Hierarchies and Nopt Structures

Alister Wilson

Abstract (Beta version)

Slow-growing Nopt structures are investigated. Comparisons are made with some well-known hierarchies. The repdigit problem is mentioned. A careful look into the start of the Grzegorczyk hierarchy is attempted. Comparing hereditary base(n) with transfinite ordinals. This paper follows on from Hyperoperations and Nopt structures.

Keywords: Nopt structures, Nept form, slow-growing nopt, Grzegorczyk hierarchy, hereditary base, ordinals, filter codes, prime factorisation trees.

Contents

Thanks very much to Wikipedia materials and Wolfram mathematica and Demonstrations project. I hope that the quality and insightful Wolfram demonstrations can be used by tertiary educational institutions as a sensible and practical way to learn about maths and share research results with the community.

Part 0 Introduction

This paper is a follow-up from the paper "Hyperoperations and Nopt Structures". It is best understood if you have read this paper where I give motivation and definition of Nopt structure. Nopt stands for Nested Operational Power Tower, defined by analogy to tetration forming exponential power towers, and pentation etc, forming nested structures of these power towers, but any increasing function on the natural numbers can be used for the operation, so that sometimes the term "power tower" may be misleading.

Nopt structures give a different feel to mathematics concerning various aspects of a recursive nature, where the emphasis is on the textual features, maths as written symbols on the page and the process of counting these symbols. You might say that nopt structures give a mathematical method to describe the visual and design aspects, geometry and recursive or fractaline properties of written or typed mathematical symbolic expressions.

This can happen in two inter-related ways, with the second being more common:

First, the re-interpretation of standard mathematical notation to give new meanings.

Second, the re-expression of standard mathematical notation to give the same mathematical meaning, in other words, without detracting from or altering previous elements of meaning, but by adding other aspects, usually concerning turning hidden, encapsulated forms of recursion into more explicit forms of recursion as well as elucidating the interplay between these two kinds of recursion.

Part 1 introduces the slow-growing nopt structure that is defined as the slowest growing nopt structure that isn't a constant nopt structure. As well, it seems appropriate to give a series of worked examples concerning the standard slow-growing hierarchy.

Some examples of iterated pentation are given to illustrate the very complex nature of pentation and differences between top-down and bottom-up bracketing. Not surprisingly, the top-down bracketing leads towards the patterns seen in hexation structures. Anyone interested in researching these structures may be curious about varying startseeds and controlseeds. The Hardy hierarchy is introduced by worked examples from the definition. Finally, the amusing repdigit problem is looked into.

In Part 2, we firstly note that when applying functional powers we need to be careful in our substitutions lest we run into the substitution paradox. Then we look at the complex nature of the Grzegorczyk hierarchy of functions starting from f(n)=n+1, and introduce the h(b,k)(n) functions that provide the simpler lower bound approximation functions to the functions produced from the Grzegorczyk hierarchy. Some of these functions are too complex to type using an equation editor, so I give some colored square diagrams that show the patterns produced by moving from f_2 to f_3 in the Grzegorczyk hierarchy and also, patterns produced by applying functional powers to the h(b,k)(n) function for small values of k.

Then, in Part 3, we take a bit of a look at the ordinal hierarchy and consider its relationship to the Wainer hierarchy. In particular, I suggest that HB3 is a good enough "model" for the ordinal hierarchy up to epsilon zero. This could be useful for creating various kinds of filters of the transfinite ordinals by thinking of a suitable function on the naturals and mapping to a portion of the ordinal hierarchy. Part 4 takes a brief look at the Height Density Problem concerning prime factorisation trees. Finally, Part 5 shows how Catalan Number Trees can be colored using the butdj method I introduced in the paper "Hyperoperations and Nopt Structures".

Part 1 Towards an understanding about some hierarchies

Section 1.1 How can you express a finite number, n, with Nopt structures?

$$n = \underbrace{1+1+\ldots+1+1}_{n} = \left.1+^{1+^{\cdot^{\cdot^{\cdot^{1+1}}}}}\right\}n$$

$$= \underbrace{1+1+\ldots+1+1}_{\underbrace{1+1+\ldots+1+1}_{n}} = \left.1+^{1+^{\cdot^{\cdot^{\cdot^{1+1}}}}}\right\}\left.1+^{1+^{\cdot^{\cdot^{\cdot^{1+1}}}}}\right\}n$$

$$= \left.\begin{matrix}\underbrace{1+1+\ldots+1+1} \\ \vdots \\ \underbrace{1+1+\ldots+1+1}_{n}\end{matrix}\right\}n = \underbrace{\left.1+^{1+^{\cdot^{\cdot^{\cdot^{1+1}}}}}\right\}\cdots\left.1+^{1+^{\cdot^{\cdot^{\cdot^{1+1}}}}}\right\}n}_{n}$$

$$= \left.\begin{matrix}\underbrace{\left.1+^{1+^{\cdot^{\cdot^{\cdot^{1+1}}}}}\right\}\cdots\left.1+^{1+^{\cdot^{\cdot^{\cdot^{1+1}}}}}\right\}n} \\ \vdots \\ \underbrace{\left.1+^{1+^{\cdot^{\cdot^{\cdot^{1+1}}}}}\right\}\cdots\left.1+^{1+^{\cdot^{\cdot^{\cdot^{1+1}}}}}\right\}n}_{n}\end{matrix}\right\}n$$

$$= \underbrace{\left.\begin{matrix}\underbrace{1+1+\ldots+1+1} \\ \vdots \\ \underbrace{1+1+\ldots+1+1}_{n}\end{matrix}\right\} \cdots \left.\begin{matrix}\underbrace{1+1+\ldots+1+1} \\ \vdots \\ \underbrace{1+1+\ldots+1+1}_{n}\end{matrix}\right\}n}_{n}$$

$$Nopt = < ordertype,\ fpt, sseed, cseed1, cseed2, ..., cseedm >$$

$$pv = prevvalue$$

$$n = < ot5, pv+1, 1, n > = < ot4, ss...ss(0), n >$$

$$= < ot6, pv+1, 1, n, 3 > = < ot5, ss...ss(0), n, 3 >$$

$$= < ot6, pv+1, 1, n, n > = < ot5, ss...ss(0), n, n >$$

$$= < ot6, ss...ss(0), n, n, n >$$

$$= < ot7, pv+1, 1, n, n, n >$$

Section 1.2 The constant nopt structure and slow-growing nopt structure

The constant nopt structure seems rather frivalous or artificial, but it is useful to know, as it still features the standard "nopt structure computational pathway" while the arithmetic involved is very simple. Also, I present this structure in order to compare with a similar nopt structure, the slow-growing nopt structure, seen on the next page.

Using the successor function from 0, we can create Constant Nopt Structures:
The "formal successor tower" is sss…ss(0)
The height of the tower is defined as the number of s's (not including the 0)
Define the seedvalue in the usual way: seedvalue=startseed=controlseed=n
Ot3 is the seedvalue by itself, so
Ot3=n
Ot4=

$$s^{s^{\cdot^{\cdot^{\cdot s(0)}}}}\}\, n = n$$

Ot5=

$$\underbrace{s^{s^{\cdot^{\cdot^{\cdot s(0)}}}}\}\ldots\}s^{s^{\cdot^{\cdot^{\cdot s(0)}}}}\}}_{n}\, n = n$$

Ot6=

$$\left.\begin{array}{c} s^{s^{\cdot^{\cdot^{\cdot s(0)}}}}\}\ldots\}s^{s^{\cdot^{\cdot^{\cdot s(0)}}}}\}\, n \\ \vdots \\ \underbrace{s^{s^{\cdot^{\cdot^{\cdot s(0)}}}}\}\ldots\}s^{s^{\cdot^{\cdot^{\cdot s(0)}}}}\}\, n}_{n} \end{array}\right\} n = n$$

The first tower has n s's, so is sss…ss(0) with n s's, s(0)=1, ss(0)=2 and sss…ss(0)=n. This value describes height of next successor tower, and so on, giving the value n for each tower in the bottom row. Therefore, the next row above has ellipsis length n (including the first n) and also evaluates to n, all the way up to the top row that has the same ellipsis length.
Clearly Ot(m)=n for all OrderTypes m>=3.

Let’s make a slight variation from this constant nopt structure:
The only difference from the previous example is that the formal successor tower finishes with 1 instead of 0.
In other words:
The “formal successor tower” is sss…ss(1)
The height of the tower is defined as the number of s’s (not including the 1)
Define the seedvalue in the usual way: seedvalue=startseed=controlseed=n
Ot3 is the seedvalue by itself, so
Ot3=n
Ot4=

$$s^{s^{\cdot^{\cdot^{s(1)}}}}\} n = n+1$$

Ot5=

$$\underbrace{s^{s^{\cdot^{\cdot^{s(1)}}}}\}\ldots\}s^{s^{\cdot^{\cdot^{s(1)}}}}\} n}_{n} = n+(n-1) = 2n-1$$

Ot6=

$$\left.\begin{matrix} s^{s^{\cdot^{\cdot^{s(1)}}}}\}\ldots\}s^{s^{\cdot^{\cdot^{s(1)}}}}\} n \\ \vdots \\ \underbrace{s^{s^{\cdot^{\cdot^{s(1)}}}}\}\ldots\}s^{s^{\cdot^{\cdot^{s(1)}}}}\} n}_{n} \end{matrix}\right\} n$$

The arithmetic from the computational pathway is a little less clear.
Let’s do it row by row from the bottom with sequences: (1)

1 n
2 nth term from < n, n+1, n+2, … > = n + (n-1) = 2n-1
3 (2n-1)th term from < n, n+1, n+2, … > = n + ((2n-1)-1) = 3n-2
4 (3n-2)nd term from < n, n+1, n+2, … > = n + ((3n-2)-1) = 4n-3
5 (4n-3)rd term from < n, n+1, n+2, … > = n + ((4n-3)-1) = 5n-4

So the new sequence is:
< n, 2n-1, 3n-2, 4n-3, 5n-4, … >
and the answer is the nth term from this sequence:
nth term from < n, 2n-1, 3n-2, 4n-3, 5n-4, … > = $n^2-(n-1)=n^2-n+1$
In summary,
Ot4 = n+1, Ot5 = 2n-1, and Ot6 = n^2-n+1

Nopt structure terminology does take a bit of time to get used to… There are the terms “formal power tower, fpt” and “theta” that are defined in a rather general way (see my other paper “Hyperoperations and Nopt structures”).
In the first case, theta=ss…ss(0) and, in the second case, theta=ss…ss(1).
The next section considers smallest possible non-constant nopt structures.
This would be where theta=prev_value+1=height+1.

Section 1.3 Constant and slow-growing

Let’s look at some NOPT structures based on Operation=successor (+1).
First of all, note that sseed and cseed are abbreviations for startseed and controlseed.
In the examples from the tables below, theta=prev_value+1=height+1.
We can see that the smallest, non-constant nopt structure, is based on sseed=2 and cseed=3. Furthermore, if we make sseed=cseed, then the smallest, non-constant nopt structure, is based on sseed=cseed=3.

