Notes concerning the complex infinite towers

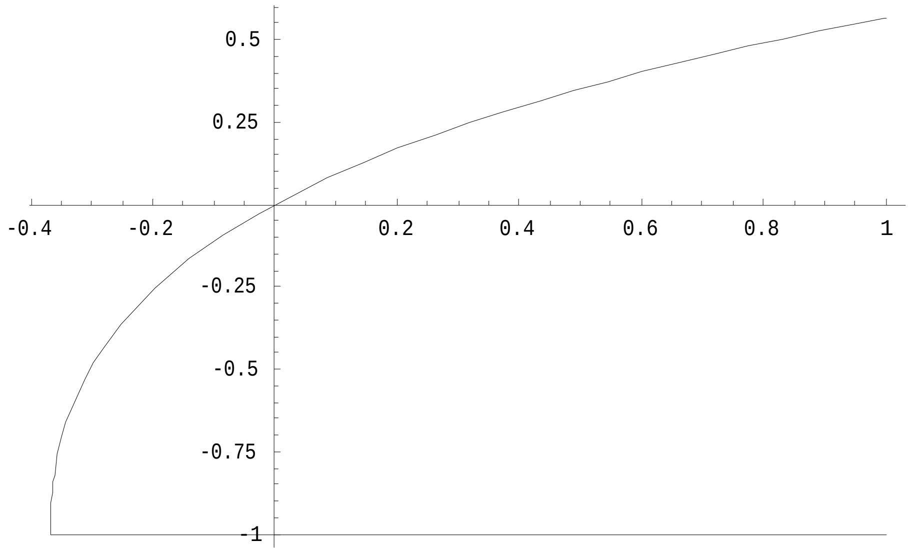
These notes refer to a comment posted by Gottfried on 11-09-07 (01:18 am) and concern the search for infinite towers, for bases $b > \sqrt[e]{e}$. We know that no real infinite towers exist in that range and, therefore, we must limit our curiosity to the possible existence of *complex infinite towers*. Gottfried succeeded in drawing a plot in the complex plan of the real and imaginary part of these hypothetical *infinite complex towers* and I think that his findings are correct. Let us start from the definition of infinite towers as:

$$h = {}^{\infty}b = b^h \qquad \to \qquad b = \sqrt[h]{h}$$

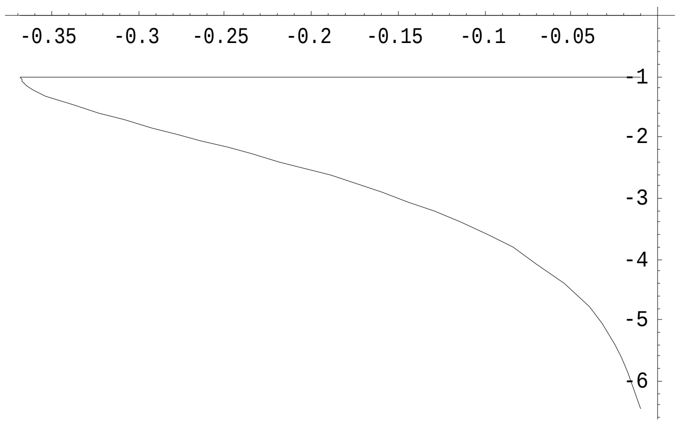
The problem is to verify if, for *b* real, we can find any complex *h*, satisfying these relations.

I propose to develop the formalism taken from the definition of the Lambert Function W, the product-logarithm (complex and multi-valued) function, which includes an infinity of complex branches and only two real ones.

