Nested tetration (by Alister Wilson)

Addition and multiplication are associative operations. This means that for:

$3\cdot n = \underbrace{3+3+...+3+3}_{n}$ and $3^n = \underbrace{3\cdot 3\cdot ...\cdot 3\cdot 3}_{n}$ every bracketing pattern gives the same answer.

But when we're dealing with exponential power towers, the choice of bracketing makes a difference.

A new notation for nested tetration uses a recurrence relation that is defined as follows:

Nested tetration recurrence relation	Corresponding normal tetration
$(3)_{[1]} = 3^{3^3}$	${}^{(3^1)}3 = 3^{3^3}$
$(3)_{[2]} = (3)_{[1]}^{(3)_{[1]}^{(3)_{[1]}}}$	${}^{(3^2)}3 = \left.3^{3^{\cdot^{\cdot^{3}}}}\right\\}3^2$
$(3)_{[n+1]} = (3)_{[n]}^{(3)_{[n]}^{(3)_{[n]}}}$	${}^{(3^{n+1})}3 = \left.3^{3^{\cdot^{\cdot^{3}}}}\right\\}3^{n+1}$

The parentheses and brackets are important for clarity and the notation is extensible.
These numbers provide a way of thinking about finite analogs of the epsilon numbers from set theory, and have some sort of amusement value.
Some examples using base number = 2:

$(2)_{[1]} = \left.2^2\right\}2^1$ $\qquad (2)_{[2]} = (2)_{[1]}^{(2)_{[1]}} = \left.(2^2)^{(2^2)}\right\}2^2$

$(2)_{[3]} = (2)_{[2]}^{(2)_{[2]}} = \left.\left((2^2)^{(2^2)}\right)^{\left((2^2)^{(2^2)}\right)}\right\}2^3$

$(3)_{[1]} = \left.3^{3^3}\right\}3$ $\qquad (3)_{[2]} = \left.(3^{3^3})^{(3^{3^3})^{(3^{3^3})}}\right\}3^2$

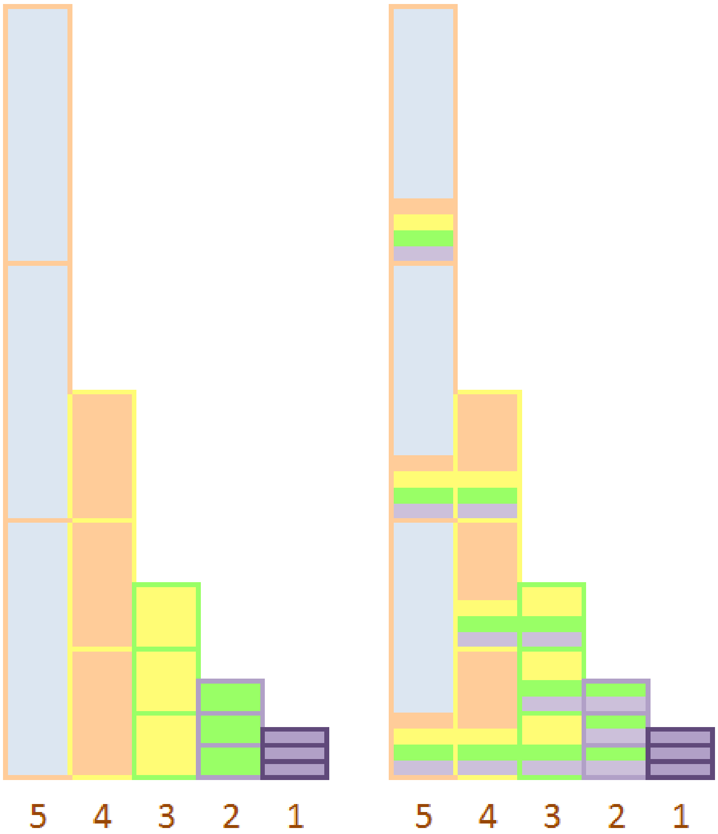
$$(3)_{[3]} = \left.\left((3^{3^3})^{(3^{3^3})^{(3^{3^3})}}\right)^{\left((3^{3^3})^{(3^{3^3})^{(3^{3^3})}}\right)^{\left((3^{3^3})^{(3^{3^3})^{(3^{3^3})}}\right)}}\right\}3^3$$

In general, an expanded notation we can use is:

$$(3)_{[n]} = \left.((3^{3^3})^{\cdots(3^{3^3}))}\right\}3^n$$

It’s an example of something that can be defined, but is more or less impossible to say anything about, hence there is almost nothing in the literature about nested tetration.

