# Notes on the beta method

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**Abstract**

Below are a series of notes on the construction of a periodic solution to tetration. This solution can be vastly generalized, but it's nice to have a hands on approach for a single case. The case we care about in these notes,is a tetration function in a strip, with a periodic structure. In such a sense–it is a tetration function on the cylinder $\mathbb{C}/2\pi i\mathbb{Z}$.

## 1 Introduction

The following is a brief series of notes on constructing $F(s)$ in which,

$$\begin{aligned}
F(0) &= 1\\
F(s+2\pi i) &= F(s)\\
F &: (-2,\infty)\to\mathbb{R} \text{ bijectively}\\
e^{F(s)} &= F(s+1)\\
F & \quad \text{is holomorphic for } |\Im(s)|<\pi \text{ and } s\notin(-\infty,-2]
\end{aligned}$$

And additionally, derived from this, it's conjugate similar $F(\overline{z})=\overline{F(z)}$. This tetration is constructed using the function $\beta(s)$:

$$\begin{aligned}
\beta(-\infty) &= 0\\
\beta(s+2\pi i) &= \beta(s)\\
\beta &: \mathbb{R}\to\mathbb{R}^+ \text{ bijectively}\\
\frac{e^{\beta(s)}}{e^{-s}+1} &= \beta(s+1)\\
\beta & \quad \text{is holomorphic for } |\Im(s)|<\pi
\end{aligned}$$

This beta function is constructed using infinite compositions. We will begin with a brief discussion of infinite compositions; but we'll pivot towards the

Taylor series approach–as this is what you are more familiar with. In the method I construct, it isn't *necessary* to use infinite compositions–but, in my opinion, it's the more natural way.

## 2 Constructing $\beta$

The technical way of constructing $\beta$ is to use the infinite composition theorem:

**Theorem 2.1.** *Let $\{H_j(s,z)\}_{j=1}^\infty$ be a sequence of holomorphic functions such that $H_j(s,z) : \mathcal{S} \times \mathcal{G} \to \mathcal{G}$ where $\mathcal{S}$ and $\mathcal{G}$ are domains in $\mathbb{C}$. Suppose there exists some $A \in \mathcal{G}$, such for all compact sets $\mathcal{N} \subset \mathcal{G}$, the following sum converges,*

$$\sum_{j=1}^\infty ||H_j(s,z) - A||_{z\in\mathcal{N},s\in\mathcal{S}} = \sum_{j=1}^\infty \sup_{z\in\mathcal{N},s\in\mathcal{S}} |H_j(s,z) - A| < \infty$$

*Then the expression,*

$$H(s) = \lim_{n\to\infty} \Omega_{j=1}^{n} H_j(s,z) \bullet z = \lim_{n\to\infty} H_1(s, H_2(s, ...H_n(s,z)))$$

*Converges uniformly for $s \in \mathcal{S}$ and $z \in \mathcal{N}$ as $n \to \infty$ to $H$, a holomorphic function in $s \in \mathcal{S}$, constant in $z$.*

A quick way to identify this is related to our problem, we need to start with our versions of $H_j$. We'll call them $q_j$ such that,

$$q_j(s,z) = \frac{e^z}{e^{j-s}+1}$$

Then, if we look at the above theorem it means since,

$$\sum_{j=1}^\infty ||q_j(s,z)||_{s\in\mathcal{S},z\in\mathcal{K}} = \sum_{j=1}^\infty \sup_{s\in\mathcal{S},z\in\mathcal{K}} |q_j(s,z)| < \infty$$

For all $\mathcal{S} \subset \{|\Im(s)| < \pi\}$ (which are compact) and for all $\mathcal{K} \subset \mathbb{C}$ (which are compact); we must have that,

$$
\begin{aligned}
q_1(s,z) &= \frac{e^z}{e^{1-s}+1} \\
q_1(s,q_2(s,z)) &= \frac{e^{\frac{e^z}{e^{2-s}+1}}}{e^{1-s}+1} \\
q_1(s,q_2(s,q_3(s,z))) &= \frac{e^{\frac{e^{\frac{e^z}{e^{3-s}+1}}}{e^{2-s}+1}}}{e^{1-s}+1} \\
&\cdots \\
\beta(s) &= \lim_{n\to\infty} q_1(s,q_2(s,q_3(s,...q_n(s,z))))
\end{aligned}
$$

And this converges uniformly on $\mathcal{S},\mathcal{K}$. Now, an important part of this, is that the $z$ variable disappears. Think of this as decaying to a value. The value $z \to 0$; so we can throw it away in the end. This is a very delicate thing that's done using advanced complex analysis. If you don't understand it, that's perfectly understandable; but it's solid $\epsilon/\delta$.