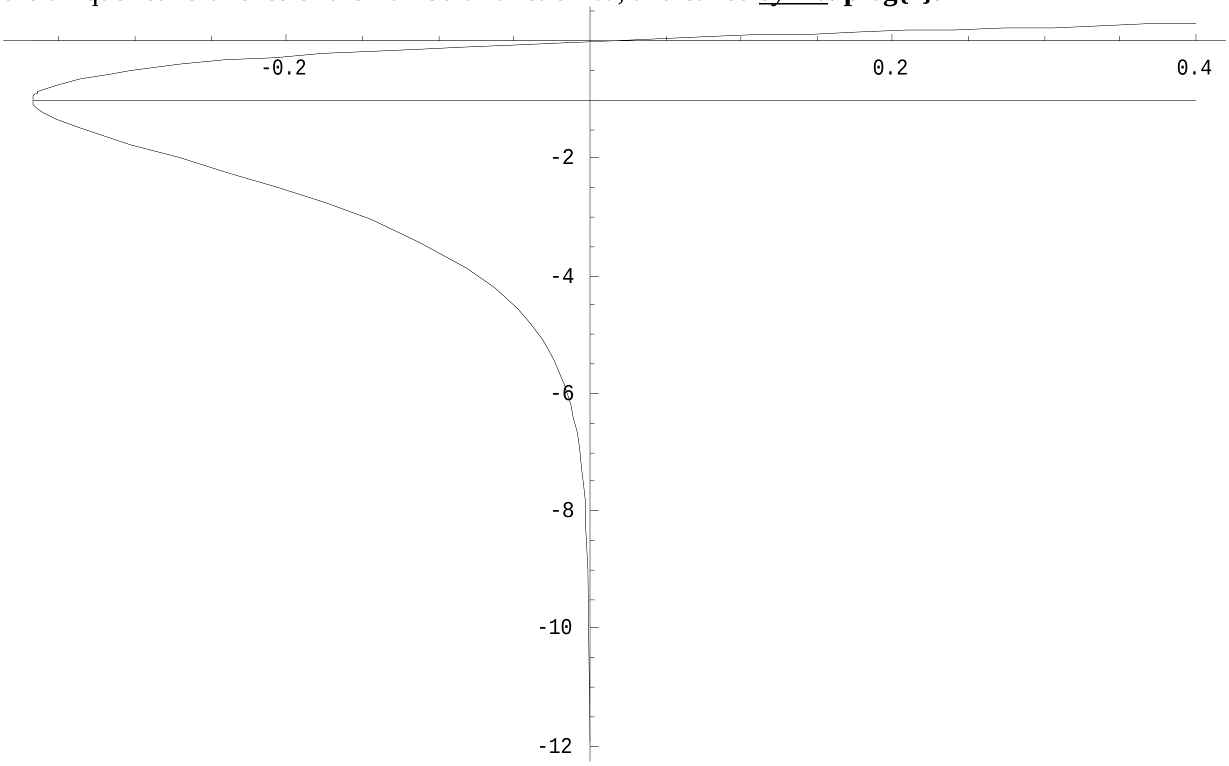
Next figure is the plot of the first real branch of the product-logarithm function, identified in *Mathematica* with operator: **ProductLog[0,*z*]** or **ProductLog[*z*]**, for $1/e < z < +\infty$ **.** The fundamental branch.



Next figure is the plot of the second real branch of the product-logarithm function, identified in *Mathematica* with operator **ProductLog[-1,*z*]**, for $1/e < z < 0$ **:**



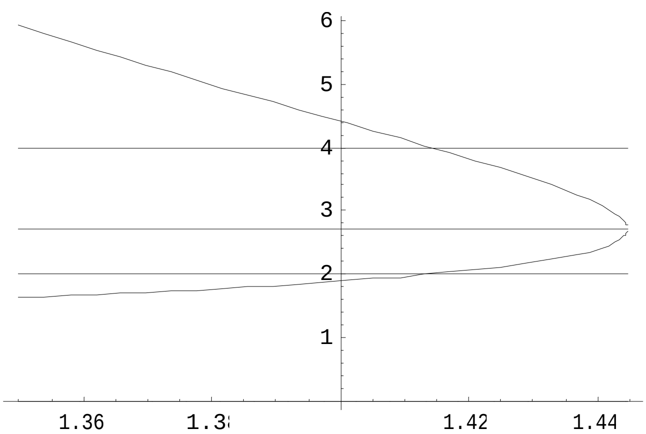
The following graph is the plot of both the above-mentioned real branches of the product-logarithm function, which jointly can be taken as the definition of a two-valued real function of z, representing the unique real branches of the Lambert Function W, and called <u>*by me*</u>: **plog{z}.**



Other complex branches of the product-logarithm function (the Lambert Function ***W***) can be made available within *Mathematica*, by using the operator **ProductLog[k,z].**

