Sums of like powers of logarithms

We want to find a general solution for the proposed function

(1) $S_p(a,b) = \log(a+1)^p + \log(a+2)^p + \log(a+3)^p + \log(a+4)^p + \ldots + \log(b)^p$

This should be in analogy to the solution of the similar problem of sums of like powers of consecutive natural numbers which can nicely be done with the bernoulli-polynomials.

We try to introduce a helper function

(2) $T_p(a) = \log(a+1)^p + \log(a+2)^p + \log(a+3)^p + \log(a+4)^p + \ldots$

initially restricted to the convergent cases, from which we expect the general solution

$$S_p(a,b) = T_p(a) - T_p(b)$$

Clearly we must begin with the convergent cases where $p < -1$ but we shall see, that the generalization to general p (with perhaps few exceptions) gives meaningful results heuristically.

We try this by the method of indefinite summation.

First we need a function which performs the increment for the argument

$$g(\log(x)) = \log(x+1)$$

Much obviously this can be solved by the function

$$g(x) = \log(1+\exp(x))$$

This can be obtained simply using Pari/GP to get

(2) $g(x) = \log(2) + 1/2\, x + 1/8\, x^2 - 1/192\, x^4 + 1/2880\, x^6 - \ldots$

Examination of the pattern of the coefficients give the most plausible explanation in terms of the Dirichlet-η "eta"-function (which is also called as "alternating zeta"[see mathworld]) at its integer arguments from 1 down to $-infinity$ which are

$\eta(1)=$	log(2) ~ 0.69314
$\eta(0)=$	½
$\eta(-1)=$	¼
$\eta(-2)=$	0
$\eta(-3)=$	-1/8
$\eta(-4)=$	0
$\eta(-5)=$	¼
…	…

With that the given coefficients for $g(x)$ have the form

~ 0.69314 = log(2)	$=\eta(1)/0!$
1/2	$=\eta(0)/1!$
1/8	$=\eta(-1)/2!$
0	$=\eta(-2)/3!$
-1/192	$=\eta(-3)/4!$
0	$=\eta(-4)/5!$
1/12880	$=\eta(-5)/6!$
…	…

and (presumably) we can write the power series for the function $g(x)$ as

(3) $$g(x) = \sum_{k=0}^{\infty} \frac{\eta(1-k)}{k!} x^k$$

We use this function to define the matrixoperator (Carlemanmatrix) ***G*** . With this we have

$$V(\log(x)) * G = V(\log(1+x))$$

Note, that the size of the vectors ***V*** and the matrix ***G*** is assumed as infinite here and also, that ***G*** is ***not*** triangular; its top-left aspect is

$$G = \begin{bmatrix} 1 & 0.693147 & 0.480453 & 0.333025 \\ 0 & 1/2 & 0.693147 & 0.720680 \\ 0 & 1/8 & 0.423287 & 0.700030 \\ 0 & 0 & 1/8 & 0.384930 \\ 0 & -1/192 & 0.00840472 & 0.118734 \\ 0 & 0 & -1/192 & 0.0126071 \\ 0 & 1/2880 & -0.000820731 & -0.00416026 \\ 0 & 0 & 1/2880 & -0.00123110 \end{bmatrix}$$

The matrix-based method of the idea of indefinite summation requires now that

$$V(\log(x)) * (G + G^2 + G^3 + \dots) = V(\log(1+x)) + V(\log(2+x)) + V(\log(3+x)) + \dots$$

is a meaningful expression and one has to search for a solution for the formal notation (see Neumann-series, Wikipedia)

$$G + G^2 + G^3 + \dots = G (I - G)^{-1}$$

where ***I*** is the identity-matrix.

Unfortunately the parenthese is not immediately invertible so we need some work-around.

