Another note on Andy's SLOG

Here I want to provide another insight into Andrew's method strictly in the view of my matrix-formalism. Here we find the connection to the method of continuum sum.

Assume the (Bell-)matrix ***B*** which performs the exponential map on a vandermonde-vector (of infinite size):

$$V(x)\sim * B = V(e^x)\sim$$

This is simply another notation for the formalisms of the needed manipulations on formal powerseries of the exponential-function which shall also allow iteration.

This matrix is very simple: it has the form of a factorially scaled Vandermonde matrix of infinite size

$$\mathbf{B} := b_{r,c=0..inf} = c^r/r!$$

and its aspect is (top-left segment):

B

1/0!	1/0!	1/0!	1/0!	1/0!	1/0!	1/0!	...
0/1!	1/1!	2/1!	3/1!	4/1!	5/1!	6/1!	...
0/2!	1/2!	4/2!	9/2!	16/2!	25/2!	36/2!	...
0/3!	1/3!	8/3!	27/3!	64/3!	125/3!	216/3!	...
0/4!	1/4!	16/4!	81/4!	256/4!	625/4!	1296/4!	...
0/5!	1/5!	32/5!	243/5!	1024/5!	3125/5!	7776/5!	...
0/6!	1/6!	64/6!	729/6!	4096/6!	15625/6!	46656/6!	...
...	...	...	...	...	...	...	...

Next consider the matrix $\mathbf{B_1} = \mathbf{B} - \mathbf{I}$ which performs

$$\begin{aligned} V(x)\sim (B - I) &= V(x)\sim * B_1 \\ &= V(e^x)\sim - V(x)\sim \end{aligned}$$

where $\mathbf{B_1}$ looks like

B1

0/0!	1/0!	1/0!	1/0!	1/0!	1/0!	1/0!	...
0/1!	0/1!	2/1!	3/1!	4/1!	5/1!	6/1!	...
0/2!	1/2!	2/2!	9/2!	16/2!	25/2!	36/2!	...
0/3!	1/3!	8/3!	21/3!	64/3!	125/3!	216/3!	...
0/4!	1/4!	16/4!	81/4!	232/4!	625/4!	1296/4!	...
0/5!	1/5!	32/5!	243/5!	1024/5!	3005/5!	7776/5!	...
0/6!	1/6!	64/6!	729/6!	4096/6!	15625/6!	45936/6!	...
...	...	...	...	...	...	...	...

More explicitely this means:

		*	**(B – I)**
	$[1, x, x^2, x^3, ...]$	=	$[1, e^x, e^{2x}, e^{3x}, ...]$ $- [1, x, x^2, x^3, ...]$

The key idea for the *slog*-function is, that we can use that matrix for the construction of a telescoping sum. We introduce the iterates of e^x (where we assume the value 0 for x):

$$(V(0) + V(1) + V(e) + V(e^e) + V(e^{e^e}) + V({}^4e) + \ldots + V({}^{n-1}e))\sim * B1 = V({}^ne)\sim - V(0)\sim$$

or

$$\left(\sum_{k=-1}^{n-1} V({}^k e)\sim\right) * (B-I) = V({}^n e)\sim - V(0)\sim$$

In table-display:

		*	**(B – I)**
k=-1	$[1, 0, 0, 0, \ldots]$	=	$-[1, 0, 0, 0, \ldots]$ $+[1, 1, 1, 1, \ldots]$
k=0	$+[1, 1, 1, 1, \ldots]$		$-[1, 1, 1, 1, \ldots]$ $+[1, e, e^2, e^3, \ldots]$
k=1	$+[1, e, e^2, e^3, \ldots]$		$-[1, e, e^2, e^3, \ldots]$ $+[1, e^e, (e^e)^2, (e^e)^3, \ldots]$
k=2	$+[1, e^e, (e^e)^2, (e^e)^3, \ldots]$		$-[1, e^e, (e^e)^2, (e^e)^3, \ldots]$ $+[1, {}^3e, ({}^3e)^2, ({}^3e)^3, \ldots]$
	$+\ldots$		$\ldots$
k=n-1	$+[1, {}^{n-1}e, ({}^{n-1}e)^2, ({}^{n-1}e)^3, \ldots]$		$-[1, {}^{n-1}e, ({}^{n-1}e)^2, ({}^{n-1}e)^3, \ldots]$ $+[1, {}^ne, ({}^ne)^2, ({}^ne)^3, \ldots]$

expressed in the sum-notation:

	*	**(B – I)**
$\sum_{k=-1}^{n-1} [1, {}^ke, ({}^ke)^2, \ldots]$	=	$[1, {}^ne, ({}^ne)^2, ({}^ne)^3, \ldots]$ $-[1, 0, 0, 0, \ldots]$

or, the result compressed,

	*	**(B – I)**
$\sum_{k=-1}^{n-1} [1, {}^ke, ({}^ke)^2, \ldots]$	=	$[0, {}^ne, ({}^ne)^2, ({}^ne)^3, \ldots]$

Now, if we attempt to do the inverse operation we see, that B_1 cannot be inverted. But the first column of B_1 is zero and also the first column of the result.

So it is "natural" to assume that we can delete the first empty column of B_1 to get the reduced matrix, call it " **B1»** " and do the inversion on that square residuum. Call that inverse matrix **C»**. .

Then we have from

	*	**B1»**
$\sum_{k=-1}^{n-1} [1, {}^ke, ({}^ke)^2, \ldots]$		$[{}^ne, ({}^ne)^2, ({}^ne)^3, \ldots]$

that the inverse operation should reproduce the original left-hand expression

	*	**B1»**	*	**C»**
$\sum_{k=-1}^{n-1} [1, {}^ke, ({}^ke)^2, \ldots]$	=	$[{}^ne, ({}^ne)^2, ({}^ne)^3, \ldots]$	=	$\sum_{k=-1}^{n-1} [1, {}^ke, ({}^ke)^2, \ldots]$

The "creative" aspect here is, that in the first column of the result we have a sum of units, which means, that the height-parameter n in ${}^n e$ is mapped to the sum of n ones.

This effect provides the principle of a counting of iterations.

We can concentrate on the first column of $\mathbf{C}$» (indicated by the square-brackets index $[0]$) and of the result only because we are only interested in the (iteration)-count:

$$[\ {}^n e,\ \ ({}^n e)^2,\ \ ({}^n e)^3, \ldots] * \mathbf{C}» [0] \qquad = \sum_{k=-1}^{n-1} 1$$

or:

$$ {}^n e\ V({}^n e)\sim \qquad * \mathbf{C}» [0] \qquad = 1+n$$

In principle we have just found a/the desired solution for the iterative "height"-function (usually called "*superlog*" oder "*slog*") by this sequence of operations.

Now let's brush it up a bit. On the lhs the vector is in principle a vandermonde vector multiplied by its argument: ${}^n e\ V({}^n e) \sim$. We can adapt it to a proper $V({}^n e)$ -vector if we prefix that vector with a unit and also append a row on top of the matrix $\mathbf{C}$» making an according matrix $\mathbf{C}$. The contents is arbitrary and simply could be zero – but there is a natural preference.

We want that the following holds:

$$V({}^n e) \sim * C\ [0] = n$$

but by the current configuration if we choose zeroes as first row for the matrix $\mathbf{C}$ we have

$$V({}^n e) \sim * C\ [0] = 1+n$$

This should be decremented by 1. So instead of a zero we insert the correction term -1 in that new row to make the *height*-("*slog*")-function giving the naturally expected measure for the zero-iteration.

So far this is only a reinterpretation / reformulation of Andrew Robbins' *slog*-construction.

It is interesting (but not yet discussed here) that this method works also for fractional n. This reminds of the principle of "continuum sums" with its fractional summation indexes. I'll see, whether I can insert some remarks about this in a later version of this article. Also the question of convergence of the occuring powerseries must be discussed. And it must be checked, that the inversion of the reduced matrix does not introduce some non-obvious inconsistencies with other common operations on formal powerseries.

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