention on $\text{tet}_\beta(s_0 + t)$ for $t \in (-\delta, \delta)$ and show that it cannot be real-valued.

Lemma 7.1 (The Non-real Lemma)**.** *For all $s_0 \in \mathbb{H}$ such that $tet_\beta(s_0) \in \mathbb{R}$ there exists $\delta > 0$ such for $t \in (-\delta, \delta)$ and $t \neq 0$ the values $tet_\beta(s_0 + t) \notin \mathbb{R}$.*

Proof. Take our asymptotic solution to tetration,

$$\beta(s) = \Omega_{j=1}^{\infty} \frac{e^z}{e^{\frac{j-s}{\sqrt{1+s}}} + 1} \bullet z$$

And note there are no lines $s_0 + t$ for $t \in \mathbb{R}$ in which this is real-valued, excepting when $s_0 \in \mathbb{R}$. Therefore, the limiting process,

$$F_n(s) = \log^{\circ n} \beta(s+n)$$

Cannot be real-valued on a line $s_0 + t$, unless $s_0 \in \mathbb{R}$; because this looks like $\beta(s_0 + t)$ for large t. Therefore the result. □

With this, we state the following theorem, which will only require a quick justification.

Theorem 7.2 (The Non-zero Theorem)**.** *The tetration function* $\text{tet}_\beta(s) \neq 0$ *for* $\Im(s) > 0$*.*

Proof. By The Non-real Lemma 7.1, we know for any point $\Im(s_0) > 0$ and $s_0 \in \mathbb{R}$, that $\text{tet}_\beta(s_0 + t)$ is not real-valued for $t \in (-\delta, \delta)$. Choose an s_0 in which $\text{tet}_\beta(s_0) = 1$. The goal is to show that $\text{tet}_\beta(s_0 - 1) = 2\pi i k$ for some $k \neq 0$.

We go by contradiction. Assume that $\text{tet}_\beta(s_0 - 1) = 0$, then $\text{tet}_\beta(s_0 - 2)$ is a singularity with a branch-point. This branching process can be done along the line $s_0 - 2 - t$ for $t \in \mathbb{R}^+$. Therefore $\text{tet}_\beta(s + s_0 - 2)$ is holomorphic for $|s| < \delta$ and $s \notin (-\delta, 0]$. This implies $\text{tet}_\beta(s_0 - 2) = -\infty$, and we have chosen the principal branch of log. The only way the principal branch of the logarithm can have a branch cut along $(-\infty, 0]$ is if it the variable is real valued on $(0, \infty)$.

But if we have used the principal branch of log, and since $\text{tet}_\beta(s + s_0 - 2)$ is holomorphic for $|s| < \delta$ and $s \notin (-\delta, 0]$; then it must be real-valued for $s \in (0, \delta)$, contradicting The Non-real Lemma 7.1. □

And with this we have constructed a tetration function holomorphic in the upper-half plane and real-valued analytic on the real-line. Therefore,

Theorem 7.3 (The Tetration Theorem)**.** *The function* tet_β *is holomorphic on* $\mathbb{C}/(-\infty, -2]$*, satisfies* $\text{tet}_\beta(0) = 1$ *and,*

$$\text{tet}_\beta(s+1) = e^{\text{tet}_\beta(s)}$$

And is given by the equation, for some $x_0 \in \mathbb{R}$*,*

$$\text{tet}_\beta(s) = \lim_{n\to\infty} \log^{\circ n} \beta(s + x_0 + n)$$

Where,

$$\beta(s) = \Omega_{j=1}^{\infty} \frac{e^z}{e^{\frac{j-s}{\sqrt{1+s}}} + 1} \bullet z$$

8 Additional properties of tet_β

In this section we'll list some properties of tet_β which are extra to the general theory. These are nice things we can say about our solution. These are largely properties which are inherited from the family β_λ. Underlining how they are inherited is the important part.