But we can sort of visualise nested tetration using Colored Square Diagrams as follows:
Represented as formulae, nested tetration looks clumsy but here are some much prettier pictures that together illustrate the nature of nested tetration:



The idea is that we are representing m nested n-groupings of n’s for m = 1, 2, 3, 4 and 5.
It looks as if n=3 but if we imagine the middle segments of the columns as a “…” then the picture is good for any natural number n>=3. Each column from the right is shrunk by a factor of 2/3, then considered as a group and then piled 3 times high. The second more colourful version emphasizes that 2nd nesting remembers the 1st grouping, the 3rd nesting remembers the 2nd nesting and 1st grouping, the 4th nesting remembers the 3rd and 2nd nestings and the initial 1st grouping and so on. So there is a fractal pattern of multilayered nestings that is the nature of nested tetration.

Visually, there is a correspondence with:

$$\ldots, \quad \varepsilon_2 = (\varepsilon_1^{\varepsilon_1^{\varepsilon_1^{\cdot^{\cdot^{\cdot}}}}}) = (\varepsilon_0^{\varepsilon_0^{\varepsilon_0^{\cdot^{\cdot^{\cdot}}}}})^{(\varepsilon_0^{\varepsilon_0^{\varepsilon_0^{\cdot^{\cdot^{\cdot}}}}})^{(\varepsilon_0^{\varepsilon_0^{\varepsilon_0^{\cdot^{\cdot^{\cdot}}}}})^{\cdot^{\cdot^{\cdot}}}}},$$

$$\varepsilon_1 = (\varepsilon_0^{\varepsilon_0^{\varepsilon_0^{\cdot^{\cdot^{\cdot}}}}}) = (\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}})^{(\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}})^{(\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}})^{\cdot^{\cdot^{\cdot}}}}}, \quad \varepsilon_0 = \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}, \quad \omega$$

$$\ldots, \quad \left.((3^{3^3})^{\ldots(3^{3^3}))}\right\} 3^3,$$

$$\left.((3^{3^3})^{\ldots(3^{3^3}))}\right\} 3^2, \qquad 3^{3^3}, \qquad 3$$

This is enough of a preamble to justify putting epsilon numbers into nopt structures but there's the fiddly aspect of whether there are $\omega, \omega+1, or\ \omega+2$ values in order to reach ε_ω depending on if we include epsilon zero and omega as well as the other epsilon numbers. Because we have a constant limit assumption, I think that $\omega+2$ is the correct choice as in the OT5 nopt structure:

$$\underbrace{\varepsilon_\omega, \ldots, \varepsilon_2, \varepsilon_1, \varepsilon_0, \omega}_{\omega+2}$$

Then we can consider…

$$\underbrace{\varepsilon_{\varepsilon_\omega}, \ldots, \varepsilon_2, \varepsilon_1, \varepsilon_0, \omega}_{\underbrace{\varepsilon_\omega, \ldots, \varepsilon_2, \varepsilon_1, \varepsilon_0, \omega}_{\omega+2}} \quad \text{and} \quad \underbrace{\varepsilon_{\varepsilon_{\varepsilon_0}}, \ldots, \varepsilon_2, \varepsilon_1, \varepsilon_0, \omega}_{\underbrace{\varepsilon_{\varepsilon_0}, \ldots, \varepsilon_2, \varepsilon_1, \varepsilon_0, \omega}_{\varepsilon_0}}$$

and so on and it is clear that the OT6 nopt structure corresponds with the famous ordinal $\varepsilon_{\varepsilon_{\varepsilon_{\ddots}}}$.