"Shifted inversion" (Walker)

A method, proposed by some authors (Peter Walker, Andrew Robbins/Tetration-Forum) is to use a cut and an extension of that matrix ***Q=(I-G)***: we assume any rectangular truncation of ***Q*** to size $n \times n+1$ and omit the first column to arrive at ***Q**** , invert this and append a first and discard the last row. Then we use this matrix as a pseudo-inverse G^+ of ***G***. Let's denote it as ***M*** for convenience in the following.

The interesting aspect is here, that heuristically the entries of ***M*** stabilize as the truncation is done with increasing n, and to see the first valid digits it suffices to use $n=32$ or $n=64$.

Having $n=32$ we get ***M*** with its top-left aspect as

$$M_{32x32}= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1.00000 & 0.577216 & -0.145632 & -0.0290711 & 0.00821534 & 0.0116269 \\ -0.500000 & -0.533859 & 0.634570 & -0.185811 & -0.0605025 & 0.00355547 \\ -0.166667 & -0.325579 & -0.386859 & 0.671304 & -0.210493 & -0.0920360 \\ -0.0416667 & -0.125274 & -0.240711 & -0.312757 & 0.695810 & -0.227607 \\ -0.00833333 & -0.0337257 & -0.0991620 & -0.191897 & -0.267035 & 0.713651 \\ -0.00138889 & -0.00685935 & -0.0284730 & -0.0814665 & -0.160312 & -0.235667 \\ -0.000198413 & -0.00117261 & -0.00592379 & -0.0245251 & -0.0689118 & -0.138200 \end{bmatrix}$$

With this we get

V(log(1))-V(log(2))	*M =	[1.0000000	0	0	0	...]
V(log(1))-V(log(3))	*M =	[2.0000000	0.69314718	0.48045301	0.33302465	...]
V(log(1))-V(log(4))	*M =	[3.0000000	1.7917595	1.6874020	1.6589936	...]
V(log(3))-V(log(4))	*M =	[1.0000000	1.0986123	1.2069490	1.3259690	...]

where the approximations to the expected values are already very well. Thus we might assume indeed the wished general expression in some range of convergence:

$$V(\log(a))-V(\log(b)) \quad *M = \quad [\sum_{k=a}^{b-1} \log(k)^0 \quad \sum_{k=a}^{b-1} \log(k)^1 \quad \sum_{k=a}^{b-1} \log(k)^2 \quad \sum_{k=a}^{b-1} \log(k)^3 \quad \dots]$$

A closer look at the most interesting, 2^{nd} column M_1 of matrix $\mathbf{M}$ suggests, that it just contains the coefficients for the composite function *log(gamma(exp(x)))*; Pari/GP gives us the taylorseries for this as

$$\begin{aligned} \log(\mathrm{gamma}(\exp(x))) = & - 0.57721566\, x \\ & + 0.53385920\, x^2 \\ & + 0.32557879\, x^3 \\ & + 0.12527414\, x^4 \\ & + 0.033725651\, x^5 \\ & + 0.0068593536\, x^6 \\ & + 0.0011726081\, x^7 \\ & + O(x^8) \end{aligned}$$

which is close enough to make us confident that we are in principle on the right track here.

> It might be worth also to note, that in the matrix-expression we have two parameters on the left hand side of the formula *(V(log(a))-V(log(b)))*, and in the *log(gamma(exp(x)))*-construction only one. This can be made compatible if we assume $a=1$, so $\log(a)=0$ and we may rewrite this as $(V(0)-V(\log(b)))^* \mathbf{M_1} = V(\log(b))^* -\mathbf{M_1}$ where the negative-signed coefficients in vector $\mathbf{M_1}$ are now exactly that of the *log(gamma(exp(x)))*-composition and the leading coefficient is zero anyway.