OrderType4

Cseed \ Sseed	1	2	3
1	1	2	3
2	21	32	43
3	21	32	43

OrderType5

Cseed \ Sseed	1	2	3
1	1	2	3
2	$\underbrace{21}_{2}$	$\underbrace{32}_{2}$	$\underbrace{43}_{2}$
3	$\underbrace{321}_{3}$	$\underbrace{432}_{3}$	$\underbrace{543}_{3}$

OrderType6

Cseed \ Sseed	1	2	3
1	1	2	3
2	$\left.\underbrace{21}_{2}\right\}2$	$\left.\underbrace{32}_{2}\right\}2$	$\left.\underbrace{43}_{2}\right\}2$
3	$\left.\underbrace{321}_{\underbrace{321}_{3}}\right\}3$	$\left.\underbrace{5432}_{\underbrace{432}_{3}}\right\}3$	$\left.\underbrace{76543}_{\underbrace{543}_{3}}\right\}3$

OrderType7

Cseed \ Sseed	1	2	3
1	1	2	3
2	$\underbrace{\left.\underbrace{21}_{2}\right\}2}_{2}$	$\underbrace{\left.\underbrace{32}_{2}\right\}2}_{2}$	$\underbrace{\left.\underbrace{43}_{2}\right\}2}_{2}$
3	$\underbrace{\left.\underbrace{321}_{\underbrace{321}_{3}}\right\}\left.\underbrace{321}_{\underbrace{321}_{3}}\right\}3}_{3}$	$\underbrace{\left.\underbrace{765432}_{\underbrace{65432}_{\underbrace{5432}_{\underbrace{432}_{3}}}}\right\}\left.\underbrace{5432}_{\underbrace{432}_{3}}\right\}3}_{3}$	

Section 1.4 The slow-growing nopt structure

Sseed=2, cseed=3, theta=prev_value+1=height+1

Ot3=2 Ot4=(3 2)

Ot5=(4 3 2)

Ot6

5 4 3 2
4 3 2
3

Ot7

7 6 5 4 3 2
6 5 4 3 2 5 4 3 2
5 4 3 2 4 3 2 3
4 3 2 3
3

Ot8

15141312111098765432
14131211109876543213121110987654 32
13121110987654321211109876543 2 111098765432
121110987654321110987654321098765432 98765432
111098765432 1098765432 98765432 8765432 765432
1098765432 98765432 8765432 765432 65432 5432
98765432 8765432 765432 65432 5432 432 3
8765432 765432 65432 5432 432 3
765432 65432 5432 432 3
65432 5432 432 3
5432 432 3
432 3
3

765432
654325432
5432 432 3
432 3
3
3

What’s the value for Ot9?
A table for working out the Ot9 value

15	32767	65535…32767…16383…8191…4095…2047…1023…511…255…127…63…31…15 13 11 9 7 5 3
14	16383	32767…16383…8191…4095…2047…1023…511…255…127…63…31…15 13 11 9 7 5 3
13	8191	16383…8191…4095…2047…1023…511…255…127…63…31…15 13 11 9 7 5 3
12	4095	8191…4095…2047…1023…511…255…127…63…31…15 13 11 9 7 5 3
11	2047	4095…2047…1023…511…255…127…63…31…15 13 11 9 7 5 3
10	1023	2047…1023…511…255…127…63…31…15 13 11 9 7 5 3
9	511	1023…511…255…127…63…31…15 13 11 9 7 5 3
8	255	511…255…127…63…31…15 13 11 9 7 5 3
7	127	255…127…63…31…15 13 11 9 7 5 3
6	63	127…63…31…15 13 11 9 7 5 3
5	31	63 61 59 57 55 53 51 49 47 45 43 41 39 37 35 33 31 29 27 25 23 21 19 17 15 13 11 9 7 5 3
4	15	31 29 27 25 23 21 19 17 15 13 11 9 7 5 3
3	7	15 13 11 9 7 5 3
2	3	7 5 3
1	1	3

And what’s the value for Ot10??
A partial table for working out the Ot10 value

65535	?	?
.	.	.
.	.	.
.	.	.
.	.	.
3	65535	65535…32767…16383…8191…4095…2047…1023…511…255…127…63…31… 15…7…3
2	15	15…7…3
1	1	3

Note that:

$$2^2 = 4, \quad 2^{2^2} = 2^4 = 16, \quad 2^{2^{2^2}} = 2^{16} = 65536$$

And:

$$2^2 - 1 = 3, \quad 2^{2^2} - 1 = 2^4 - 1 = 15, \quad 2^{2^{2^2}} - 1 = 2^{16} - 1 = 65535$$

I conjecture that the pattern continues and so transposing the 1st and 3rd columns we get:

1	2	3	4	5	6	…	65535
$2^2 - 1 = 3$	$2^{2^2} - 1 = 15$	$2^{2^{2^2}} - 1 = 65535$	$2^{2^{2^{2^2}}} - 1 = {}^{5}2 - 1$	${}^{6}2 - 1$	${}^{7}2 - 1$	…	?

There seems to be a regular pattern.
Maybe the 65535th value in the table is ${}^{(65535+1)}2 - 1 = {}^{65536}2 - 1 = {}^{2^{16}}2 - 1$

In summary, although the slow-growing nopt structure starts off slow, it speeds up so that each increment in ordertype corresponds with the next hyperoperation from the hyperoperation hierarchy. Here is a summary table of the values above:

Ot3	Ot4	Ot5	Ot6	Ot7	Ot8	Ot9	Ot10
2	3	4	5	7	15	65535	(2^^65536)-1

Section 1.5 The slow-growing hierarchy

The following definition of the slow-growing hierarchy is taken from wikipedia.

In computability theory, computational complexity theory and proof theory, the slow-growing hierarchy is an ordinal-indexed family of slowly increasing functions:

$g_\alpha : \mathrm{N} \to \mathrm{N}$

where N is the set of natural numbers, {0, 1, 2, 3, ...}.
It contrasts with the fast-growing hierarchy.

Definition

Let μ be a large countable ordinal such that a fundamental sequence is assigned to every limit ordinal less than μ.

The **slow-growing hierarchy** of functions $g_\alpha : \mathrm{N} \to \mathrm{N}$,
for α < μ, is then defined as follows:

$g_0(n) = 0$

$g_{\alpha+1}(n) = g_\alpha(n) + 1$

$g_\alpha(n) = g_{\alpha[n]}(n)$ if α is a limit ordinal.

Here $\alpha[n]$ denotes the n^{th} element of the fundamental sequence assigned to the limit ordinal α.

The wikipedia article on the Fast-growing hierarchy describes a standardized choice for fundamental sequence for all $\alpha < \varepsilon_0$

Limit ordinals and their fundamental sequences:

$\omega \quad <1,2,3,\ldots n,\ldots>$

Here $\omega[n]$ denotes the n^{th} element of the fundamental sequence assigned to the limit ordinal ω.

$g_0(n) = 0$

$g_1(n) = g_{0+1}(n) = g_0(n) + 1 = 0 + 1 = 1$

$g_2(n) = g_{1+1}(n) = g_1(n) + 1 = 1 + 1 = 2$

$g_3(n) = g_{2+1}(n) = g_2(n) + 1 = 2 + 1 = 3$

$g_k(n) = g_{(k-1)+1}(n) = g_{(k-1)}(n) + 1 = (k-1) + 1 = k$

$$g_{\omega}(n) = g_{\omega[n]}(n) = g_n(n) = n$$

$$g_{\omega+1}(n) = g_{\omega}(n) + 1 = n + 1$$
$$g_{\omega+2}(n) = g_{(\omega+1)+1}(n) = g_{\omega+1}(n) + 1 = (n+1) + 1 = n + 2$$
$$g_{\omega+k}(n) = g_{(\omega+(k-1))+1}(n) = g_{\omega+(k-1)}(n) + 1 = (n + (k-1)) + 1 = n + k$$

The next limit ordinal and associated fundamental sequence is:

$$\omega.2 \quad < \omega+1, \omega+2, \omega+3, \ldots \omega+n, \ldots >$$

$$g_{\omega.2}(n) = g_{\omega.2[n]}(n) = g_{\omega+n}(n) = n + n = 2n$$
$$g_{\omega.2+1}(n) = g_{\omega.2}(n) + 1 = 2n + 1$$
$$g_{\omega.2+2}(n) = g_{(\omega.2+1)+1}(n) = g_{(\omega.2+1)}(n) + 1 = (2n+1) + 1 = 2n + 2$$
$$g_{\omega.2+k}(n) = 2n + k$$

The next limit ordinal and associated fundamental sequence is:

$$\omega.3 \quad < \omega.2+1, \omega.2+2, \omega.2+3, \ldots \omega.2+n, \ldots >$$

$$g_{\omega.3}(n) = g_{\omega.3[n]}(n) = g_{\omega.2+n}(n) = 2n + n = 3n$$

$$g_{\omega.4}(n) = g_{\omega.4[n]}(n) = g_{\omega.3+n}(n) = 3n + n = 4n$$

$$g_{\omega.k}(n) = g_{\omega.k[n]}(n) = g_{\omega.(k-1)+n}(n) = (k-1)n + n = k.n$$

The “next” limit ordinal and associated fundamental sequence is:

$$\omega^2 = \omega.\omega \quad < \omega, \omega.2, \omega.3, \ldots \omega.n, \ldots >$$
$$g_{\omega^2}(n) = g_{\omega^2[n]}(n) = g_{\omega.n}(n) = n.n = n^2$$

The “next” limit ordinal and associated fundamental sequence is:

$$\omega^3 = \omega^2.\omega \quad < \omega^2, \omega^2.2, \omega^2.3, \ldots \omega^2.n, \ldots >$$
$$g_{\omega^3}(n) = g_{\omega^3[n]}(n) = g_{\omega^2.n}(n) = n^2.n = n^3$$

The "next" limit ordinal and associated fundamental sequence is:

$$\omega^{\omega} \quad < \omega, \omega^2, \omega^3, \ldots \omega^n, \ldots >$$

$$g_{\omega^{\omega}}(n) = g_{\omega^{\omega}[n]}(n) = g_{\omega^n}(n) = n^n$$

$$g_{\omega^{\omega}+1}(n) = g_{\omega^{\omega}}(n) + 1 = n^n + 1$$

$$\omega^{\omega}.2 \quad < \omega^{\omega} + \omega, \omega^{\omega} + \omega^2, \omega^{\omega} + \omega^3, \ldots \omega^{\omega} + \omega^n, \ldots >$$

$$g_{\omega^{\omega}.2}(n) = g_{\omega^{\omega}.2[n]}(n) = g_{\omega^{\omega}+\omega^n}(n) = n^n + n^n = 2n^n$$

$$\omega^{\omega+1} = \omega^{\omega}.\omega \quad < \omega^{\omega}, \omega^{\omega}.2, \omega^{\omega}.3, \ldots \omega^{\omega}.n, \ldots >$$

$$g_{\omega^{\omega+1}}(n) = g_{\omega^{\omega+1}[n]}(n) = g_{\omega^{\omega}.n}(n) = n^n.n = n^{n+1}$$

$$\omega^{\omega.2} \quad < \omega^{\omega+1}, \omega^{\omega+2}, \omega^{\omega+3}, \ldots \omega^{\omega+n}, \ldots >$$

$$g_{\omega^{\omega.2}}(n) = g_{\omega^{\omega.2}[n]}(n) = g_{\omega^{\omega+n}}(n) = n^{n+n} = n^{2n}$$

$$\omega^{\omega^2} = \omega^{\omega.\omega} \quad < \omega^{\omega}, \omega^{\omega.2}, \omega^{\omega.3}, \ldots \omega^{\omega.n}, \ldots >$$

$$g_{\omega^{\omega^2}}(n) = g_{\omega^{\omega^2}[n]}(n) = g_{\omega^{\omega.n}}(n) = n^{n.n} = n^{n^2}$$

$$\omega^{\omega^{\omega}} \quad < \omega^{\omega}, \omega^{\omega^2}, \omega^{\omega^3}, \ldots \omega^{\omega^n}, \ldots >$$

$$g_{\omega^{\omega^{\omega}}}(n) = g_{\omega^{\omega^{\omega}}[n]}(n) = g_{\omega^{\omega^n}}(n) = n^{n^n}$$

$$\varepsilon_0 \quad < \omega, \omega^{\omega}, \omega^{\omega^{\omega}}, \ldots {}^{n}\omega, \ldots >$$

$$g_{\varepsilon_0}(n) = g_{\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}[n]}(n) = g_{\left.\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}\right\}n}(n) = g_{({}^{n}\omega)}(n) = \left.n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}\right\}n = {}^{n}n$$

Section 1.6 Iterated pentation structures

Bottom-up bracketing examples

$n \uparrow^3 3$

$$\underbrace{n^{\cdot^{\cdot^{n}}}\Big)\, n^{\cdot^{\cdot^{n}}}\Big)\, n}_{3}$$

Example A: $(n \uparrow^3 3) \uparrow^3 3$

$$\underbrace{n^{\cdot^{\cdot^{n}}}\Big)\, n^{\cdot^{\cdot^{n}}}\Big)\, n^{\cdot^{\cdot^{n^{\cdot^{\cdot^{n}}}\big) n^{\cdot^{\cdot^{n}}}\big) n}}}\Bigg)\, n^{\cdot^{\cdot^{n}}}\Big)\, n^{\cdot^{\cdot^{n}}}\Big)\, n^{\cdot^{\cdot^{n^{\cdot^{\cdot^{n}}}\big) n^{\cdot^{\cdot^{n}}}\big) n}}}\Bigg)\, n^{\cdot^{\cdot^{n}}}\big)\, n^{\cdot^{\cdot^{n}}}\big)\, n}_{3}$$

$3 \uparrow^3 n$

$$\underbrace{3^{\cdot^{\cdot^{3}}}\Big) \ldots \Big)\, 3^{\cdot^{\cdot^{3}}}\Big)\, 3}_{n}$$