We start with our function $\beta(s)$ which we write as,

$$\beta(s) = \underset{j=1}{\overset{\infty}{\Omega}} \frac{e^z}{e^{\frac{j-s}{\sqrt{1+s}}} + 1} \bullet z$$

In which,

$$\log \beta(s+1) - \beta(s) = \mathcal{O}(e^{-|s|^{1/2}})$$

But even better, we have that,

$$\text{tet}_\beta(s) - \beta(s + x_0) = \mathcal{O}(e^{-|s|^{1/2}}) \text{ as } |s| \to \infty \text{ while } |\arg(s)| < \theta$$

And so we can expect this with higher-order derivatives too, as the convergence is uniform for $|s| \to \infty$ while $|\arg(s)| < \theta$. Now differentiating β is fairly easy; especially if we view this as a manner of computing Taylor coefficients of $g(w)$. We'll skip a few steps here, but the algebraic relationship as $\Re(s) \to \infty$,

$$\beta'(s) \sim \frac{\beta'(s)e^{\beta(s)}}{e^{\frac{s}{\sqrt{1+s}}} + 1} + \frac{e^{\beta(s)}e^{\frac{-s}{\sqrt{1+s}}}}{\left(e^{\frac{-s}{\sqrt{1+s}}} + 1\right)^2} \left(\frac{d}{ds} \frac{s}{\sqrt{1+s}}\right)$$

Means that,

$$\beta'(s) \sim \underset{j=1}{\overset{\infty}{\Omega}} \frac{ze^{\beta(s-j)}}{e^{\frac{j-s}{\sqrt{1+s}}} + 1} + \frac{e^{\beta(s-j)}e^{\frac{j-s}{\sqrt{1+s}}}}{\left(e^{\frac{j-s}{\sqrt{1+s}}} + 1\right)^2} \left(\frac{d}{ds} \frac{s}{\sqrt{1+s}}\right) \bullet z$$

Which is eventually non-zero for large enough s. As such, we can expect that,

$$\text{tet}'_\beta(s) \neq 0 \text{ for } |s| > R$$

And hereupon, since $\text{tet}'_\beta(s) \neq 0$ we can derive that $\text{tet}'_\beta(s-1) \neq 0$. Where,

$$\text{tet}'_\beta(s-1) = \frac{\text{tet}'_\beta(s)}{\text{tet}_\beta(s)} \neq 0$$

Which implies that $\text{tet}'_\beta(s) \neq 0$ everywhere $\text{tet}_\beta(s) \neq 0$ which excludes the point -1 but at -1 we know that $\text{tet}_\beta(-2)$ is a singularity, so this isn't a problem. Which is something really advantangeous to know, but to most studies of tetration, is rather apparent. This implies, yet again, that $\text{tet}'_\beta(x) > 0$ for all $x \in (-2, \infty)$. This means that tet_β is a bijection of $(-2, \infty) \to \mathbb{R}$.

We can have a similar discussion with higher-order derivatives as well. It doesn't work out as nice, but is still worth while. Now, again we know that higher order derivatives of β are also non-zero. As such, we get that,

$$\text{tet}_\beta^{(n)}(s) \neq 0 \text{ for } |s| > R_n$$

This matters particularly for the real-line. Where it says that for large enough X_n, then for all $x > X_n$ we know that $\text{tet}_\beta^{(n)}(x) > 0$. This tells us, eventually, each of our derivatives will be monotone. This is a slightly weaker criterion than all of its derivatives being monotone.

We'd like to take a quick moment to discuss the inverse function slog_β of tet_β. We specifically refer to the slog_β function which takes $\mathbb{R} \to (-2, \infty)$ bijectively. This function will be analytic by the implicit function theorem.

Take $\mathcal{N}$ a neighborhood of zero in which slog_β is holomorphic. Then,

$$\text{slog}_\beta(e^z) = \text{slog}_\beta(z) + 1$$

This allows us to analytically continue slog_β to the set $\mathcal{S}$ in which,

$$\mathcal{S} = \bigcup_{n=0}^{\infty} \exp^{\circ n}(\mathcal{N})$$

Now, the orbits of the exponential map on an arbitrary neighborhood are dense in the complex plane. Which is the equivalent statement that the Julia set of exp is all of $\mathbb{C}$ (Again, we cite [1, 2, 7]). This amounts to $\overline{\mathcal{S}} = \mathbb{C}$. As such, we know that slog_β is holmorphic almost everywhere in $\mathbb{C}$; upto a measure zero set in $\mathbb{C}$.

This gives us a clear language that,

$$\exp^{\circ s}(z) = \text{tet}_\beta(s + \text{slog}_\beta(z))$$

Is holomorphic on a domain $\mathbb{P}$ in which $(s, z) \in \mathbb{P}$ and $\mathbb{C}^2/\mathbb{P}$ is a measure-zero set in $\mathbb{C}^2$. This function satisfies the functional equation,

$$\exp^{\circ s}(\exp^{\circ s'}(z)) = \exp^{\circ s + s'}(z)$$

For appropriately chosen s and s'. This constructs what we'd think of as an appropriate fractional iteration of exponentiation; which satisfies the exponent law and takes real-values to real-values. Upon which the identity value $z \mapsto z$ is given at $s = 0$ and $\text{tet}_\beta(s)$ given at $z = 1$.