This is to say that,

$$
\lim_{n\to\infty} q_1(s,q_2(s,q_3(s,...q_n(s,z)))) = \lim_{n\to\infty} q_1(s,q_2(s,q_3(s,...q_n(s,z))))\Big|_{z=0}
$$

A naive way of seeing this is because $q_n(s,z) \to 0$ as $n \to \infty$. Now, recall that, by definition:

$$
q_j(s-1,z) = q_{j+1}(s,z)
$$

And that, if we define:

$$
q_0(s,z) = \frac{e^z}{e^{-s}+1}
$$

Then,

$$
\begin{aligned}
\beta(s+1) &= \lim_{n\to\infty} q_1(s+1,q_2(s+1,q_3(s,...q_n(s+1,z)))) \\
&= \lim_{n\to\infty} q_0(s,q_1(s,q_2(s,...q_{n-1}(s,z)))) \\
&= q_0(s,\lim_{n\to\infty} q_1(s,q_2(s,...q_{n-1}(s,z)))) \\
&= q_0(s,\beta(s)) \\
&= \frac{e^{\beta(s)}}{e^{-s}+1}
\end{aligned}
$$

Which gives the functional equation we so desired. And inherently shows, this function will be holomorphic everywhere $q_j(s,z)$ is holomorphic. As each of these functions have singularities at $s_0 = j + (2k+1)\pi i$ for $k \in \mathbb{Z}$. We must have $\beta$ is holomorphic on $|\Im(s)| < \pi$. It's real valued by its expression. And by its functional equation tends to zero as $\Re(s) \to -\infty$.

And lastly it is $2\pi i$ periodic in $s$ because each $q_j$ is $2\pi i$ periodic in $s$.

# 3 The Taylor series approach

Here is where Sheldon's insight comes in handy. We're going to make the variable change that $w = e^s$. We are now looking at:

$$r_j(w,z) = \frac{we^z}{w + e^j}$$

which satisfies $r_j(w/e, z) = r_{j+1}(w,z)$. And we can talk about,

$$g(w) = \lim_{n\to\infty} r_1(w, r_2(w, ...r_n(w,z)))$$

which satisfies,

$$\begin{aligned} g(0) &= 0 \\ g(ew) &= \frac{we^{g(w)}}{w+1} \\ g & \quad \text{is holomorphic for } |w| < e \end{aligned}$$

The thing about this is though; we can discover taylor series at $w = 0$. This is what I like to call a "mock Schröder" equation. It almost satisfies an inverse Schröder equation; except it has a multiplier $w/w+1$. Either way we can iteratively derive a Taylor series through the formula:

$$g^{(n)}(0) = \frac{1}{e^n} \frac{d^n}{dw^n}\Big|_{w=0} \frac{we^{g(w)}}{w+1}$$

I attribute this to you, Sheldon. Because you did this exact same change of variables but with the $\phi$ method. I had never really thought of it like that.

As to the rigor to this. I prove all of my holomorphy discussions using infinite compositions. If you avoid this, you have to prove that the Taylor series converges. I don't know how to do this without using infinite compositions. I can do it on case by case manner. But why would I do that, when I can do it universally with infinite compositions. Again, my use of infinite compositions is $\epsilon/\delta$ solid. This isn't what's on trial right now. What's on trial is the tetration.

So let's move on to tetration.

## 4 Ψ–Schroder's function

The goal of this section, is to solve the schroder equation. The equation we are looking to solve is,

$$\Psi(ew) = e^{\Psi(w)}$$

Such that, there is a function $u(w)$ which satisfies $\lim_{\Re(w)\to\infty} u(w) = 0$, and:

$$\Psi(w) = g(w) + u(w)$$

This isn't typically of what you think of when you think of Schröder's functional equation. It is not about a fixed point; it's simply the solution–without being attached to a fixed point.

Now, the real question, how do you find $u$? I won't go through the nitty gritty rigor right now. I'm painting in broadstrokes at the moment, but,

$$\begin{aligned} u^0 &= 0 \\ u^{n+1}(w) &= -\log(1 + 1/w) + \log(1 + \frac{u^n(ew)}{g(ew)}) \end{aligned}$$

In which we can (with a good amount of work; and in the right context) show that:

$$u^n(\infty) = 0$$

Now we're just taking a limit as $n \to \infty$, but we're normal around a point. Think, all our values are in a bounded neighborhood. We're using a, kind of, bolzano weierstrass theorem. Additionally we can apply Banach's fixed point theorem in a clever way. This is most definitely the most difficult part of my paper. But there exists $0 < \mu < 1$ in which we get the inequality:

$$||u^{n+1}(w) - u^n(w)||_{\mathcal{B}} \leq \mu ||u^n(w) - u^{n-1}(w)||_{\mathcal{B}}$$

For a compact set $\mathcal{B} \subset \{|w| > R,\ w \notin (-\infty, -R)\}$–and this is the supremum norm. The constant $\mu$ is calculated fairly implicitly–but rigorously. If we examine the expressions:

$$\begin{aligned} u^{n+1}(w) - u^n(w) &= \log\Big(\frac{g(ew) + u^n(ew)}{g(ew) + u^{n-1}(ew)}\Big) \\ |u^{n+1}(w) - u^n(w)| &= \Big|\log\Big(\frac{g(ew) + u^n(ew)}{g(ew) + u^{n-1}(ew)}\Big)\Big| \\ ||u^{n+1}(w) - u^n(w)||_{\mathcal{B}} &\leq A||u^n(ew) - u^{n-1}(ew)||_{\mathcal{B}} \end{aligned}$$