Example B: $(3 \uparrow^3 n) \uparrow^3 3$

$$\underbrace{\underbrace{3^{\cdot^{\cdot^{3}}}\Big) \ldots \Big)\, 3^{\cdot^{\cdot^{3}}}\Big)\, 3}_{n}{}^{\cdot^{\cdot^{\underbrace{3^{\cdot^{\cdot^{3}}}\big) \ldots \big) 3^{\cdot^{\cdot^{3}}}\big) 3}_{n}}}}\Bigg)\, \underbrace{3^{\cdot^{\cdot^{3}}}\Big) \ldots \Big)\, 3^{\cdot^{\cdot^{3}}}\Big)\, 3}_{n}{}^{\cdot^{\cdot^{\underbrace{3^{\cdot^{\cdot^{3}}}\big) \ldots \big) 3^{\cdot^{\cdot^{3}}}\big) 3}_{n}}}}\Bigg)\, \underbrace{3^{\cdot^{\cdot^{3}}}\big) \ldots \big) 3^{\cdot^{\cdot^{3}}}\big) 3}_{n}}_{3}$$

$3 \uparrow^3 3$

$$\underbrace{3^{\cdot^{\cdot^{3}}}\Big)\, 3^{\cdot^{\cdot^{3}}}\Big)\, 3}_{3}$$

Example C: $(3 \uparrow^3 3) \uparrow^3 n$

$$\underbrace{3^{\cdot^{\cdot^{3}}}\Big)\, 3^{\cdot^{\cdot^{3}}}\Big)\, 3^{\cdot^{\cdot^{3^{\cdot^{\cdot^{3}}}\big) 3^{\cdot^{\cdot^{3}}}\big) 3}}}\Bigg) \ldots \Bigg)\, 3^{\cdot^{\cdot^{3}}}\Big)\, 3^{\cdot^{\cdot^{3}}}\Big)\, 3^{\cdot^{\cdot^{3^{\cdot^{\cdot^{3}}}\big) 3^{\cdot^{\cdot^{3}}}\big) 3}}}\Bigg)\, 3^{\cdot^{\cdot^{3}}}\big)\, 3^{\cdot^{\cdot^{3}}}\big)\, 3}_{n}$$

Top-down bracketing examples

Example D:
$n\uparrow^3 (3\uparrow^3 3)$

$$3\uparrow^3 3 = \underbrace{3^{\cdot^{\cdot^{3}}}\Big)3^{\cdot^{\cdot^{3}}}\Big)3}_{3} \qquad n\uparrow^3 m = \underbrace{n^{\cdot^{\cdot^{n}}}\Big)\ldots\Big)n^{\cdot^{\cdot^{n}}}\Big)n}_{m}$$

$$n\uparrow^3 (3\uparrow^3 3) = \underbrace{n^{\cdot^{\cdot^{n}}}\Big)\ldots\Big)n^{\cdot^{\cdot^{n}}}\Big)n}_{\underbrace{3^{\cdot^{\cdot^{3}}}\Big)3^{\cdot^{\cdot^{3}}}\Big)3}_{3}}$$

Example E:
$3\uparrow^3 (n\uparrow^3 3)$

$$3\uparrow^3 n = \underbrace{3^{\cdot^{\cdot^{3}}}\Big)\ldots\Big)3^{\cdot^{\cdot^{3}}}\Big)3}_{n} \qquad n\uparrow^3 3 = \underbrace{n^{\cdot^{\cdot^{n}}}\Big)n^{\cdot^{\cdot^{n}}}\Big)n}_{3}$$

$$3\uparrow^3 (n\uparrow^3 3) = \underbrace{3^{\cdot^{\cdot^{3}}}\Big)\ldots\Big)3^{\cdot^{\cdot^{3}}}\Big)3}_{\underbrace{n^{\cdot^{\cdot^{n}}}\Big)\ldots\Big)n^{\cdot^{\cdot^{n}}}\Big)n}_{3}}$$

Example F:
$3\uparrow^3 (3\uparrow^3 n)$

$$3\uparrow^3 n = \underbrace{3^{\cdot^{\cdot^{3}}}\Big)\ldots\Big)3^{\cdot^{\cdot^{3}}}\Big)3}_{n}$$

$$3\uparrow^3 (3\uparrow^3 n) = \underbrace{3^{\cdot^{\cdot^{3}}}\Big)\ldots\Big)3^{\cdot^{\cdot^{3}}}\Big)3}_{\underbrace{3^{\cdot^{\cdot^{3}}}\Big)\ldots\Big)3^{\cdot^{\cdot^{3}}}\Big)3}_{n}}$$

These examples show the kinds of patterns involved with iterating pentation in different ways.

Section 1.7 Varying control and start seeds in formal addition towers

Usually, the startseed and controlseed are the same in a Nopt structure.
If we make them slightly different how does that change things?
In the standard definition of nopt structure, the seedvalue in the triple defining the nopt structure always refers to the startseed and the controlseed value.
First notice that the nopt structures <ot(m), 1+…+1, n> where n=startseed=controlseed, are the constant nopt structures equal to n.
What about the nopt structures <ot(m), 2+…+2, n> where n=startseed=controlseed?

Ot(4)

$$2+\ldots+2\,\}\,n=2n$$

Ot(5)

$$\underbrace{2+^{\cdot^{\cdot^{+2}}}\}\ldots\}\,2+^{\cdot^{\cdot^{+2}}}\}\,n}_{n}=2(\ldots2(2(2n))..)\,\}=2^{n-1}n$$

versus

$$\underbrace{2+^{\cdot^{\cdot^{+2}}}\}\ldots\}\,2+^{\cdot^{\cdot^{+2}}}\}\,n}_{n+1}=2^{n}n$$

Ot(6)

$\left.\begin{matrix}2+^{\cdot^{\cdot^{+2}}}\}\ldots\}\,2+^{\cdot^{\cdot^{+2}}}\}\,n\\ \vdots\\ \underbrace{2+^{\cdot^{\cdot^{+2}}}\}\ldots\}\,2+^{\cdot^{\cdot^{+2}}}\}\,n}_{n}\end{matrix}\right\}n$	Then the first row above the “n” is: $2^{n-1}n$
$\left.\begin{matrix}2+^{\cdot^{\cdot^{+2}}}\}\ldots\}\,2+^{\cdot^{\cdot^{+2}}}\}\,n\\ \vdots\\ \underbrace{2+^{\cdot^{\cdot^{+2}}}\}\ldots\}\,2+^{\cdot^{\cdot^{+2}}}\}\,n}_{n+1}\end{matrix}\right\}n+1$	Then the first row above the “n+1” is: $2^{n}n$

Section 1.8 The Hardy Hierarchy and Caicedo variant

What is the Hardy hierarchy?
The following definition of the Hardy hierarchy is taken from wikipedia.
In computability theory, computational complexity theory and proof theory, the Hardy hierarchy, named after G.H. Hardy, is an ordinal-indexed family of functions
$h_\alpha : \mathrm{N} \to \mathrm{N}$
Where N is the set of natural numbers, {0, 1, 2, 3, …}
It is related to the FGH and SGH. The hierarchy was first described in
Hardy's 1904 paper, "A theorem concerning the infinite cardinal numbers".

Definition

Let μ be a large countable ordinal such that a fundamental sequence is assigned to every limit ordinal less than μ. The **Hardy hierarchy** of functions $h_\alpha : \mathrm{N} \to \mathrm{N}$ for $\alpha < \mu$,
is then defined as follows:

$$h_0(n) = n$$

$$h_{\alpha+1}(n) = h_\alpha(n+1)$$

$$h_\alpha(n) = h_{\alpha[n]}(n)$$ if α is a limit ordinal.

Here $\alpha[n]$ denotes the n^{th} element of the fundamental sequence assigned to the limit ordinal α. A standardized choice of fundamental sequence for all $\alpha \le \varepsilon_0$ is described in the article on the fast-growing hierarchy.

[Caicedo (2007) defines a modified Hardy hierarchy of functions H_α by using the standard fundamental sequences, but with α[*n*+1] (instead of $\alpha[n]$) in the third line of the above definition.]

$$h_0(n) = n$$

$$h_1(n) = h_{0+1}(n) = h_0(n+1) = n+1$$

$$h_2(n) = h_{1+1}(n) = h_1(n+1) = (n+1)+1 = n+2$$

$$h_3(n) = h_{2+1}(n) = h_2(n+1) = (n+1)+2 = n+3$$

$$h_k(n) = h_{(k-1)+1}(n) = h_{(k-1)}(n+1) = (n+1)+(k-1) = n+k$$

$\omega \quad <1,2,3,\ldots n,\ldots>$

$$h_\omega(n) = h_{\omega[n]}(n) = h_n(n) = n+n = 2n$$

$$h_{\omega+1}(n) = h_{\omega}(n+1) = 2(n+1)$$

$$h_{\omega+2}(n) = h_{\omega+1}(n+1) = 2((n+1)+1) = 2(n+2)$$

$$h_{\omega+k}(n) = h_{\omega+(k-1)}(n+1) = 2((n+(k-1))+1) = 2(n+k)$$

$$\omega.2 \quad < \omega+1, \omega+2, \omega+3, \ldots \omega+n, \ldots >$$

$$h_{\omega.2}(n) = h_{\omega.2[n]}(n+1) = h_{\omega+n}(n+1) = 2(n+n) = 2.2n = 2^2 n$$

$$h_{\omega.2+1}(n) = h_{\omega.2}(n+1) = 2^2(n+1)$$

$$h_{\omega.2+2}(n) = h_{\omega.2+1}(n+1) = 2^2((n+1)+1) = 2^2(n+2)$$

$$h_{\omega.2+k}(n) = h_{\omega.2+(k-1)}(n+1) = 2^2((n+(k-1))+1) = 2^2(n+k)$$

$$\omega.3 \quad < \omega.2+1, \omega.2+2, \omega.2+3, \ldots \omega.2+n, \ldots >$$

$$h_{\omega.3}(n) = h_{\omega.2+n}(n+1) = 2^2(n+n) = 2^2(2n) = 2^3 n$$

$$h_{\omega.3+1}(n) = h_{\omega.3}(n+1) = 2^3(n+1)$$

$$h_{\omega.3+2}(n) = h_{\omega.3+1}(n+1) = 2^3((n+1)+1) = 2^3(n+2)$$

$$h_{\omega.3+k}(n) = h_{\omega.3+(k-1)}(n+1) = 2^3((n+(k-1))+1) = 2^3(n+k)$$

$$\omega.4 \quad < \omega.3+1, \omega.3+2, \omega.3+3, \ldots \omega.3+n, \ldots >$$

$$h_{\omega.4}(n) = h_{\omega.4[n]}(n) = h_{\omega.3+n}(n) = 2^3(n+n) = 2^3(2n) = 2^4 n$$

$$h_{\omega.k}(n) = h_{\omega.k[n]}(n) = h_{\omega.(k-1)+n}(n) = 2^{(k-1)}(n+n) = 2^{(k-1)}(2n) = 2^k n$$

$$\omega^2 = \omega.\omega \quad < \omega, \omega.2, \omega.3, \ldots \omega.n, \ldots >$$

$$h_{\omega^2}(n) = h_{\omega.\omega[n]}(n) = h_{\omega.n}(n) = 2^n n$$

$$\omega^3 = \omega^2.\omega \quad < \omega^2, \omega^2.2, \omega^2.3, \ldots \omega^2.n, \ldots >$$

$$h_{\omega^2+1}(n) = h_{\omega^2}(n+1) = 2^{(n+1)}(n+1)$$

$$h_{\omega^2+2}(n) = 2^{(n+2)}(n+2)$$

$$h_{\omega^2+k}(n) = 2^{(n+k)}(n+k)$$

$$h_{\omega^2+\omega}(n) = 2^{(n+n)}(n+n) = 2^{2n}2n$$

$$h_{\omega^2+\omega+1}(n) = h_{\omega^2+\omega}(n+1) = 2^{2(n+1)}2(n+1)$$

$$h_{\omega^2+\omega+k}(n) = 2^{2(n+k)}2(n+k)$$

$$h_{\omega^2+\omega.2}(n) = h_{(\omega^2+\omega.2)[n]}(n) = h_{\omega^2+\omega+n}(n) = 2^{2(2n)}2(2n)$$

$$h_{\omega^2.2}(n) = h_{\omega^2+\omega.\omega}(n) = 2^{2^n n}2^n n$$

I guess that:

$$h_{\omega^2.3}(n) = h_{\omega^2.2+\omega.\omega}(n) = 2^{2^{2^n n}2^n n}2^{2^n n}2^n n$$

(Caicedo variant)

$$h_0(n) = n$$

$$h_{\alpha+1}(n) = h_\alpha(n+1)$$

$h_\alpha(n) = h_{\alpha[n+1]}(n)$ if α is a limit ordinal.

$$\omega \quad < 1,2,3,\ldots n,\ldots >$$

$$h_\omega(n) = h_{\omega[n+1]}(n) = h_{n+1}(n) = n + (n+1) = 2n+1$$

Section 1.9 The repdigit problem

The rather classic repdigit (repeated digit) problem is how long can a string of digits go?
Or, how long is a piece of string?
But the condition is, you can only use SPN numbers to describe this number.
For example, the numbers 11, 22, 33, 44, 55, 66, 77, 88, 99 are repdigit numbers and are rather short, being only of length 2. They are suitable as startseed numbers for longer repdigit numbers. For example, we have:

Eg1… 11111111111 } 11
The seed (11, eleven) describes the number of digits in the number to the left of the braces.
And this number can be described in words:
11,111,111,111 is 11 billion, 111 million, 111 thousand, one hundred and eleven.