If we fix z; this produces a holomorphic function in s excepting branch cuts; and vice versa. Where the restriction $s, z \in \mathbb{R}^+$ implies $\exp^{\circ s}(z) \in \mathbb{R}^+$. This produces a family of functions ripe to construct pentation...

9 In Conclusion

To conclude this paper we broach the idea of doing this for more exotic functions. We ask if for other transcendental functions $h(z) : \mathbb{C} \to \mathbb{C}$, the asymptotic approach works to construct a super-function $H(z)$ such that $h(H(z)) = H(z+1)$; so H satisfies the inverse Abel equation. Constructing an arbitrary function,

$$\rho_\lambda(s) = \mathop{\Omega}_{j=1}^{\infty} \frac{h(z)}{e^{\lambda(j-s)} + 1} \bullet z$$

Which satisfies,

$$\rho_\lambda(s+1) = \frac{h(\rho_\lambda(s))}{e^{-\lambda s} + 1}$$

is not a difficult task, if the domains of h are well behaved. But pulling back with iterates $h^{\circ -n}$ is a very careful procedure. Upon which, we were lucky with e^z because log is a well behaved inverse. And despite the rapid growth of e^z, we were able to do this; where rapid growth is actually very beneficial. In essence, this method is more effective for rapid growing functions than it is for slowly growing functions.

Least of all, with these functions ρ_λ we can describe asymptotically what H should look like. Wherein, the equation,

$$h^{-1}(\rho_\lambda(s+1)) - \rho_\lambda(s) = \mathcal{O}(e^{-\lambda s}) \text{ as } |s| \to \infty \; \Re(\lambda s) > 0$$

Is certainly viable (so long as we have a decently well behaved function h^{-1} at ∞). But without a decently behaved inverse h^{-1}, the most we'd be able to say is that ρ_λ is a solution to the asymptotic inverse Abel equation–expressing the same thing but in a more implicit manner. The equation above being the frank way.

The author foresees no problem in utilizing this asymptotic method for $h(z) = b^z$ for $b > e^{1/e}$, where the iterates of h are unbounded here. He imagines this would follow little differently than the case for $b = e$; subtracting minor details. The complex plane $e^{\omega z}$ for $\omega \in \mathbb{C}$ is a different story though–it may be tractable, as long as its iterates are unbounded; though the complexity of the logarithms sounds like a serious headache.

The author also knows no way of understanding the dynamics of $\text{tet}_\beta(s-n)$ for $\Im(s) > 0$. This equates to the repeated application of the logarithm; for varying branches of log. The author is somewhat convinced this tetration $\text{tet}_\beta \neq \text{tet}_K$, Kneser's tetration. Where in this regard, he expects the iterated log's on $\text{tet}_\beta(s)$ may converge to varying fixed points, or diverge like the Julia set of the log map; and $\lim_{|s|\to\infty} \text{tet}_K(s) = L$ for $\pi/2 \leq \arg(s) \leq \pi$.

With this, I conjecture that $\lim_{n\to\infty} \text{tet}_\beta(s-n) \to L_{\Im s}, \infty$; where $L_{\Im s}$ is a fixed point $e^L = L$. And ∞ means that $\text{tet}_\beta(s) \in \mathcal{J}$ for $\mathcal{J}$ the Julia set of log; upon which repeated applications don't converge. Infer, we interpret ∞ as non-normality, and $L_{\Im s}$ as normality, and convergence towards a fixed point.

I, further, do not expect this solution to be Kneser's tetration because the behaviour as $\Im(s) = t \to \infty$ of tet_β should be ∞; as it should look like $\beta(it) + \mathcal{O}(e^{-\frac{it}{\sqrt{1+it}}})$, which $\beta(it)$ should tend to infinity (again the author isn't certain here, it just looks like it might work this way).

We thank the reader for their time, and their willingness to get to the bottom of this paper.

Appendix

We've attached here a proof of Theorem 1.1.