The recurrence relation from above is repeated in the table below:

Nested tetration with base = 3	Nested tetration for natural number, base = b
$(3)_{[1]} = 3^{3^3}$	$(b)_{[1]} = b \uparrow\uparrow b$
$(3)_{[2]} = (3)_{[1]}^{(3)_{[1]}^{(3)_{[1]}}}$	$(b)_{[2]} = (b \uparrow\uparrow b) \uparrow\uparrow b$
$(3)_{[3]} = (3)_{[2]}^{(3)_{[2]}^{(3)_{[2]}}}$	$(b)_{[3]} = ((b \uparrow\uparrow b) \uparrow\uparrow b) \uparrow\uparrow b$
$(3)_{[n+1]} = (3)_{[n]}^{(3)_{[n]}^{(3)_{[n]}}}$	$(b)_{[n+1]} = \left.(b)_{[n]}^{\cdot^{\cdot^{\cdot (b)_{[n]}}}}\right\} b$

The recurrence relation does pentation with bottom-up bracketing:

$$(3)_{[1]} = {}^{3}3, \quad (3)_{[2]} = {}^{3}({}^{3}3), \quad (3)_{[3]} = {}^{3}({}^{3}({}^{3}3)), \ldots$$

Now we can explore the recurrence relation a little:

$$(3)_{[1]} = 3^{3^3} = {}^{3}3$$

$$((3)_{[1]})_{[1]} = ({}^{3}3)_{[1]} = {}^{({}^{3}3)}({}^{3}3)$$

$$(((3)_{[1]})_{[1]})_{[1]} = {}^{({}^{({}^{3}3)}({}^{3}3))}({}^{({}^{3}3)}({}^{3}3))$$

And so on…

Also, suppose we consider the sequence:

$$3^{3^3}, \quad \left.((3^{3^3})^{\cdots^{(3^{3^3}))}}\right\} 3^{3^3}\ nestings, \quad \left.((3^{3^3})^{\cdots^{(3^{3^3}))}}\right\} \left.((3^{3^3})^{\cdots^{(3^{3^3}))}}\right\} 3^{3^3}\ nestings$$

so the right hand value is counting the number of nestings.

How does this relate to the nested tetration recurrence relation?

The relevant subsequence from the recurrence relation is:

$$(3)_{[1]}, \quad (3)_{\left[(3)_{[1]}\right]}, \quad (3)_{\left[(3)_{\left[(3)_{[1]}\right]}\right]}, \ldots$$

there may be other interesting things to say about the nested tetration recurrence relation.

Anyway it’s so esoteric that it’s verging on silliness. Never mind. Such is life. I guess.

The number of structural bracketing patterns for the nested tetration recurrence relation on n symbols is C(n), the nth Catalan number:

2 elements, C(2)=1 possibility

$$(a)_{[a]}$$

3 elements, C(3)=2 possibilities

$$((a)_{[a]})_{[a]} \qquad (a)_{\left[(a)_{[a]}\right]}$$

4 elements, C(4)=5 possibilities

$$(((a)_{[a]})_{[a]})_{[a]} \qquad ((a)_{\left[(a)_{[a]}\right]})_{[a]} \qquad ((a)_{[a]})_{\left[(a)_{[a]}\right]}$$

$$(a)_{\left[((a)_{[a]})_{[a]}\right]} \qquad (a)_{\left[(a)_{\left[(a)_{[a]}\right]}\right]}$$

Some more examples based on these structural patterns…

2 elements, C(2)=1 possibility

$$(a)_{[a]}$$

Examples

$$(2)_{[1]} = {}^{2}2 = 2^2 = 4 \qquad (2)_{[2]} = (2)_{[1]}^{\;(2)_{[1]}} = (2^2)^{(2^2)} = 4^4$$