We can derive that for large enough $R$ that $A < 1 + \epsilon$ for $\epsilon \to 0$. But for large enough $R$,

$$||u^n(ew) - u^{n-1}(ew)||_{\mathcal{B}} \leq q||u^n(w) - u^{n-1}(w)||_{\mathcal{B}}$$

Where $1/e < q < 1$ in which, $q \to 1/e$ as $R \to \infty$. So if we write $qA = \mu$, we can expect this to be less than 1 for large enough $R$ and appropriate compact sets.

This means this sequence of functions will converge on compact sets for large enough $|w| > R$ and $w \notin (-\infty, -R)$. This gives us a Schröder function in the "neighborhood" of infinity on the Riemann Sphere–avoiding the negative real-axis.

There's a deeper trick to this result, but it's difficult to explain without more familiarity with advanced complex analysis . So let's just stick to broad brushstrokes. This gives us a function:

$$\begin{aligned}
\Psi(w) &: \{w \in \mathbb{C} \,|\, |w| > R\}/(-\infty, -R] \to \mathbb{C} \\
\Psi(x) &: \mathbb{R}_{x>R} \to \mathbb{R}^+ \\
\Psi(ew) &= e^{\Psi(w)}
\end{aligned}$$

And again, by consequence, $\Psi(\overline{w}) = \overline{\Psi(w)}$. This will be the basis to construct our Abel function.

# 5 $F$ – The Abel function

We're going to start with the function,

$$f(s) = \Psi(e^s)$$

Which satisfies the equation:

$$f(s+1) = e^{f(s)}$$

for all $\Re(s) > \log(R)$ and $|\Im(s)| < \pi$. Now, we can pull this back with logarithms on $s \in \mathbb{R}^+$; and it'll be at least analytic on the real line. The trouble is, how do we ensure that it is holomorphic for $0 < \Im(s) < \pi$? This is again a nuanced argument; which I would present, but it may be confusing. I'll just stick to broad brush strokes; and say that $\log^{\circ n} \Psi(w) = \Psi(w/e^n)$ is a normal family for $\Im(w) > 0$; and so singularities that appear are sparse–we can use this to show that there are in fact no singularities.

We can associate $\Psi(0^+) = L$ and $\Psi(0^-) = \overline{L}$; where we take $w/e^n \to 0$ for $\Im(w) > 0$ or $< 0$. But $\Psi(w/e^n) \to \infty$ for $w \in \mathbb{R}^+$. This equates to the upper/lower half planes of Kneser–and the real line which is a branchcut/riddled with singularities.

And the key to remember, is we are calculating something small. We're not really calculating iterated logs. We're calculating a small number which makes $\beta(s) + \tau(s)$ be tetration; where beta already looks like tetration. As this will satisfy the functional equation; if $\tau$ blows up, then it's not really satisfying the equation is it?

The trouble which could arise would arise way off in the left half plane. But $\tau(-\infty)$ in the strip $0 < \Im(s) < \pi$ should decay to $L$ the first fixed point in the upper half plane. So that,

$$\begin{aligned}
\beta(-\infty) &= 0 \\
\exp(\beta(-\infty) + \tau(-\infty)) &= \beta(-\infty) + \tau(-\infty) \\
\exp(\tau(-\infty)) &= \tau(-\infty) \\
\tau(-\infty) &= L
\end{aligned}$$

This is seen because the process reduces to a fixed point here. So, the math stays consistent. On the real line this doesn't happen because the logarithm is not normal on the real line. But it is normal in the upper/lower half planes. This is largely why Kneser works; it needs to decay to L at positive imaginary infinity; because the principal branch of the log decays to this fixed point in the upper half plane. These are the fatou sets of the principal brach of log.

We're in a similar situation here, but instead of a half plane, we're in a strip; but the upper portion of the strip is the upperhalf plane fatou set (at least for small values). This allows us to find a value $s_0 \approx 2$ in which,

$$F(s) = f(s + s_0)$$

And it's our desired tetration. It will be as we wrote in the beginning:

$$\begin{aligned}
F(0) &= 1 \\
F(s + 2\pi i) &= F(s) \\
F &: (-2, \infty) \to \mathbb{R} \text{ bijectively} \\
e^{F(s)} &= F(s+1) \\
F & \quad \text{is holomorphic for } |\Im(s)| < \pi \text{ and } s \notin (-\infty, -2]
\end{aligned}$$

## 6 In Conclusion

I hope I can better explain the deeper intricacies of how this method works in the future. Because this just scratches the surface of the math. I am going rather slowly here; I moved really fast in my preprint paper mathematically, largely to get it all out quickly. Infinite compositions can be really confusing, but they're the life blood of my work.

I appreciate your time and considerations.
Regards, James.