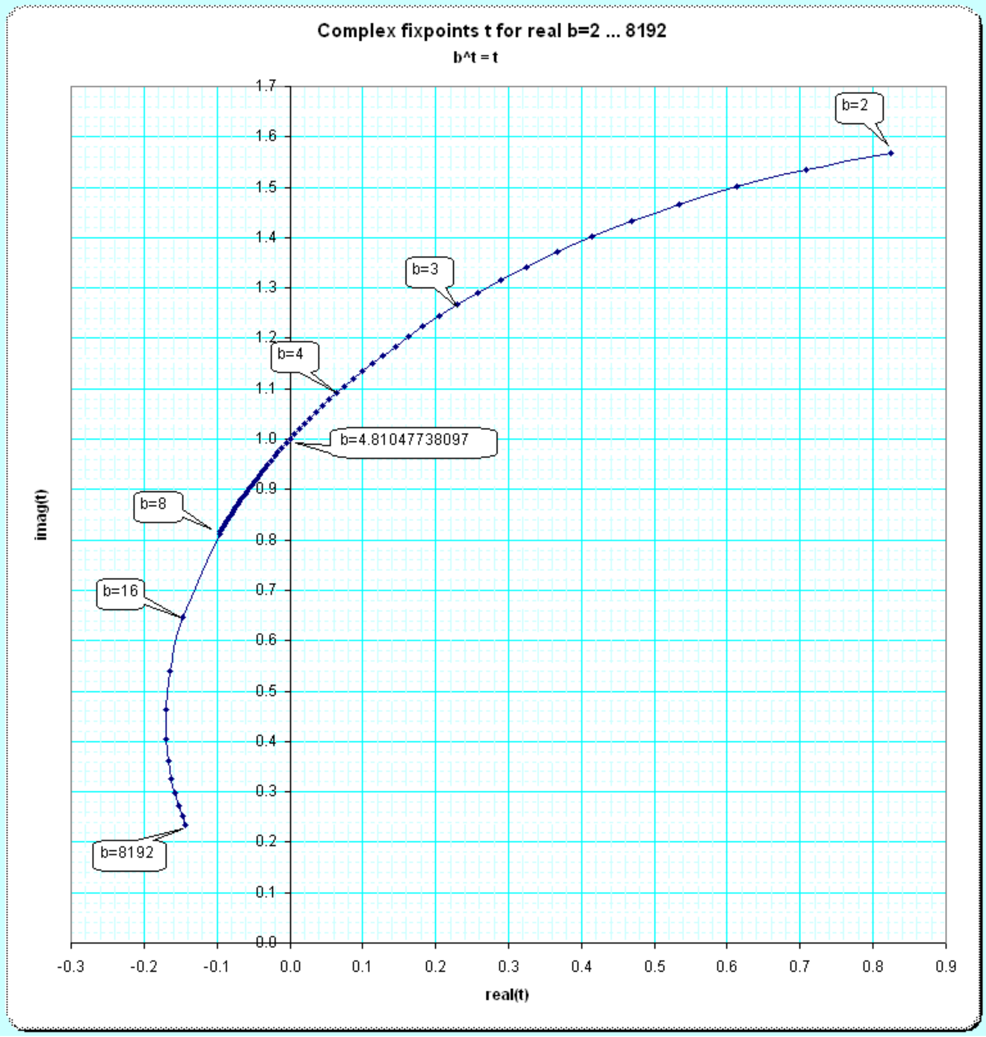
The following plot of the "real" infinite towers (please see the <u>two</u> "branches") was obtained by using the following formula, to be intended as the "logical union" of the ProductLog's of order 0 and -1:

$$^{\infty}b = h(b) = \frac{\operatorname{plog}(-\ln(b))}{-\ln b}$$



Axis b is horizontal and axis $h = {}^{\infty}b$ is vertical. The plot puts in evidence (as an example) the two values of ${}^{\infty}b$, for $b = \sqrt{2}$, which are $h = {}^{\infty}\left(\sqrt{2}\right) = \{2, 4\}$. The maximum possible values of b for obtaining *real infinite towers* is $\eta = \sqrt[e]{e} = 1.444667861..$. Let us try ... harder!

We may suppose that, after the famous η (i.e.: for $b > \sqrt[e]{e}$), the solutions are complex, as shown by Gottfried. As a matter of fact, he found the following nice plot of the real and imaginary part of the complex infinite towers, for $b > \eta$:



The results obtained by Gottfried seem perfectly all right. In fact, by using the following expressions, we obtain (with ***either*** the "plog" *Mathematica* operators, **ProductLog[z]** and **ProductLog[-1,z]**) :