3 elements, C(3)=2 possibilities

$$((a)_{[a]})_{[a]} \qquad (a)_{\left[(a)_{[a]}\right]}$$

Examples for $((a)_{[a]})_{[a]}$

$$((2)_{[1]})_{[1]} = {}^{4}4 \qquad ((2)_{[2]})_{[1]} = {}^{(4^4)}(4^4)$$

$$((2)_{[1]})_{[2]} = (4)_{[2]} = \left.(4)_{[1]}^{\cdot^{\cdot^{\cdot^{(4)_{[1]}}}}}\right\}4 = ({}^{4}4)^{({}^{4}4)^{({}^{4}4)^{({}^{4}4)}}} = {}^{4}({}^{4}4)$$

$$((2)_{[2]})_{[2]} = (4^4)_{[2]} = \left.(4^4)_{[1]}^{\cdot^{\cdot^{\cdot^{(4^4)_{[1]}}}}}\right\}4^4 = \left.\left({}^{(4^4)}(4^4)\right)^{\cdot^{\cdot^{\cdot^{\left({}^{(4^4)}(4^4)\right)}}}}\right\}4^4 = {}^{(4^4)}({}^{(4^4)}(4^4))$$

Examples for $(a)_{\left[(a)_{[a]}\right]}$

$$(2)_{[(2)_{[1]}]} = (2)_{[4]} = ((2^2)^{\cdots(2^2))}\Big\}2^4 = \left((4^4)^{(4^4)}\right)^{\left((4^4)^{(4^4)}\right)}$$

$$(2)_{[(2)_{[2]}]} = (2)_{[4^4]} = ((2^2)^{\cdots(2^2))}\Big\}2^{4^4}$$

4 elements, C(4)=5 possibilities

$(((a)_{[a]})_{[a]})_{[a]}$ $\quad ((a)_{\left[(a)_{[a]}\right]})_{[a]}$ $\quad ((a)_{[a]})_{\left[(a)_{[a]}\right]}$

$(a)_{\left[((a)_{[a]})_{[a]}\right]}$ $\quad (a)_{\left[(a)_{\left[(a)_{[a]}\right]}\right]}$

Examples for $(((a)_{[a]})_{[a]})_{[a]}$

$(((2)_{[1]})_{[1]})_{[1]} = {}^{({}^{4}4)}({}^{4}4)$ $\quad (((2)_{[2]})_{[1]})_{[1]} = {}^{\left({}^{(4^4)}(4^4)\right)}\left({}^{(4^4)}(4^4)\right)$

$$(((2)_{[1]})_{[2]})_{[1]} = {}^{({}^{4}({}^{4}4))}({}^{4}({}^{4}4))$$

$$(((2)_{[2]})_{[2]})_{[1]} = {}^{({}^{(4^4)}({}^{(4^4)}(4^4)))}({}^{(4^4)}({}^{(4^4)}(4^4)))$$

$$(((2)_{[1]})_{[1]})_{[2]} = {}^{({}^{4}4)}\left({}^{({}^{4}4)}({}^{4}4)\right)$$

$$(((2)_{[2]})_{[1]})_{[2]} = {}^{\left({}^{(4^4)}(4^4)\right)}\left({}^{\left({}^{(4^4)}(4^4)\right)}\left({}^{(4^4)}(4^4)\right)\right)$$

Examples for $((a)_{[(a)_{[a]}]})_{[a]}$

$$((2)_{[(2)_{[1]}]})_{[1]} = {}^{\left(\left.((2^2)^{\dots(2^2))}\right\}2^4\right)}\left(\left.((2^2)^{\dots(2^2))}\right\}2^4\right)$$

$$= \left(\left.((2^2)^{\dots(2^2))}\right\}2^4\right)^{\cdot^{\cdot^{\cdot^{\left(\left.((2^2)^{\dots(2^2))}\right\}2^4\right)}}}}\left.\vphantom{\int_0^1}\right\}\left(\left.((2^2)^{\dots(2^2))}\right\}2^4\right)$$

$$((2)_{[(2)_{[1]}]})_{[2]} = {}^{\left(\left.((2^2)^{\dots(2^2))}\right\}2^4\right)}\left({}^{\left(\left.((2^2)^{\dots(2^2))}\right\}2^4\right)}\left(\left.((2^2)^{\dots(2^2))}\right\}2^4\right)\right)$$

$$((2)_{[(2)_{[2]}]})_{[1]} = {}^{\left(\left.((2^2)^{\dots(2^2))}\right\}2^{4^4}\right)}\left(\left.((2^2)^{\dots(2^2))}\right\}2^{4^4}\right)$$