Eg2… 2222222222222222222222 } 22
And 2222222222222222222222 = 2,222,222,222,222,222,222,222
And to read it out requires some knowledge of names for large numbers.
However if I type out a 23 digit number with a 1 and 22 trailing zeros we know that
10,000,000,000,000,000,000,000 > 2,222,222,222,222,222,222,222
10^22 > 22…22 } 22

In a similar way:

10^33 > 33…33 } 33 10^44 > 44…44 } 44 10^55 > 55…55 } 55
10^66 > 66…66 } 66 10^77 > 77…77 } 77 10^88 > 88…88 } 88

And finally: 10^99 > 99…99 } 99
Actually with this number, we can see that: [99…99 } 99] + 1 = 10^99
Using terminology from NOPT structures we can say that: 11…11 } 11 has ordertype4, the fpt=11…11, and startseed=11. The term fpt="formal power tower" is a "template" for producing a number. The startseed completes the fpt so that 11…11 } 11 is a well-defined number. For nopt structures with ordertypes greater than four, ie, ot>=5, then we need to use control seeds as well. For convenience, we may as well have startseed=controlseed.
Now let's look at some examples with ot5 (repdigits=2, 3, 5) and ot6 (repdigits=7, 8):

Ot5:

A) $\underbrace{22...22\}\ldots\}22...22\}22}_{22}$ B) $\underbrace{33...33\}\ldots\}33...33\}33}_{33}$ C) $\underbrace{55...55\}\ldots\}55...55\}55}_{55}$

Ot6:

D) $\left.\begin{array}{c} 77...77\}\ldots\}77...77\}77 \\ \vdots \\ \underbrace{77...77\}\ldots\}77...77\}77}_{77} \end{array}\right\}77$ E) $\left.\begin{array}{c} 88...88\}\ldots\}88...88\}88 \\ \vdots \\ \underbrace{88...88\}\ldots\}88...88\}88}_{88} \end{array}\right\}88$

Exercise: Find NEPT approximations for the examples A) and B) above.

Example A)

$$22...22\}22 \cong 10^{22},\quad 22...22\}22...22\}22 \cong 10^{10^{22}},\ \ldots$$

$$\underbrace{22...22\}\ldots\}22...22\}22}_{22} \cong 10^{\cdot^{\cdot^{10^{22}}}}\Big\}22 \cong 10^{\cdot^{\cdot^{10}}}\Big\}22 = {}^{22}10$$

Example B)

$$33...33\}33 \cong 10^{33},\quad 33...33\}33...33\}33 \cong 10^{10^{33}},\ \ldots$$

$$\underbrace{33...33\}\ldots\}33...33\}33}_{33} \cong 10^{\cdot^{\cdot^{10^{33}}}}\Big\}33 \cong 10^{\cdot^{\cdot^{10}}}\Big\}33 = {}^{33}10$$

Part 2 Picturing the Grzegorczyk hierarchy

Section 2.1 The Substitution Paradox

Consider the function $f(n) = 2^n n$. This function maps n to a number that uses n in 2 different ways. First, the number 2 is multiplied by itself n times. Then the result is added to itself n times. Ok, pretty standard. Now consider taking a functional power.

$f^2(n) = f(f(n)) = ...$

So we're using the function f as a template and inputting f(n) instead of n.

$f(f(n)) = 2^{2^n n} 2^n n$ I think this is clear, and is the answer as to the functional form of the second functional power of the function f.

Now consider writing the function f in 2 other different ways.

A $f(n) = \underbrace{2*2*...*2*2}_{n}*n$

B $f(n) = \underbrace{2*2*...*2*2*n}_{n+1}$

Now it is clear that f(n)=A=B.

Now consider the second functional power in both cases.

A $f(f(n)) = \underbrace{2*2*...*2*2}_{\underbrace{2*2*...*2*2}_{n}*n}*\underbrace{2*2*...*2*2}_{n}*n$

This equals $2^m.2^n.n \quad where \quad m = \underbrace{2*2*...*2*2}_{n}*n = 2^n n$

So $2^m.2^n.n = 2^{2^n n}2^n n$, the same answer as above.

But, what about case B? We should get the same answer, right?

B $f(n) = \underbrace{2*2*...*2*2*n}_{n+1}$

$f(f(n)) = \underbrace{2*2*...*2*2*\underbrace{2*2*...*2*2*n}_{n+1}}_{\underbrace{2*2*...*2*2*n}_{n+1}+1} = \underbrace{\underbrace{2*2*...*2*2}_{m}*\underbrace{2*2*...*2*2*n}_{n+1}}_{2^n n+1}$

$So \quad m + (n+1) = 2^n n + 1, \quad m + n = 2^n n, \quad m = 2^n n - n$

$f(f(n)) = \underbrace{2*2*...*2*2}_{(2^n n - n)+n}*n = \underbrace{2*2*...*2*2}_{2^n n}*n = 2^{2^n n} n \quad \ldots!!$

So, it seems we have a kind of a paradox! The way out seems to be this:

$f(f(n)) = \underbrace{2*2*...*2*2*(\underbrace{2*2*...*2*2*n}_{n+1})}_{2^n n+1} = \underbrace{2*2*...*2*2}_{2^n n}*(\underbrace{2*2*...*2*2*n}_{n+1})$

$= 2^{2^n n}.2^n n \quad ok.$

So the paradox is resolved by making sure that n is a free variable, that by parenthesising the "n" allows it to be substituted, whilst treating it as a self-contained unit and counting it as such. Also, we don't want to twice-count the 2's in (2*2*…*2*2*n). The symbols inside this expression are already counted by the n+1, and the $2^n n+1$ effectively counts the remaining 2's plus the expression (2*2*…*2*2*n) considered as a single unit.

Exercise: Check whether the substitution paradox appears in the following example. The function is $f(n) = 2n + n = 3n$ and there are 2 variant forms as above:

A $f(n) = \underbrace{2+2+...+2+2}_{n} + n$ B $f(n) = \underbrace{2+2+...+2+2+n}_{n+1}$

Section 2.2 The h(b,k)(n) functions

The function "h(2,k)" can be defined with base number "2" and "k" the kth hyperoperation. So, please note that the "(2,k)" is a label. For example:

Define a function "$h(2,2)$" as follows:

$$h(2,2)(0) = n, \quad h(2,2)(n+1) = 2 * h(2,2)(n)$$

Then $h(2,2)(n) = \underbrace{2*2*...*2*2}_{n} * n = 2^n n$

Define a function "$h(2,3)$" as follows:

$$h(2,3)(0) = n, \quad h(2,3)(n+1) = 2^{h(2,3)(n)}$$

So with the function "h(2,3)", the "2" is a base number and the 3 refers to the 3rd hyperoperation, that is, exponentiation.

Similarly we can define:

$$h(3,3)(0) = n, \quad h(3,3)(n+1) = 3^{h(3,3)(n)}$$

And we can use the 4th hyperoperation, that is, tetration:

$$h(2,4)(0) = n, \quad h(2,4)(n+1) = {}^{h(2,4)(n)}2$$

And using the base number 3, we have:

$$h(3,4)(0) = n, \quad h(3,4)(n+1) = {}^{h(3,4)(n)}3$$

In general, we have:
The function "h(b,k)" defined with base number "b" and "k" the kth hyperoperation.
All h(b,k) functions are defined to be "n" at the input of 0.
For example:

$$h(2,2)(n) = \underbrace{2*2*...*2*2*n}_{n+1} = \underbrace{2*2*...*2*2}_{n} * n = 2^n n$$

$$h(2,3)(n) = \left.2^{2^{\cdot^{\cdot^{2^n}}}}\right\}(n+1) = \left.2^{2^{\cdot^{\cdot^{2}}}}\right\}n \; n$$

So $h(2,3)(n)$ is a tower (or "stack") of 2's (n high) together with an n on top and the bracketing is top-down from the very top "n" all the way down to the bottom "2".
For ease of reference, let's label these expressions:

$$A = \left.2^{2^{\cdot^{\cdot^{2^n}}}}\right\}(n+1) \quad and \quad B = \left.2^{2^{\cdot^{\cdot^{2}}}}\right\}n \; n$$

If we want to iterate the function described in expression A we should use the equivalent form in expression B, as in expression B, the n's are now "free variables" and can be substituted for, in functional iterations (as noted in Section 2.1).

Another example:

$$h(3,3)(n) = 3^{3^{\cdot^{\cdot^{3^n}}}} \Big\} (n+1) = 3^{3^{\cdot^{\cdot^{3}}}} \Big\}^n n$$

Now let's compare h(2,2)(n) with 2^n:

$$h(2,2)(n) = 2^n n \; > \; 2^n = 2 \uparrow n \quad when \quad n \geq 2$$

And we can compare h(2,3)(n) with 2^^n:

$$h(2,3)(n) = 2^{2^{\cdot^{\cdot^{2^n}}}} \Big\} (n+1) \; > \; 2^{2^{\cdot^{\cdot^{2}}}} \Big\} n \; = {}^n 2 = 2 \uparrow\uparrow n \quad when \quad n \geq 2$$

For example, when $n \geq 2$,

$$h(2,3)(2) = 2^{2^n} \Big\} 3 \; > \; 2^2 \Big\} 2 \quad h(2,3)(3) = 2^{2^{2^n}} \Big\} 4 \; > \; 2^{2^2} \Big\} 3$$

$$h(2,3)(4) = 2^{2^{2^{2^n}}} \Big\} 5 \; > \; 2^{2^{2^2}} \Big\} 4$$

In general, we have:

$$h(2,k)(n) > 2 \uparrow^{k-1} n \quad when \quad n \geq 2$$

Now consider the Grzegorczyk hierarchy (see wikipedia article "Fast-growing hierarchy"):

$$f_0(n) = n+1 \quad and \quad f_1(n) = f_0^n(n) = 2n \quad and \ldots$$

$$f_2(n) = f_1^n(n) = 2^n n$$

There is 1 "2" touching the base line, and the highest tower has height 2.

$$f_2^2(n) = (2^{2^n n})(2^n n)$$

There are 2 "2's" touching the base line, and the highest tower has height 3.

$$f_2^3(n) = 2^{(2^{2^n n})(2^n n)}(2^{2^n n})(2^n n)$$

There are 3 "2's" touching the base line, and the highest tower has height 4.

$$f_2^4(n) = 2^{2^{(2^{2^n n})(2^n n)}(2^{2^n n})(2^n n)} 2^{(2^{2^n n})(2^n n)}(2^{2^n n})(2^n n)$$

There are 4 "2's" touching the base line, and the highest tower has height 5.

If we let:

$$a_1 = 2^n n \;,\; a_2 = 2^{2^n n} = 2^{a_1} \;,\; a_3 = 2^{(2^{2^n n})(2^n n)} = 2^{a_2 a_1} \;, \ldots$$

$$f_2^n(n) = 2^{a_{n-1} \cdots a_3 a_2 a_1} \cdot \cdots \cdot 2^{a_3 a_2 a_1} 2^{a_2 a_1} 2^{a_1} 2^n n$$

There are n "2's" touching the base line, and the highest tower has height (n+1).

Therefore, a lower approximation for this function is h(2,3)(n):

$$f_3(n) = f_2^n(n) > h(2,3)(n) = 2^{2^{\cdot^{\cdot^{2^n}}}} \Big\} (n+1) \; > \; 2^{2^{\cdot^{\cdot^{2}}}} \Big\} n \; = {}^n 2 = 2 \uparrow^2 n \quad when \quad n \geq 2$$

In general:

$$f_{k+1}(n) = f_k^n(n) > h(2,k+1)(n) > 2\uparrow^k n \quad when \quad n \geq 2$$

(A reminder: Note that in the function $h(2,k+1)$, the "k+1" refers to the $(k+1)^{th}$ hyperoperation, and that, in turn, corresponds with $(k+1)-2=k-1$ Knuth arrows.)