Theorem 9.1. *Let $\{H_j(s,z)\}_{j=1}^\infty$ be a sequence of holomorphic functions such that $H_j(s,z) : \mathcal{S} \times \mathcal{G} \to \mathcal{G}$ where $\mathcal{S}$ and $\mathcal{G}$ are domains in $\mathbb{C}$. Suppose there exists some $A \in \mathcal{G}$, such for all compact sets $\mathcal{N} \subset \mathcal{G}$, the following sum converges,*

$$\sum_{j=1}^{\infty} ||H_j(s,z) - A||_{z\in\mathcal{N}, s\in\mathcal{S}} < \infty$$

Then the expression,

$$H(s) = \lim_{n\to\infty} \Omega_{j=1}^{n} H_j(s,z) \bullet z = \lim_{n\to\infty} H_1(s, H_2(s, ...H_n(s,z)))$$

Converges uniformly for $s \in \mathcal{S}$ and $z \in \mathcal{N}$ as $n \to \infty$ to H, a holomorphic function in $s \in \mathcal{S}$, constant in z.

Proof. The first thing we show is for all $\epsilon > 0$, there exists some N, such when $m \geq n > N$,

$$|\Omega_{j=n}^{m} H_j(s,z) \bullet z - A| < \epsilon$$

For z in $\mathcal{N} \subset \mathcal{G}$ (where A is in the open component of $\mathcal{N}$), and $s \in \mathcal{S}$. This then implies as we let $m \to \infty$, the tail of the infinite composition stays bounded. Forthwith, the infinite composition becomes a normal family, and proving convergence becomes simpler. We provide a quick proof of this inequality.

Set $||H_j(s,z) - A||_{\mathcal{S},\mathcal{N}} = \rho_j$. Pick $\epsilon > 0$, and choose N large enough so when $n > N$,

$$\rho_n < \epsilon$$

Denote: $\phi_{nm}(s,z) = \Omega_{j=n}^{m} H_j(s,z) \bullet z = H_n(s, H_{n+1}(s, ...H_m(s,z)))$. We go by induction on the difference $m - n = k$. When $k = 0$ then,

$$||\phi_{nn}(s,z) - A||_{\mathcal{S},\mathcal{N}} = ||H_n(s,z) - A||_{\mathcal{S},\mathcal{N}} = \rho_n < \epsilon$$

Assume the result holds for $m - n < k$, we show it holds for $m - n = k$. Observe,

$$\begin{aligned}
||\phi_{nm}(s,z) - A||_{\mathcal{S},\mathcal{N}} &= ||H_n(s, \phi_{(n+1)m}(s,z)) - A||_{\mathcal{S},\mathcal{N}} \\
&\leq ||H_n(s,z) - A||_{\mathcal{S},\mathcal{N}} \\
&= \rho_n < \epsilon
\end{aligned}$$

Which follows by the induction hypothesis because $\phi_{(n+1)m}(s,z) \subset \mathcal{N}$–it's in a neighborhood of A which is in $\mathcal{N}$. That is $m - n - 1 < k$.

The next step is to observe that $\Omega_{j=1}^{m} H_j(s,z)$ is a normal family as $m \to \infty$, for $z \in \mathcal{N}$ and $s \in \mathcal{S}$. This follows because the tail of this composition is bounded. We can say $||\,\Omega_{j=1}^{m} H_j(s,z)||_{\mathcal{S},\mathcal{N}} < M$ for all m.

Since $\phi_m(s,z) = \Omega_{j=1}^{m} H_j(s,z) \bullet z$ are a normal family for all compact sets $\mathcal{N} \subset \mathcal{G}$; there is some constant $M \in \mathbb{R}^+$ and $L \in \mathbb{R}^+$ such,

$$||\frac{d^k}{dz^k}\phi_m(s,z)||_{\mathcal{S},\mathcal{N}} \leq M \cdot k! \cdot L^k$$

To see this, take $|z - A| < 2\delta$ and observe,

$$\frac{d^k}{dz^k}\phi_m(s,z) = \frac{k!}{2\pi i}\int_{|\xi - A| = 2\delta} \frac{\phi_m(s,\xi)}{(\xi - z)^{k+1}}\,d\xi$$

So that, taking the supremum norm across $|z - A| \leq \delta$

$$\begin{aligned}
||\frac{d^k}{dz^k}\phi_m(s,z)||_{\mathcal{S},|z-A|\leq\delta} &\leq \frac{k!}{2\pi}\int_{|\xi - A| = 2\delta} \frac{||\phi_m(s,\xi)||_{\mathcal{S}}}{|\xi - z|_{|z-A|\leq\delta}^{k+1}}\,d\xi \\
&\leq \frac{k!}{2\pi}\int_{|\xi - A| = 2\delta} \frac{M}{\delta^{k+1}}\,d\xi \\
&\leq \frac{Mk!}{\delta^{k+1}}
\end{aligned}$$

Where we've used the bound $|\xi - z| \geq \delta$ when $|\xi - A| = 2\delta$ and $|z - A| \leq \delta$. This bound can be derived regardless of $\mathcal{N}$ for varying M and L.