The "Hurwitz-logarithm-sums"

But this matrix-representation allows to proceed one more step. For any finite difference between a and b we use $V(\log(a)) - V(\log(b))$ which cancels the coefficient at x^0 in the resulting power series. But if we want omit the second coefficient, then we ask for the infinite series

$V(\log(a)) * M_1 = \log(a+1) + \log(a+2) + \ldots$
$V(\log(a)) * M_2 = \log(a+1)^2 + \log(a+2)^2 + \ldots$

$V(\log(a)) * M_p = \log(a+1)^p + \log(a+2)^p + \ldots$

which are also a divergent series. However, for $a=1$ we have the expression via the derivatives of the $\zeta()$- (zeta)-function, in particular

$V(\log(1)) * M_1 = \log(1+1) + \log(1+2) + \ldots = \zeta'(0)$

and for the rhs we have a finite value of $-0.91893\ldots$

More generally: the sums of the consecutive p^{th} powers of the logarithms can be expressed by the according p^{th} derivatives of the zeta at zero, so

$$\sum_{k=0}^{\infty} \log(1+k)^p = \sum_{k=0}^{\infty} \frac{\log(1+k)^p}{(1+k)^0} = (-1)^p \zeta^{(p)}(s)_{|s=0}$$

That first few derivatives are approximately

[-1/2, -0.91893853, -2.0063565, -6.0047112, -23.997103, -120.00023]

Only the first two numbers have "simple" representations; we have for instance

$-0.91893853\ldots = -\tfrac{1}{2}\log(2\pi)$

More of this may be found using Mathematica at *wolframalpha.com*

That derivatives are also expressible as infinite series, for instance a well converging one is

$$\zeta^{(p)}(s)_{|s=0} = (-1)^p \left(\sum_{k=0}^{\inf} \frac{\gamma_{p+k}}{k!} \right) - p!$$

where the coefficients γ are the Stieltjes-constants beginning with $\gamma_0=0.5772156...$ (also known as "Euler-Mascheroni-constant").

The first row of the matrix ***M*** should then be filled with the following constants which are thus well defined for any size of truncation of ***M***:

[-0.5, 0.918939, -2.00636, +6.00471, -23.9971, +120.000, ...]

and defining a function $T_1(x)$ using the entries from the 2^{nd} column of ***M***

$$T_1(\log(x)) = \sum_{r=0}^{\infty} m_{r,1} x^r = \sum_{k=0}^{\infty} \log(x+k)$$

we can immediately define the function $S_1(\log(a),\log(b))$ for the sum of the like powers of logs

$$S_1(\log(a),\log(b)) = T_1(\log(a)) - T_1(\log(b)) = \sum_{k=0}^{(b-1)-a} \log(a+k)$$

We have even more: completely analoguously we have the same with positive integer exponents p

$$T_p(\log(x)) = \sum_{r=0}^{\infty} m_{r,p} x^r = \sum_{k=0}^{\infty} \log(x+k)^p$$

and

$$S_p(\log(a),\log(b)) = T_p(\log(a)) - T_p(\log(b)) = \sum_{k=0}^{(b-1)-a} \log(a+k)^p$$

The final form of ***M*** is now (only the top-left part is shown):

M=

-0.50000000	0.91893853	-2.0063565	6.0047112	...
-1.0000000	0.57721566	-0.14563169	-0.029071090	...
-0.50000000	-0.53385920	0.63456997	-0.18581113	...
-0.16666667	-0.32557879	-0.38685888	0.67130432	...
-0.041666667	-0.12527414	-0.24071138	-0.31275691	...
-0.0083333333	-0.033725651	-0.099162025	-0.19189685	...
-0.0013888889	-0.0068593536	-0.028473038	-0.081466529	...
-0.00019841270	-0.0011726081	-0.0059237927	-0.024525073	...
...	...	...	...	...

Fractional bounds for the sums

Just like with the bernoulli/zeta-polynomials we have the option to generalize this to fractional values of a and b, and also to non-integer/non-natural differences between a and b.

However, we do not get polynomials (which have finite number of coefficients) but infinite series, so no "exact" values in simple terms of known constants so far.