Section 2.3 Leading up to the functions f3, f4, f5 and f6 from the GH

In this section we consider the functions that lead up to f_3, f_4, f_5 and f_6. Notice that at each stage, an approximation based on the h(2,k) function is used.

We know the lower bound approximation function for f_3 is $h(2,3)(n)$.
We can now approximate the functional powers of f_3 with functional powers of $h(2,3)(n)$ in order to reach the next function in the hierarchy f_4:
Let's rename the lower bound approximation function for the f_3 function as "g":

$$f_3(n) \approx h(2,3)(n) = g(n) = \left.2^{2^{\cdot^{\cdot^{2^n}}}}\right\}(n+1) = \left.2^{\cdot^{\cdot^{2}}}\right\}n^{[n]}$$

and the first iterate of this function…

$$g^2(n) = g(g(n)) = ...$$

$$\left.2^{\cdot^{\cdot^{2}}}\right\} \left[\left.2^{\cdot^{\cdot^{2}}}\right\} n^{[n]}\right] \quad \left.2^{\cdot^{\cdot^{2}}}\right\} n^{[n]}$$

The square brackets signify a number (or substack) that is being appended to the top of a stack while retaining the top down bracketing. It is easier to see the pattern using colored square diagrams (see Figure 2.3.2 below).

The next function to consider, is the approximation function for the f_4 function.
We can approximate the functional powers of f_4 with functional powers of $h(2,4)(n)$ in order to reach the next function in the hierarchy, f_5:

$$g(n) = \underbrace{2^{\cdot^{\cdot^{2}}}\Big)\ldots 2^{\cdot^{\cdot^{2}}}\Big)}_{n} n = h(2,4)(n)$$

and the first iterate of this function:

$$g^2(n) = \underbrace{2^{\cdot^{\cdot^{2}}}\Big)\ldots 2^{\cdot^{\cdot^{2}}}\Big)}_{\underbrace{2^{\cdot^{\cdot^{2}}}\big)\ldots 2^{\cdot^{\cdot^{2}}}\big)}_{n} n} \underbrace{2^{\cdot^{\cdot^{2}}}\Big)\ldots 2^{\cdot^{\cdot^{2}}}\Big)}_{n} n$$

The fractaline patterns produced from the iterates of f2, and the iterates of the lower approximation functions of f3, f4 and f5 are shown in the pictures below.

The figure 2.3.1 below shows how you can iteratively transition from $f_2(n) = h(2,2)(n) = 2^n n$ towards f_3 by iterating $f_2(n) = h(2,2)(n)$

Figure 2.3.1
$f_2(n) = f_1^n(n) = h(2,2)(n)$ Iterate the function $f_2(n)$

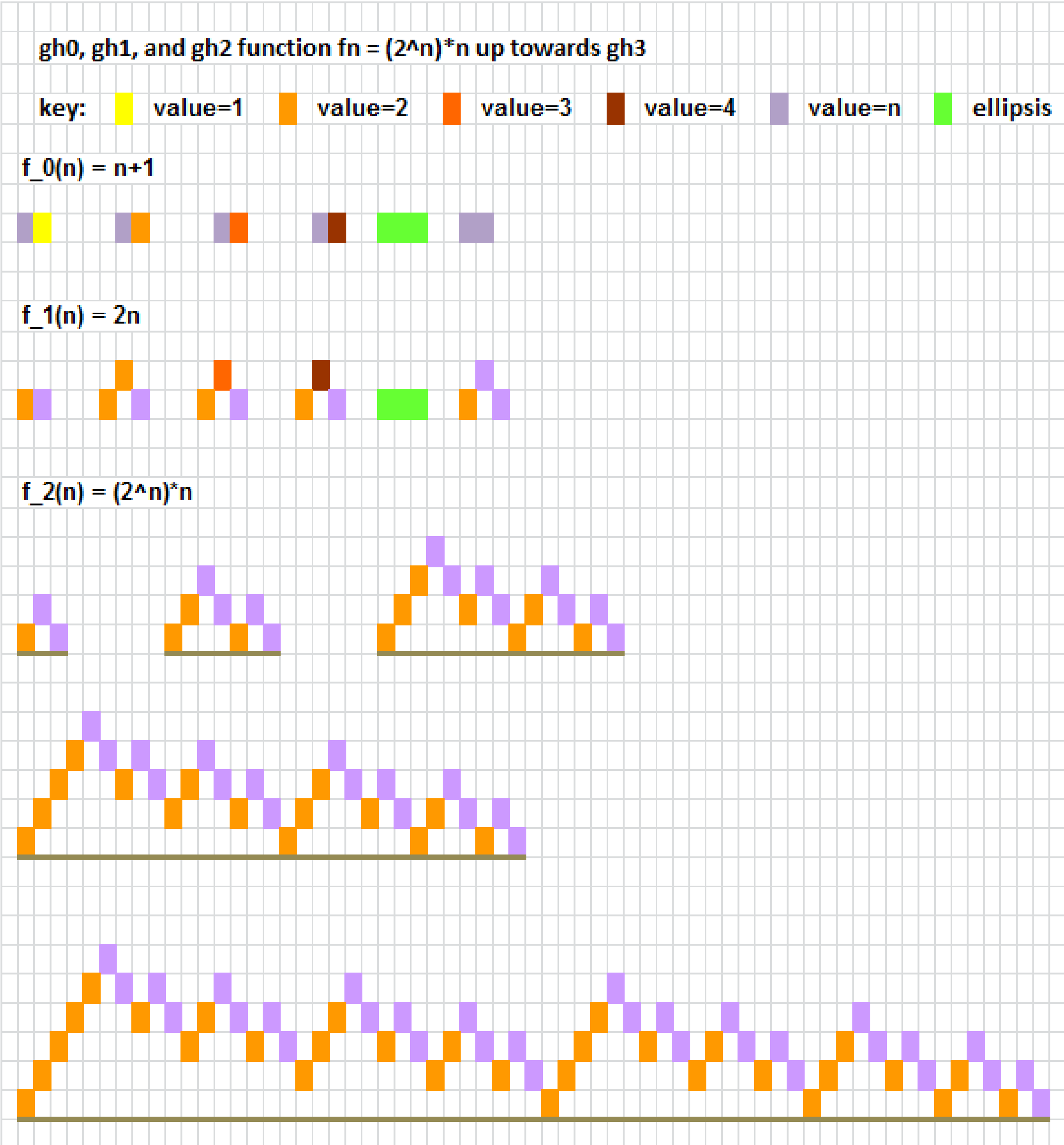


Figure 2.3.2 From $f_3 \approx h(2,3)(n)$ towards f_4 by iterating $g(n) = h(2,3)(n)$

$f_3(n) = f_2^n(n) > h(2,3)(n)$ Iterate the lower approximation $g(n) = h(2,3)(n)$

Figure 2.3.3 From $f_4 \approx h(2,4)(n)$ towards f_5 by iterating $g(n) = h(2,4)(n)$

$f_4(n) = f_3^n(n) > h(2,4)(n)$ Iterate the lower approximation $g(n) = h(2,4)(n)$

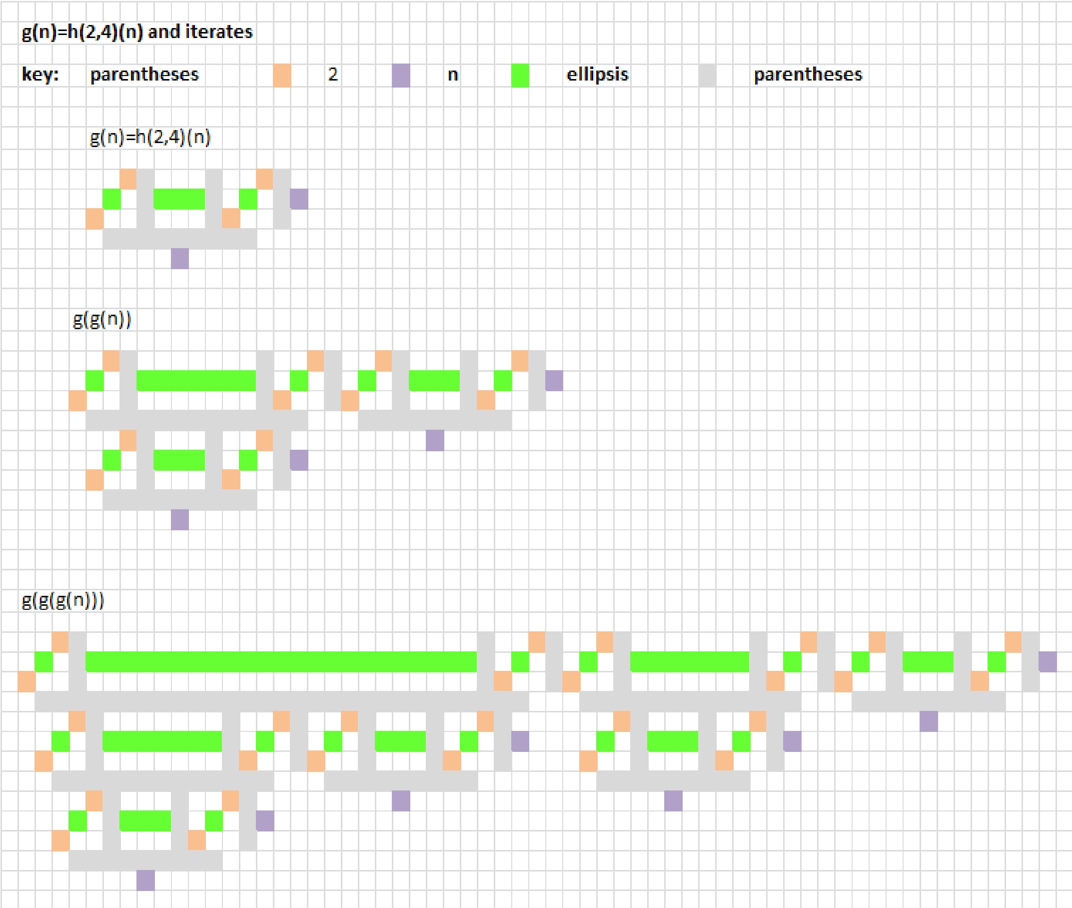


Figure 2.3.4 From $f_5 \approx h(2,5)(n)$ towards f_6 by iterating $g(n) = h(2,5)(n)$

$f_5(n) = f_4^n(n) > h(2,5)(n)$ Iterate the lower approximation $g(n) = h(2,5)(n)$

Section 2.4 An exact formula for f_4 from the GH.

We can find an expression for $f_2^n(n)$:
If we let:

$$a_1 = 2^n n \ , \ a_2 = 2^{2^n n} = 2^{a_1} \ , \ a_3 = 2^{(2^{2^n n})(2^n n)} = 2^{a_2 a_1} \ , \ldots$$

$$f_3(n) = f_2^n(n) = 2^{a_{n-1}\cdots a_3 a_2 a_1} \cdot \cdots . 2^{a_3 a_2 a_1} 2^{a_2 a_1} 2^{a_1} a_1 \qquad (*)$$

The formula (*) is an exact formula for f_3. However, finding an exact expression for the function f_4, is messy. This probably works, but it's not that clear.

$$f_3(f_3(n)) = \underbrace{2^{a_{g_2(n)-1}\cdots a_3 a_2 a_1} \cdot \cdots . 2^{a_3 a_2 a_1} 2^{a_2 a_1} 2^{a_1} a_1}_{\underbrace{2^{a_{g_1(n)-1}\cdots a_3 a_2 a_1} \cdot \cdots . 2^{a_3 a_2 a_1} 2^{a_2 a_1} 2^{a_1} a_1 \ \Rightarrow g_2(n)}_{n \ \Rightarrow g_1(n)}}$$

$$f_3^3(n)) = \underbrace{2^{a_{g_3(n)-1}\cdots a_3 a_2 a_1} \cdot \cdots . 2^{a_3 a_2 a_1} 2^{a_2 a_1} 2^{a_1} a_1}_{\underbrace{2^{a_{g_2(n)-1}\cdots a_3 a_2 a_1} \cdot \cdots . 2^{a_3 a_2 a_1} 2^{a_2 a_1} 2^{a_1} a_1 \ \Rightarrow g_3(n)}_{\underbrace{2^{a_{g_1(n)-1}\cdots a_3 a_2 a_1} \cdot \cdots . 2^{a_3 a_2 a_1} 2^{a_2 a_1} 2^{a_1} a_1 \ \Rightarrow g_2(n)}_{n \ \Rightarrow g_1(n)}}}$$

$$f_3^n(n)) = \underbrace{2^{a_{g_n(n)-1}\cdots a_3 a_2 a_1} \cdot \cdots . 2^{a_3 a_2 a_1} 2^{a_2 a_1} 2^{a_1} a_1}_{\underbrace{2^{a_{g_{n-1}(n)-1}\cdots a_3 a_2 a_1} \cdot \cdots . 2^{a_3 a_2 a_1} 2^{a_2 a_1} 2^{a_1} a_1 \ \Rightarrow g_n(n)}_{\underbrace{\vdots}_{\underbrace{2^{a_{g_1(n)-1}\cdots a_3 a_2 a_1} \cdot \cdots . 2^{a_3 a_2 a_1} 2^{a_2 a_1} 2^{a_1} a_1 \ \Rightarrow g_2(n)}_{n \ \Rightarrow g_1(n)}}}}$$

