On the first derivative of the n-th tetration of $f(x)$

Luca Onnis

October 2020

Abstract

In this document will be studied the recursive properties of the first derivative of the n-th tetration of a real function $f(x)$. Note that as the hyper-exponent grows, the first derivative of the tetration became more and more complicated; in this paper I'll prove by induction a formula in order to compute it. The new function will depend on the hyper-exponent n and on $f(x)$. In mathematics, tetration is an operation based on iterated, or repeated, exponentiation. It is the next hyperoperation after exponentiation, but before pentation. The word was coined by Reuben Louis Goodstein from tetra- (four) and iteration. ${}^{a}n$ represent the a-th tetration of n , or:

$$n^{n^{n^{\cdots}}}\Big\}\, a \text{ times}$$

Or using Knuth's notation: $a \uparrow\uparrow n = {}^{n}a$.

1 Introduction

Consider the first derivative of the function:

$$f(x) = (\sin x)^{(\sin x)^{(\sin x)}}$$

The question requires lots of computation, but with our formula we can conclude that:

$$\frac{d}{dx}f(x) = \frac{d}{dx}{}^{3}\sin(x) = (\sin x)^{(\sin x)^{(\sin x)}}(\sin x)^{(\sin x)}(\cot x)\Big\{(\sin x)[(\log(\sin x))^2 + \log(\sin x)] + 1\Big\}$$

As the second example, consider the first derivative of this function:

$$\frac{d}{dx}x^{x^{x^{x}}} = \frac{d}{dx}{}^{4}x = x^{x^{x^{x}}}x^{x^{x}}\frac{1}{x}\Big\{x^{x}x\Big[[\log(x)]^3 + [\log(x)]^2\Big] + x^{x}\log(x) + 1\Big\}$$

2 Theorem 1

Let $f(x)$ a real function with $n \in \mathbb{N}$, $n \geq 2$, furthermore let $f'(x)$ be the first derivative of $f(x)$, hence:

$$\frac{d}{dx}{}^n f(x) = {}^n f(x)^{n-1} f(x) \frac{f'(x)}{f(x)} \left\{ \sum_{j=0}^{n-2} \left[\prod_j^{n-2} {}^j f(x) \right] \left[\log\Big(f(x)\Big) \right]^{n-j-1} + 1 \right\}$$

I'll prove this by induction on n. Consider the $n = 2$ case.

$$\frac{d}{dx} f(x)^{f(x)} = \frac{d}{dx} e^{\log f(x)^{f(x)}} = \frac{d}{dx} e^{f(x)\log f(x)}$$

And applying the definition of the derivative of a product of functions we'll have:

$$\frac{d}{dx} f(x)^{f(x)} = f(x)^{f(x)}\left\{ f(x)\frac{f'(x)}{f(x)} + f^1(x)\log\Big(f(x)\Big)\right\} = f(x)^{f(x)} f^1(x)\Big[\log\Big(f(x)\Big)+1\Big]$$

And using our formula we'll have:

$$\frac{d}{dx}{}^2 f(x) = {}^2 f(x) f(x) \frac{f'(x)}{f(x)}\left\{\sum_{j=0}^{0}\left[\prod_0^0 j_{f(x)}\right]\left[\log\Big(f(x)\Big)\right]^{2-j-1} + 1\right\} = f(x)^{f(x)} f'(x)\Big[\log\Big(f(x)\Big)+1\Big]$$

Now consider the induction case $(n + 1)$:

$$\frac{d}{dx}{}^{n+1} f(x) = \frac{d}{dx} f(x)^{{}^n f(x)}$$

For the definition of tetration.

$$\frac{d}{dx}{}^{n+1}f(x) = f(x)^{{}^n f(x)-1}\left\{{}^n f(x) f'(x) + f(x)\log\Big(f(x)\Big)\frac{d}{dx}{}^n f(x)\right\}$$

But we can write $\frac{d}{dx}{}^n f(x)$ using the induction hypothesis:

$$d = \frac{f(x)^{{}^n f(x)}}{f(x)}\left\{{}^n f(x) f'1(x) + f(x) log\Big(f(x)\Big) {}^n f(x)^{n-1} f(x)\frac{f'(x)}{f(x)}\left\{\sum_{j=0}^{n-2}\left[\prod_j^{n-2}{}^j f(x)\right]\left[\log\Big(f(x)\Big)\right]^{n-j-1}+1\right\}\right.$$

Where $d = \frac{d}{dx}{}^{n+1}f(x)$.
Furthermore $f(x)^{{}^n f(x)} = {}^{n+1}f(x)$, so:

$$d = {}^{n+1}f(x)^n f(x)\frac{f'(x)}{f(x)}\left\{log\Big[f(x)\Big]^{n-1}f(x)\left(\sum_{j=0}^{n-2}\left[\prod_j^{n-2}{}^j f(x)\right]\left[log\Big(f(x)\Big)\right]^{n-j-1}\right)+{}^{n-1}f(x)\log\Big[f(x)\Big]+1\right\}$$

$$\frac{d}{dx}{}^{n+1}f(x) = {}^{n+1}f(x)^n f(x)\frac{f'(x)}{f(x)}\left\{\sum_{j=0}^{n-1}\left[\prod_j^{n-1}{}^j f(x)\right]\left[\log\Big(f(x)\Big)\right]^{n-j}+1\right\}$$

Because ${}^{n-1}f(x)\log\Big[f(x)\Big]$ multiplies all the terms of the summation and the ${}^{n-1}f(x)log\Big[f(x)\Big]$ term at the end is the last of the sum, or:

$$\sum_{j=n-1}^{n-1}\Big[\prod_{j}^{n-1}{}^{j}f(x)\Big]\Big[\log\Big(f(x)\Big)\Big]^{n-j}$$

So the conclusion is that:

$$\frac{d}{dx}{}^{n}f(x) = {}^{n}f(x)^{n-1}f(x)\frac{f'(x)}{f(x)}\Big\{\sum_{j=0}^{n-2}\Big[\prod_{j}^{n-2}{}^{j}f(x)\Big]\Big[\log\Big(f(x)\Big)\Big]^{n-j-1}+1\Big\}$$

is true for all real function and for all $n \geq 2$

3 Conclusions

We can use this result to compute this complicated and long derivative faster.