$$h(2) = \frac{\text{plog}(-\ln 2)}{-\ln 2} = 0.824679 \pm 1.56743\,\mathbf{i}$$

$$h(3) = \frac{\text{plog}(-\ln 3)}{-\ln 3} = 0.22975 \pm 1.26645\,\mathbf{i}$$

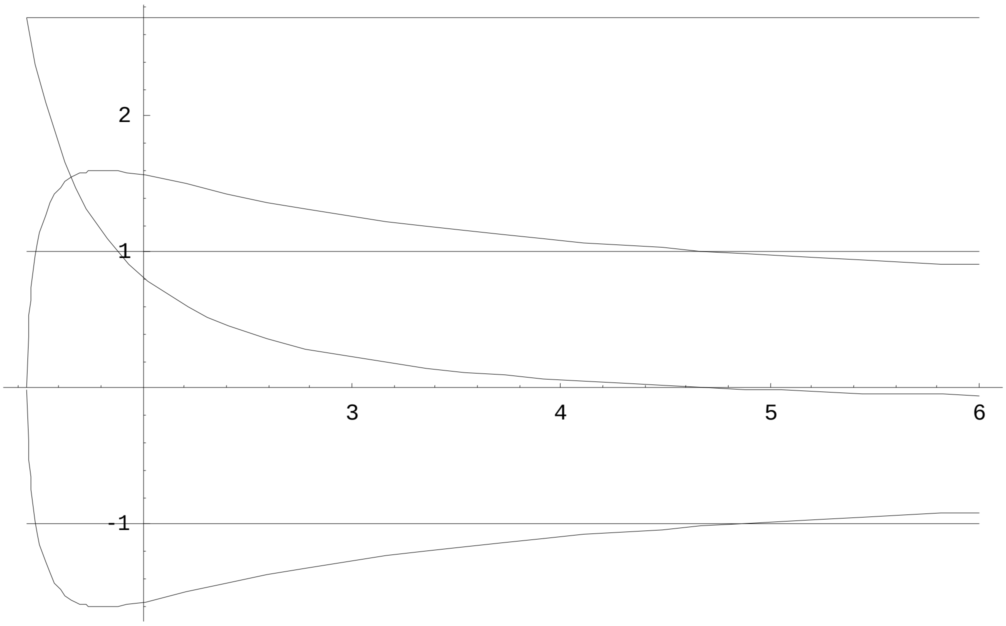
$$h(4.81047738097) = \frac{\text{plog}(-\ln 4.81047738097)}{-\ln 4.81047738097} = 2.28399\ 10^{-8} \pm \mathbf{i} \approx \pm \mathbf{i}$$

$$h(8192) = \frac{\text{plog}(-\ln 8192)}{-\ln 8192} = -0.143181 \pm 0.235041\,\mathbf{i}$$

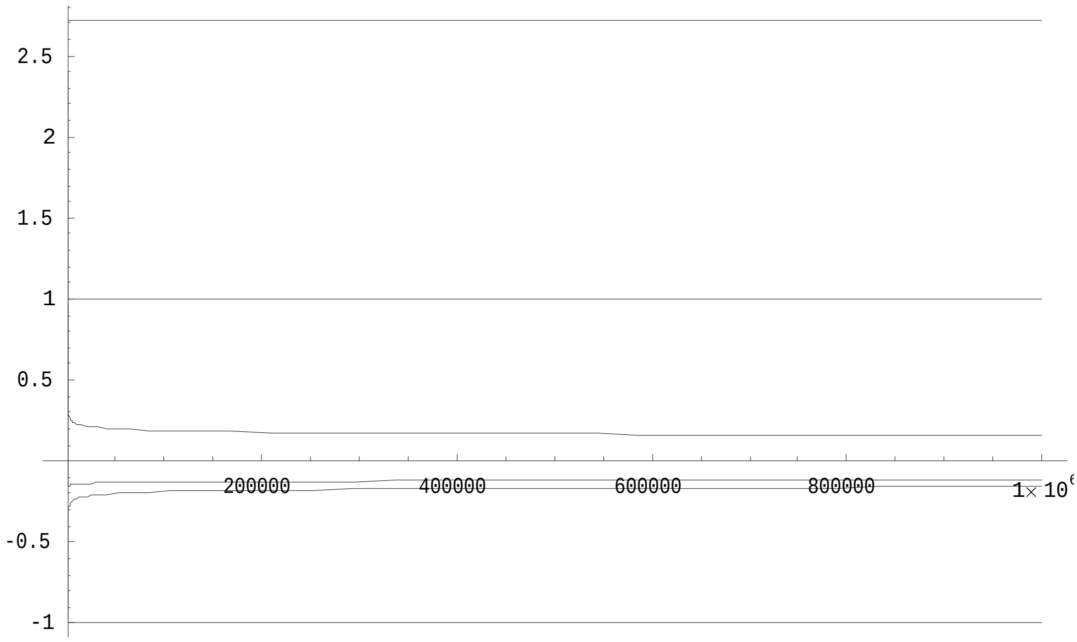
Conjectural conclusion : "After η", the "infinite towers" corresponding to any real base b are all complex, with two complex conjugate values for each value of the real base b .

Is it so?? If it is so, the Gottfried's plot must have a symmetrical image.

In fact, the plots of the real and imaginary (conjugate) parts of $h(b)$, for $b > \eta$ are as follows:



We can see that, for $b = 4.81047738097$, we have $\mathrm{Im}(h) = \pm 1,\ \mathrm{Re}(h) = 0$, within the limitations of the graphic precision. Nevertheless, for $b \to +\infty$, the infinite tower does not vanish.



Is this acceptable ? Can this be proved ?

GFR
21th September, 2007