Part 3 A thought about the ordinal hierarchy

Section 3.1 Hereditary Base(n) comparisons with n = 2, 3, 4, 10

Table 3.1.1

Base10 number	HB2	HB3	HB4	HB10
1	1	1	1	1
2	2	2	2	2
3	2+1	3	3	3
4	2^2	3+1	4	4
5	2^2+1	3+2	4+1	5
6	2^2+2	3.2	4+2	6
7	2^2+2+1	3.2+1	4+3	7
8	2^{2+1}	3.2+2	4.2	8
9	$2^{2+1}+1$	3^2	4.2+1	9
10	$2^{2+1}+2$	3^2+1	4.2+2	10
11	$2^{2+1}+2+1$	3^2+2	4.2+3	10+1
12	$2^{2+1}+2^2$	3^2+3	4.3	10+2
13	$2^{2+1}+2^2+1$	3^2+3+1	4.3+1	10+3
14	$2^{2+1}+2^2+2$	3^2+3+2	4.3+2	10+4
15	$2^{2+1}+2^2+2+1$	$3^2+3.2$	4.3+3	10+5
16	2^{2^2}	$3^2+3.2+1$	4^2	10+6
17	$2^{2^2}+1$	$3^2+3.2+2$	4^2+1	10+7
18	$2^{2^2}+2$	$3^2.2$	4^2+2	10+8
19	$2^{2^2}+2+1$	$3^2.2+1$	4^2+3	10+9
20	$2^{2^2}+2^2$	$3^2.2+2$	4^2+4	10.2
21	$2^{2^2}+2^2+1$	$3^2.2+3$	4^2+4+1	10.2+1
22	$2^{2^2}+2^2+2$	$3^2.2+3+1$	4^2+4+2	10.2+2
23	$2^{2^2}+2^2+2+1$	$3^2.2+3+2$	4^2+4+3	10.2+3
24	$2^{2^2}+2^{2+1}$	$3^2.2+3.2$	$4^2+4.2$	10.2+4
25	$2^{2^2}+2^{2+1}+1$	$3^2.2+3.2+1$	$4^2+4.2+1$	10.2+5
26	$2^{2^2}+2^{2+1}+2$	$3^2.2+3.2+2$	$4^2+4.2+2$	10.2+6
27	$2^{2^2}+2^{2+1}+2+1$	3^3	$4^2+4.2+3$	10.2+7
28	$2^{2^2}+2^{2+1}+2^2$	3^3+1	$4^2+4.3$	10.2+8
29	$2^{2^2}+2^{2+1}+2^2+1$	3^3+2	$4^2+4.3+1$	10.2+9
30	$2^{2^2}+2^{2+1}+2^2+2$	3^3+3	$4^2+4.3+2$	10.3
31	$2^{2^2}+2^{2+1}+2^2+2+1$	3^3+3+1	$4^2+4.3+3$	10.3+1
32	2^{2^2+1}	3^3+3+2	$4^2.2$	10.3+2
33	$2^{2^2+1}+1$	$3^3+3.2$	$4^2.2+1$	10.3+3
34	$2^{2^2+1}+2$	$3^3+3.2+1$	$4^2.2+2$	10.3+4

Section 3.2 Comparing HB3, Ordhb3w, Ordhb3n and a filter code

Table 3.2.1

Base10 number	HB3	Ordhb3w	Ordhb3n	FilterCode
0	0	0	0	0
1	1	1	1	1
2	2	2	2	2
3	3	ω	n	00 00
4	3+1	$\omega+1$	$n+1$	01
5	3+2	$\omega+2$	$n+2$	02
6	3.2	$\omega.2$	$2n$	10
7	3.2+1	$\omega.2+1$	$2n+1$	11
8	3.2+2	$\omega.2+2$	$2n+2$	12
9	3^2	ω^2	n^2	20
10	3^2+1	ω^2+1	n^2+1	21
11	3^2+2	ω^2+2	n^2+2	22
12	3^2+3	$\omega^2+\omega$	n^2+n	000 000 000
13	3^2+3+1	$\omega^2+\omega+1$	n^2+n+1	001
14	3^2+3+2	$\omega^2+\omega+2$	n^2+n+2	002
15	$3^2+3.2$	$\omega^2+\omega.2$	n^2+2n	010
16	$3^2+3.2+1$	$\omega^2+\omega.2+1$	n^2+2n+1	011
17	$3^2+3.2+2$	$\omega^2+\omega.2+2$	n^2+2n+2	012
18	$3^2.2$	$\omega^2.2$	$2n^2$	020
19	$3^2.2+1$	$\omega^2.2+1$	$2n^2+1$	021
20	$3^2.2+2$	$\omega^2.2+2$	$2n^2+2$	022
21	$3^2.2+3$	$\omega^2.2+\omega$	$2n^2+n$	100 100
22	$3^2.2+3+1$	$\omega^2.2+\omega+1$	$2n^2+n+1$	101
23	$3^2.2+3+2$	$\omega^2.2+\omega+2$	$2n^2+n+2$	102
24	$3^2.2+3.2$	$\omega^2.2+\omega.2$	$2n^2+2n$	110
25	$3^2.2+3.2+1$	$\omega^2.2+\omega.2+1$	$2n^2+2n+1$	111
26	$3^2.2+3.2+2$	$\omega^2.2+\omega.2+2$	$2n^2+2n+2$	112
27	3^3	ω^ω	n^n	120
28	3^3+1	$\omega^\omega+1$	n^n+1	121
29	3^3+2	$\omega^\omega+2$	n^n+2	122
30	3^3+3	$\omega^\omega+\omega$	n^n+n	200 200
31	3^3+3+1	$\omega^\omega+\omega+1$	n^n+n+1	201
32	3^3+3+2	$\omega^\omega+\omega+2$	n^n+n+2	202
33	$3^3+3.2$	$\omega^\omega+\omega.2$	n^n+2n	210
34	$3^3+3.2+1$	$\omega^\omega+\omega.2+1$	n^n+2n+1	211
35	$3^3+3.2+2$	$\omega^\omega+\omega.2+2$	n^n+2n+2	212
36	3^3+3^2	$\omega^\omega+\omega^2$	n^n+n^2	220
37	3^3+3^2+1	$\omega^\omega+\omega^2+1$	n^n+n^2+1	221
38	3^3+3^2+2	$\omega^\omega+\omega^2+2$	n^n+n^2+2	222

Section 3.3 Comparing HB4, Ordhb4w and a filter code

Table 3.3.1

Base10 number	HB4	Ordhb4w	FilterCode
0	0	0	0
1	1	1	1
2	2	2	2
3	3	3	3
4	4	ω	00 00
5	4+1	$\omega+1$	01
6	4+2	$\omega+2$	02
7	4+3	$\omega+3$	03
8	4.2	$\omega.2$	10
9	4.2+1	$\omega.2+1$	11
10	4.2+2	$\omega.2+2$	12
11	4.2+3	$\omega.2+3$	13
12	4.3	$\omega.3$	20
13	4.3+1	$\omega.3+1$	21
14	4.3+2	$\omega.3+2$	22
15	4.3+3	$\omega.3+3$	23
16	4^2	ω^2	30
17	4^2+1	ω^2+1	31
18	4^2+2	ω^2+2	32
19	4^2+3	ω^2+3	33
20	4^2+4	$\omega^2+\omega$	000 000
21	4^2+4+1	$\omega^2+\omega+1$	001
22	4^2+4+2	$\omega^2+\omega+2$	002
23	4^2+4+3	$\omega^2+\omega+3$	003
24	$4^2+4.2$	$\omega^2+\omega.2$	010
25	$4^2+4.2+1$	$\omega^2+\omega.2+1$	011
26	$4^2+4.2+2$	$\omega^2+\omega.2+2$	012
27	$4^2+4.2+3$	$\omega^2+\omega.2+3$	013
28	$4^2+4.3$	$\omega^2+\omega.3$	020
29	$4^2+4.3+1$	$\omega^2+\omega.3+1$	021
30	$4^2+4.3+2$	$\omega^2+\omega.3+2$	022
31	$4^2+4.3+3$	$\omega^2+\omega.3+3$	023
32	$4^2.2$	$\omega^2.2$	030
33	$4^2.2+1$	$\omega^2.2+1$	031
34	$4^2.2+2$	$\omega^2.2+2$	032
35	$4^2.2+3$	$\omega^2.2+3$	033
36	$4^2.2+4$	$\omega^2.2+\omega$	100 100
37	$4^2.2+4+1$	$\omega^2.2+\omega+1$	101
38	$4^2.2+4+2$	$\omega^2.2+\omega+2$	102

Section 3.4 Another kind of traversing of the ordinal hierarchy

The idea of defining a traversal through a part of Ord and then assigning filter codes so that filtering can take place in stages is shown in the table below. Note that transfinite ordinals are obtained by replacing 2's in the tree expressions with w's.

Table 3.4.1 HB2 (+1) and applying the filter code one time

(+1) style HB2 and filter code		Applying the filter code one time	
2	0	2	0
$2+1$	1	2^2	00
2^2	00	2^{2+1}	10
2^2+1	01	2^{2^2}	000
2^{2+1}	10	2^{2^2+1}	010
$2^{2+1}+1$	11	$2^{2^{2+1}}$	100
2^{2^2}	000	$2^{2^{2+1}+1}$	110
$2^{2^2}+1$	001	…	
2^{2^2+1}	010		
$2^{2^2+1}+1$	011		
$2^{2^{2+1}}$	100		
$2^{2^{2+1}}+1$	101		
$2^{2^{2+1}+1}$	110		
$2^{2^{2+1}+1}+1$	111		
…			

The table above shows "(+1) style HB2".
We could also use "(.2) style HB2" or "(^2) style HB2" as in the table below.
The table below shows a binary code that can point to values from a tree structure.

Using a method and a code to point to values from a tree structure.

Table 3.4.2 HB2 (+1), (w+1), (w.2) and (w^2)

Code	HB2 (+1)	w+1	w.2	w^2
0	2	ω	ω	ω
1	$2+1$	$\omega+1$	$\omega.2$	ω^2
00	2^2	ω^ω	ω^ω	ω^ω
01	2^2+1	$\omega^\omega+1$	$\omega^\omega.2$	$(\omega^\omega)^2=\omega^{\omega.2}$
10	2^{2+1}	$\omega^{\omega+1}$	$\omega^{\omega.2}$	ω^{ω^2}
11	$2^{2+1}+1$	$\omega^{\omega+1}+1$	$\omega^{\omega.2}.2$	$(\omega^{\omega^2})^2=\omega^{\omega^2.2}$
000	2^{2^2}	ω^{ω^ω}	ω^{ω^ω}	ω^{ω^ω}
001	$2^{2^2}+1$	$\omega^{\omega^\omega}+1$	$\omega^{\omega^\omega}.2$	$(\omega^{\omega^\omega})^2=\omega^{\omega^\omega.2}$
010	2^{2^2+1}	$\omega^{\omega^\omega+1}$	$\omega^{\omega^\omega.2}$	$\omega^{(\omega^\omega)^2}=\omega^{\omega^{\omega.2}}$
011	$2^{2^2+1}+1$	$\omega^{\omega^\omega+1}+1$	$\omega^{\omega^\omega.2}.2$	$(\omega^{(\omega^\omega)^2})^2=\omega^{\omega^{\omega.2}.2}$
100	$2^{2^{2+1}}$	$\omega^{\omega^{\omega+1}}$	$\omega^{\omega^{\omega.2}}$	$\omega^{\omega^{\omega^2}}$
101	$2^{2^{2+1}}+1$	$\omega^{\omega^{\omega+1}}+1$	$\omega^{\omega^{\omega.2}}.2$	$(\omega^{\omega^{\omega^2}})^2=\omega^{\omega^{\omega^2}.2}$
110	$2^{2^{2+1}+1}$	$\omega^{\omega^{\omega+1}+1}$	$\omega^{\omega^{\omega.2}.2}$	$\omega^{(\omega^{\omega^2})^2}=\omega^{\omega^{\omega^2.2}}$
111	$2^{2^{2+1}+1}+1$	$\omega^{\omega^{\omega+1}+1}+1$	$\omega^{\omega^{\omega.2}.2}.2$	$(\omega^{(\omega^{\omega^2})^2})^2=\omega^{\omega^{\omega^2.2}.2}$

The set “code” is the set of all codestrings, c, as in the first column of the table. Define the standard projection functions to identify values in the codestring leftwards from the rightmost position:

$P_i : code \mapsto \{0,1\} \; : P_i(c) = i^{th} \; value \; in \; c$

Now define some filters, (or sieves) that allow focus on different regions of the hierarchy:

$filter\,1 = code - \{c : P_1(c) = 1\}$

$filter\,2 = code - \{c : P_1(c) = 1 \vee P_2(c) = 1\}$

$filter\,3 = code - \{c : P_1(c) = 1 \vee P_2(c) = 1 \vee P_3(c) = 1\}$

These filters can be applied to any of the HB2(+1), (w+1), (w.2) or (w^2) subsequences.