Examples for $((a)_{[a]})_{[(a)_{[a]}]}$

Recall that: $(2)_{[1]} = {}^2 2 = 2^2 = 4$ and $(2)_{[2]} = (2)_{[1]}^{(2)_{[1]}} = (2^2)^{(2^2)} = 4^4$

$$((2)_{[1]})_{[(2)_{[1]}]} = (4)_{[4]} = ((({}^4 4)\uparrow^2 4)\uparrow^2 4)\uparrow^2 4 = {}^4({}^4({}^4({}^4 4)))$$

$$= ((4^{4^{4^4}})^{\cdot^{\cdot^{\cdot^{(4^{4^{4^4}}))}}}}\left.\vphantom{\int_0^1}\right\} 4\ nestings = ((4^{4^{4^4}})^{\cdot^{\cdot^{\cdot^{(4^{4^{4^4}}))}}}}\left.\vphantom{\int_0^1}\right\}4^4$$

Remembering that, in general, an expanded notation we can use is:

$(3)_{[n]} = \left.((3^{3^3})^{\dots(3^{3^3}))}\right\}3^n$ so $(4^4)_{[n]} = \left.\left(\left({}^{(4^4)}(4^4)\right)^{\dots\left({}^{(4^4)}(4^4)\right))}\right.\right\}(4^4)^n$

Therefore:

$$((2)_{[2]})_{[(2)_{[2]}]} = (4^4)_{[4^4]} = \left.\left(\left({}^{(4^4)}(4^4)\right)^{\dots\left({}^{(4^4)}(4^4)\right))}\right.\right\}(4^4)^{(4^4)}$$

Examples for $(a)_{[((a)_{[a]})_{[a]}]}$

$$(2)_{[((2)_{[1]})_{[1]}]} = (2)_{[{}^{4}4]} = ((2^2)^{\dots(2^2)}) \Big\} 2^{({}^{4}4)}$$

$$(2)_{[((2)_{[2]})_{[2]}]} = (2)_{[{}^{(4^4)}({}^{(4^4)}(4^4))]} = ((2^2)^{\dots(2^2)}) \Big\} 2^{\left({}^{(4^4)}({}^{(4^4)}(4^4))\right)}$$

These expressions are complex and can be tricky to follow. If we refer to the notational definitions described earlier in this paper, we can consider the expression from the last equality above in terms of the number of nestings and the number of 2's in the nested bracketing exponential power tower. The number of 2's in the left tower is the number to the right of the brace, that is:

$$2^{\left({}^{(4^4)}({}^{(4^4)}(4^4))\right)}$$

The number of nestings in the tower of 2's to the left of the brace is the exponent of the above expression, that is:

$$\left({}^{(4^4)}({}^{(4^4)}(4^4))\right) = \left((4\uparrow 4)\uparrow\uparrow(4\uparrow 4)\right)\uparrow\uparrow(4\uparrow 4)$$

I think this shows quite well how some describable large natural numbers can be very complex to comprehend. The numbers usually described in number theory are, of course, much less complex and a lot more suitable for arithmetic manipulations and combinatorial interpretations. Anyway, I think it's interesting that the idea of nested tetration is possible to partially understand, at least at the level of symbolic manipulations.

Example for $(a)_{\left[(a)_{[(a)_{[a]}]}\right]}$

Recall that:

$$(2)_{[(2)_{[1]}]} = (2)_{[4]} = ((2^2)^{\dots(2^2)}) \Big\} 2^4 = \left((4^4)^{(4^4)}\right)^{\left((4^4)^{(4^4)}\right)}$$

$$(2)_{\left[(2)_{[(2)_{[1]}]}\right]} = ((2^2)^{\dots(2^2)}) \Big\} \left[2^{\left((4^4)^{(4^4)}\right)^{\left((4^4)^{(4^4)}\right)}} \text{"2's"}\right]$$

Therefore…

$$= ((2^2)^{\dots(2^2)}) \Big\} \left[\left((4^4)^{(4^4)}\right)^{\left((4^4)^{(4^4)}\right)} \textit{nestings}\right]$$

The first equality gives the height of the nested power tower in terms of the number of 2's and the second equality expresses the depth of the same power tower in terms of the number of nestings and this number is actually the exponent of the height number.