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\\ ========== Pari/GP ============================================

  g_x = vectorv(n+1,r,aeta(2-r)/(r-1)!);
     G1 = matfromser(g_x);
           M = VE(dV(1,n+1)-G1,n,-n)^-1 ;
     M = matrix(n,n,r,c,if(r>1,M[r-1,c]));

%box >tst VE(M,n,8)

       pse = sumalt(k=0,(-1)^k*1/(1+k)^x)  \\ powerseries for  eta(x)
       psz = 1/(1-2^(1-x))*pse             \\ powerseries for zeta(x)
       pszInc = psz + 1/(1-x)              \\ powerseries for zetaInc(x)
       pczInc = polcoeffs(pszInc,n)        \\ coeffs for zetaInc(x)

       M_0 = pczInc * dFac(1) *dJ          \\ equals derivatives of zeta at 0
  M[1,] = M_0              // insert derivtives to allow correct results for

\\ check: (compare)  abs(M_0[1+k]== abs(zetadiff(0,k))
{ zeta_d(x=2,d=0) = local(a,b,c);  \\ zeta and d'th deriv. of zeta at small x
      a = sum(k=0,n-1-d,x^k*pczInc[1+k+d]*binomial(d+k,d));
      b = 1/(1-x)^(d+1)) ;
      c = d!*(a-b);
    return(c) ; }

 zeta_d_at0(d) = (-1)^d*sum(k=0,n-1, Stieltjes[1+d+k]/k!) - d!

 %box >tst ESum(0.0)*(dV(log(1))-dV(log(4)))*VE(M,n,8)
 log(2)^3+log(3)^3
```

Using the ZETA-matrix

A concurring approach is the following which uses the ***ZETA***-matrix for the summation-part of the problem. Here we do the following.

The basic function, *g(x) = log(1+exp(x))*, can be implemented by *g(x)=log(1 + 1 + (exp(x)-1)))* by the use of the Stirling-matrices 2^{nd} and 1^{st} kind:

*V(log(x)) * fS2F = V(exp(log(x))-1) = V(x-1)*
*V(x-1) * P~ = V(x)*
*V(x) * fS1F = V(log(x+1))*

or

*V(log(x)) * fS2F * P~ * fS1F = V(log(x+1))*

The partial products in this expression are not all well-converging; as the mercator-series for the logarithm has very limited range of convergence the product P~ * fS1F converges slowly, and also the result of

*V(log(x)) * fS2F * P~ = V(x)*

allows *|x-1|<1* only for the next step

*V(x) * fS1F = V(log(x+1))*

However, eulersummation can accelerate convergence in such a way, that we can use that matrices with size of *32 x 32* or *64x64* and have ten or more correct digits. The process of eulersummation can be reduced to a simple additional diagonal factor (denoted here as matrix-function ***ESum()*** with an appropriate parameter indicating the order of the eulersummation, for example *2.0*) between the leading factors and ***fS1F***:

*V(x) * ESum(2.0)* fS1F = V(log(x+1))*

(In the following I'll omit that reference for readability)

If we want the sum of the two consecutive logarithms then we could write

*V(log(x)) * fS2F * (P +P²)~ * fS1F = V(log(x+1)) + V(log(x+2))*

and the generalization for the case of infinite summation would then be:

*V(log(x)) * fS2F * (P +P² + P³ + P⁴ + ...)~ * fS1F = V(log(x+1)) + V(log(x+2)) + ...*

The parenthese is the geometric series of the matrix-argument ***P***; in "sums of like powers" I solved the problem how this series can meaningfully be summed, we get the ***ZETA***-matrix, which is a close relative of the matrix of Bernoulli-polynomials.

*V(log(x)) * fS2F * ZETA~ * fS1F = V(log(x+1)) + V(log(x+2)) + ...*

And in fact, we get for the inner product-expression

*fS2F * ZETA~ * fS1F = M*