Part 4 The Height Density Problem

Section 4.1 Introduction to the Height Density Problem

The following paragraphs are taken verbatim from Andydude's tetration website:

"There is a problem posted on a wall of the math club at the University of Maryland, by the author Andrew Snowden. He apparently teaches both there and at Princeton, and poses many interesting problems in mathematics.
The problem set was simply called Some More Problems, and contained problems from algebra and number theory to real and complex analysis. The second problem posted caught my attention, as it related to nested exponentials, so I will reproduce it here:
A natural number n may be factored as $n = p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n}$ where the p_n are distinct prime numbers and a_n are natural numbers.
Since the a_n are natural numbers, they may be factored in such a manner as well.
This process may be continued, building a "factorization tree" until all the top numbers are 1. Thus any question that can be asked of trees (i.e. the height of a tree, the number of nodes in a tree, etc) may be asked of our natural number n.
This problem is about the height of n which we denote $h(n)$.
Define:

$$D_n = \lim_{N\to\infty} \frac{1}{N} |\{k \le N : h(k) \ge n\}|$$

D_n is sort of the density of numbers with height at least n.
It is obvious that $D_1 = 1$ since all numbers have height at least 1.
Show that $\frac{1}{2} < ({}^n 2) D_n \le 3$

Let a be the average height of a natural number (ie if you were to pick many numbers at random their height would average out to a). Using the previous part and other methods, give bounds on a.
The best bounds [Andrew Snowden has] are 1.42333 < a < 1.4618 "

I did some brief empirical checking about this curious, but difficult problem.

$$D_1 = \lim_{N\to\infty} \frac{1}{N} |\{k \le N : h(k) \ge 1\}| = \lim_{N\to\infty} \frac{1}{N} |\{1,2,\ldots,N\}| = \lim_{N\to\infty} \frac{1}{N} N = 1$$

Look at N=100. (also, see section 5.2)

D_2 and D_3

$$\textit{with } N = 100, \quad D_2 = \frac{38}{100} \approx 40\% \quad \textit{and } D_3 = \frac{4}{100} \approx 4\%$$

D_4

The first number with height=4 is:

$$65{,}536 = 2^{2^{2^{2}}}$$

Consider these numbers:

$$2^{2^{2}} = 16, \quad 2^{2^{2}}.3 = 48, \quad 3^{2^{2}} = 81$$

So the next number with height=4 is:

$$2^{2^{2^{2}}}.3 = 65{,}536 * 3 = 196{,}608$$

And very approximately, with N=196,608

$$D_4 \approx \frac{1}{65{,}536} \approx 0.0015\% \quad or \quad D_4 \approx \frac{2}{196{,}608} \approx 0.001\%$$

In summary, we have some approximate values for Dn:

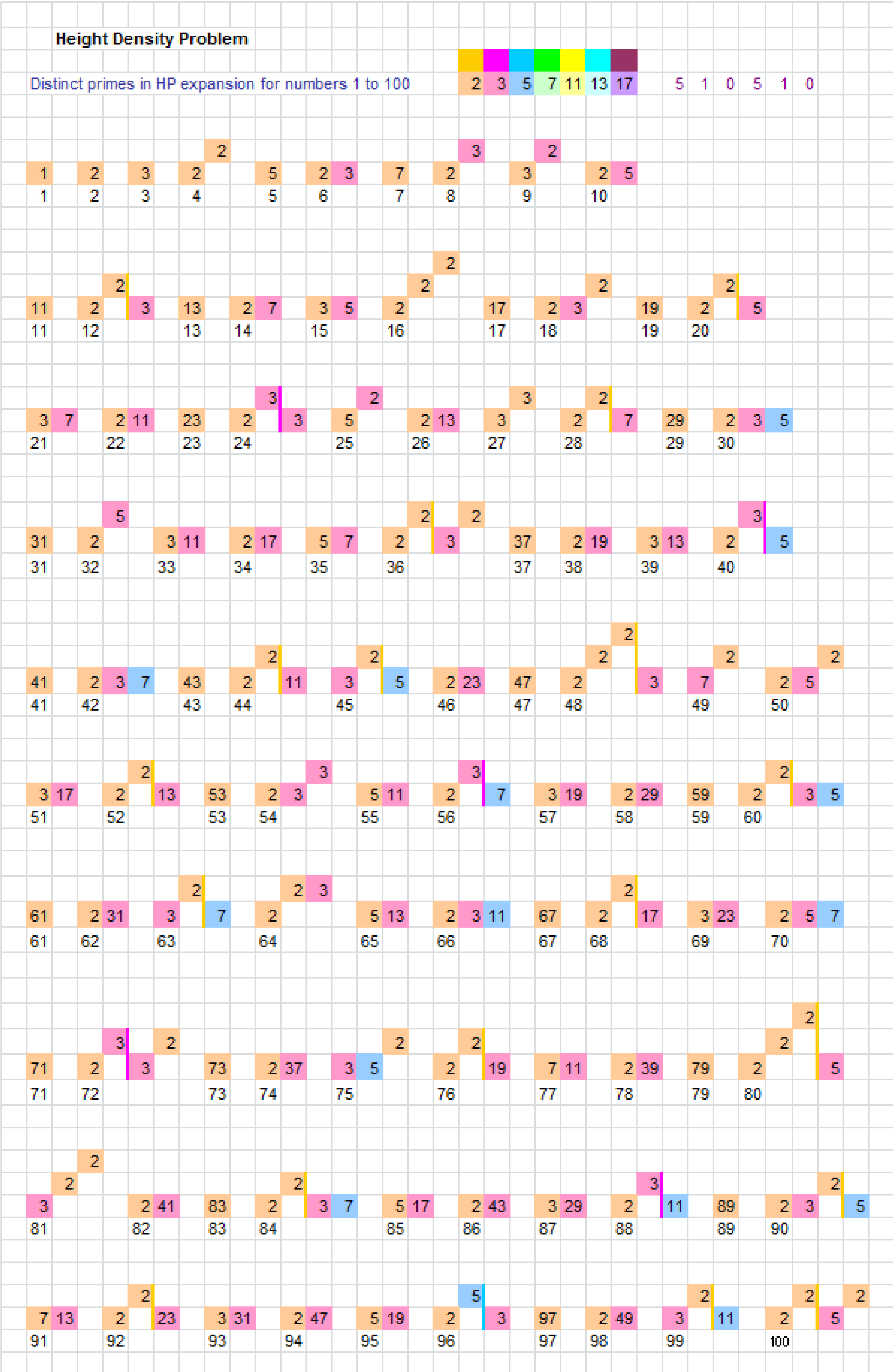
	D1	D2	D3	D4
Dn	100%	40%	4%	0.001%=A
2^^n	2	2^2=4	2^(2^2)=16	2^16=65,536=B
$({}^{n}2)D_n$	2*1=2	4*0.4=1.6	16*0.04=0.64	A*B=0.65536

The table suggests that the bounds in the formula:

$$\frac{1}{2} < ({}^{n}2)D_n \leq 3$$

may be about right,… it seems to me, not easy to make progress on this problem, probably due to inherent computational intractability.

In the next section, I have made some pretty representations using colored square diagrams of the prime factorisation trees that correspond with the first 100 natural numbers.


Height Density Problem
Distinct primes in HP expansion for numbers 1 to 100
2 3 5 7 11 13 17
5 1 0 5 1 0