Secondly, using Taylor's theorem,

$$\begin{aligned}
\phi_{m+1}(s,z) - \phi_m(s,z) &= \phi_m(s, H_{m+1}(s,z)) - \phi_m(s,z) \\
&= \sum_{k=1}^{\infty} \frac{d^k}{dz^k}\phi_m(s,z)\frac{(H_{m+1}(s,z)-z)^k}{k!} \\
&= (H_{m+1}(s,z)-z)\sum_{k=1}^{\infty} \frac{d^k}{dz^k}\phi_m(s,z)\frac{(H_{m+1}(s,z)-z)^{k-1}}{k!}
\end{aligned}$$

So that, setting $z = A$,

$$\begin{aligned}
||\phi_{m+1}(s,A) - \phi_m(s,A)||_{s\in\mathcal{S}} &\leq ||H_{m+1}(s,A) - A||_{s\in\mathcal{S}} \sum_{k=1}^{\infty} ML^k ||H_{m+1}(s,A) - A||^{k-1} \\
&\leq ||H_{m+1}(s,A) - A||_{\mathcal{S}} \frac{ML}{1-q}
\end{aligned}$$

For $L||H_{m+1}(s,A) - A||_{\mathcal{S}} \leq q < 1$, which is true for large enough $m > N$. Setting $C = \frac{ML}{1-q}$. Applying from here,

$$||\phi_{m+1}(s,A) - \phi_m(s,A)||_{s\in\mathcal{S}} \leq C||H_{m+1}(s,A) - A||_{s\in\mathcal{S}}$$

This is a convergent series per our assumption. Choose N large enough, so that when $m, n > N$,

$$\sum_{j=n}^{m-1} ||H_{j+1}(s,A) - A||_{s\in\mathcal{S}} < \frac{\epsilon}{C}$$

Then,

$$\begin{aligned}
||\phi_m(s,A) - \phi_n(s,A)||_{s\in\mathcal{S}} &\leq \sum_{j=n}^{m-1} ||\phi_{j+1}(s,A) - \phi_j(s,A)||_{s\in\mathcal{S}} \\
&\leq C\sum_{j=n}^{m-1} ||H_{j+1}(s,A) - A||_{s\in\mathcal{S}} \\
&< \epsilon
\end{aligned}$$

So we can see $\phi_m(s)$ must be uniformly convergent for $s \in \mathcal{S}$, and therefore defines a holomorphic function $H(s)$ as $m \to \infty$.

This tells us,

$$H(s) = \Omega_{j=1}^{\infty} H_j(s,z) \bullet z\Big|_{z=A}$$

Converges and is holomorphic. To show this function equals,

$$\underset{j=1}{\overset{\infty}{\Omega}} H_j(s,z) \bullet z$$

For all $z \in \mathcal{G}$; simply notice that,

$$\underset{j=m}{\overset{\infty}{\Omega}} H_j(s,z) \bullet z$$

Is arbitrarily close to A as we let m grow (which was shown at the beginning of this proof). Then,

$$\begin{aligned}
\underset{j=1}{\overset{\infty}{\Omega}} H_j(s,z) \bullet z &= \underset{j=1}{\overset{m-1}{\Omega}} H_j(s,z) \bullet \underset{j=m}{\overset{\infty}{\Omega}} H_j(s,z) \bullet z \\
&= \lim_{m\to\infty} \underset{j=1}{\overset{m-1}{\Omega}} H_j(s,z) \bullet \lim_{m\to\infty} \underset{j=m}{\overset{\infty}{\Omega}} H_j(s,z) \bullet z \\
&= \underset{j=1}{\overset{\infty}{\Omega}} H_j(s,z) \bullet z \Big|_{z=A}
\end{aligned}$$

This concludes our proof.

□

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Regards, James

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