Part 5 Using the butdj coloring method and colored square diagrams

Figure 5.1 Catalan Number Trees {1,2,5,14,42}

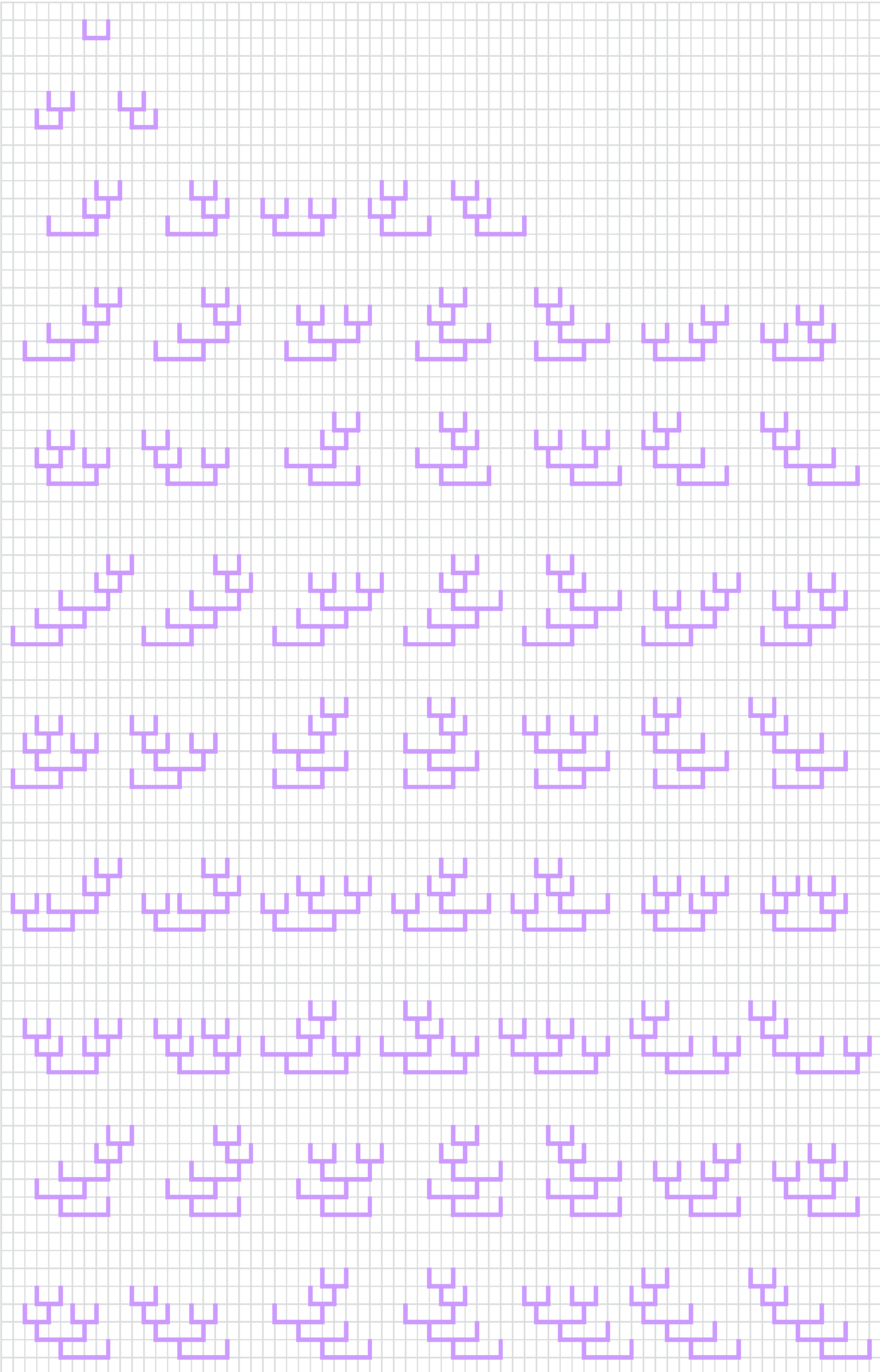


Figure 5.2 Butdj Coloring of Catalan Number Trees {1,2,5,14,42}

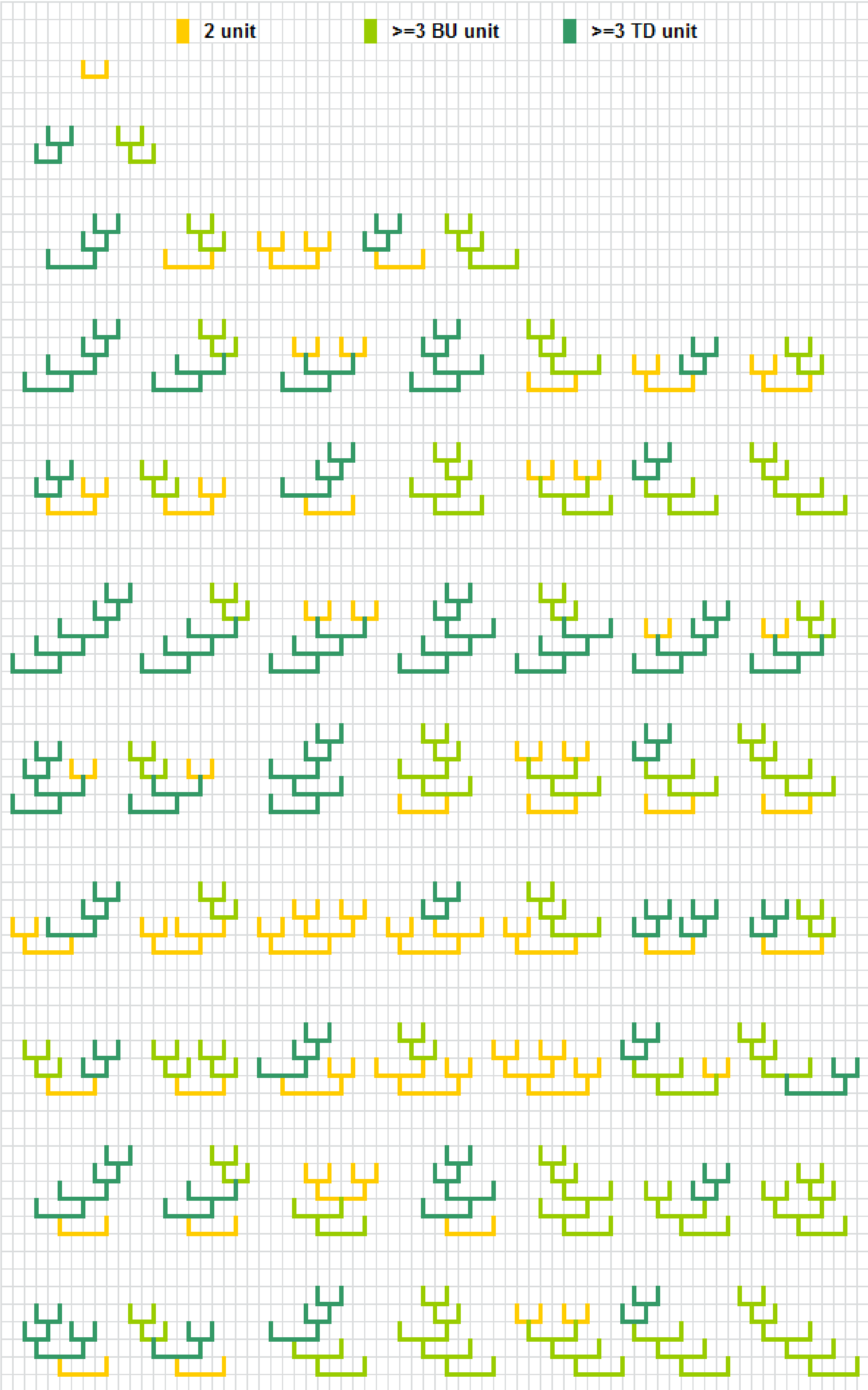


Figure 5.3 Nested Knuth Arrow Towers {Ordertype=4,5,6}

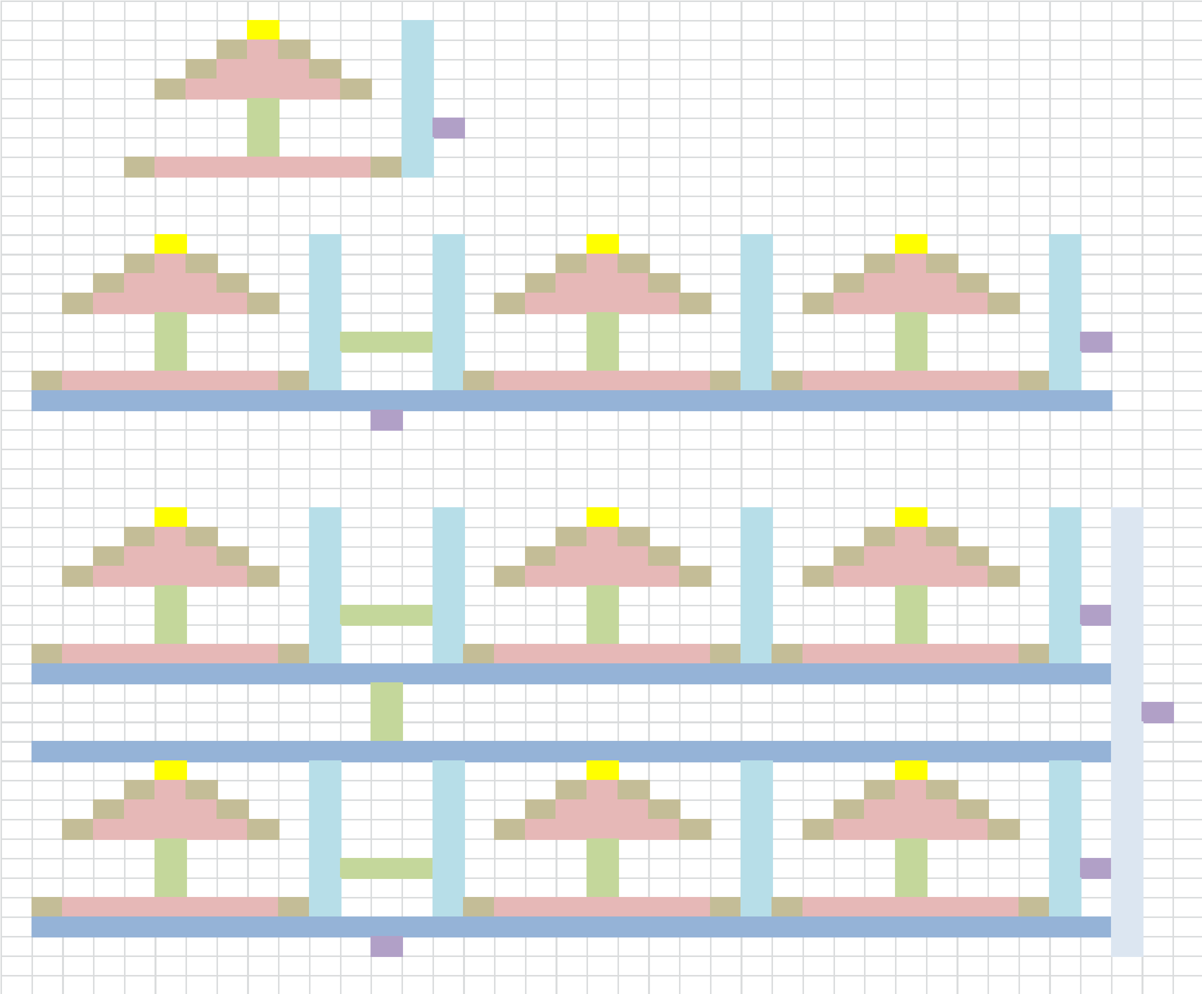
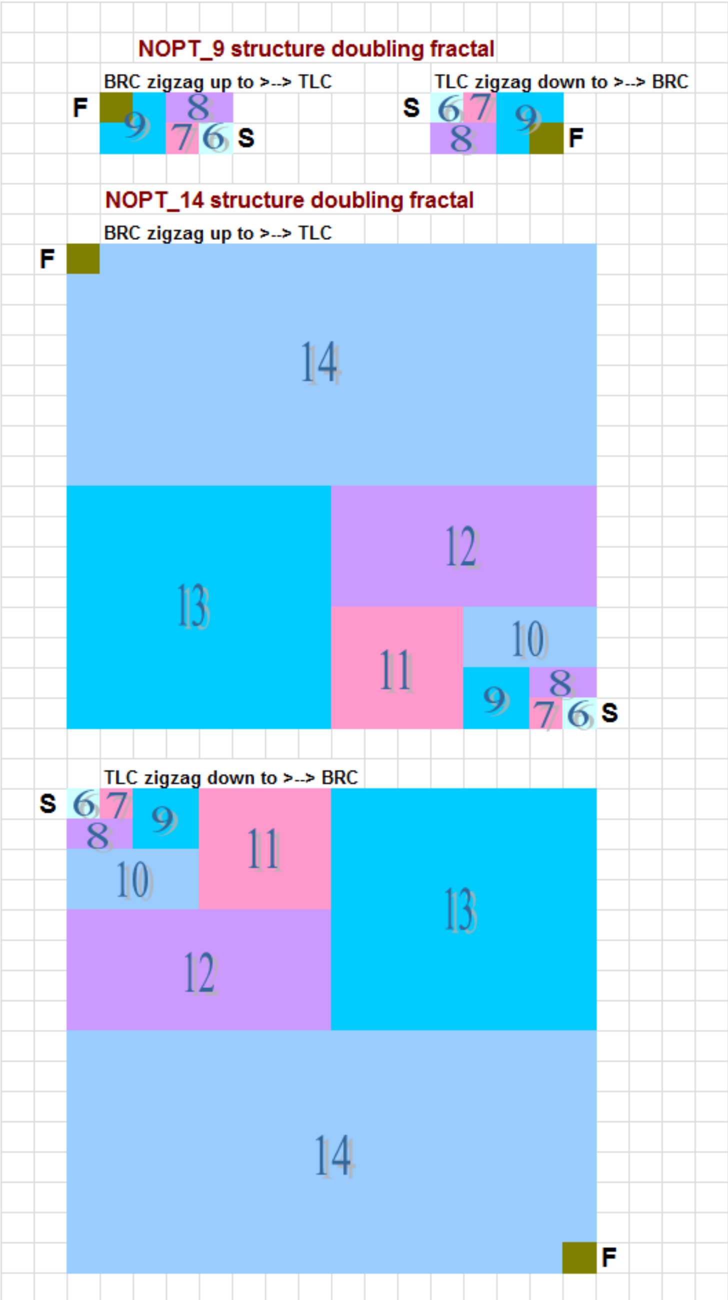


Figure 5.4 Doubling Fractal associated with Nopt structures

If we read a number such as 2048, we read from left to right and the most significant digit is in the "top left corner". For a number such as

$\underbrace{1000...0000}_{100 zeros} = \underbrace{1000...0000}_{101 digits} = 10^{100}$ the most significant digit is on the left and the

explanatory information is often placed below (see Section 1.9 on the repdigit problem) This is the kind of motivation for choosing the "BRC zigzag up to TLC" style of doubling fractal to guide the components of a Nopt structure.
Then the "answer" is at the TLC position, but the "build up" to the "answer" starts from the BRC position.



Part 6 Further Reading and Weblinks

I tend to recommend a searching approach to resources following two complementary approaches: searching on phrases to see how other authors are using language you are currently thinking about and detailed reading when you have enough time, energy, desire and willpower to do this. In particular, there are various websites and papers with ideas and concepts that may gel with varying degrees of success with the ideas in this paper.

Weblinks

Big Number Central
Jonathan Bowers
http://www.polytope.net/hedrondude/bnc.htm

Large numbers
Robert Munafo
http://www.mrob.com/pub/math/largenum.html

Introduction to Nept and Nopt structures
Mike Smith (aka Alister Wilson, Dolti Fantara)
http://math.eretrandre.org/tetrationforum

Hyperoperations and Nopt structures
Alister Wilson

Tetration and higher-order operations on transfinite ordinals
Quickfur
http://math.eretrandre.org/tetrationforum
http://www.teamikaria.org

Exponentials reiterated
R. Arthur Knoebel
Department of Mathematics, New Mexico State University

Parabola Volume 40, Issue 1 (2004)
On Iterated Exponentiation – the Hyperexponentials
Sean Stewart

Catalan Numbers
Tom Davis

Making and Understanding Large Numbers
Peter William Hurford
http://www.greatplay.net/essays/table-of-contents